GAUHATI UNIVERSITY
Institute of Distance and Open Learning

M101

Real Analysis and Lebesgue Measure

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Gauhati University

Institute of Distance and Open Learning

M.A/M.Sc Syllabus in Mathematics

Issued by
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Institute of Distance and Open Learning
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Assam : India

The M.A/ M.Sc syllabus of IDOL has been restructured for the semester system on the basis of the guidelines of the UGC. There are four semesters in two years; each semester comprising of five papers. In the 3rd semester 3 papers are common to all and the other two papers are optional. In the 4th semester two papers are common and three papers are optional. Questions will be set from each unit, proper weightage will be given unit wise and marks from each unit is shown accordingly. Examination will be held at the end of every semester. Each examination paper will carry 64 marks. In each paper there will be an internal assessment of 16 marks.

Semester	**Credit**
Semester I	
Paper- M101: Real Analysis and Lebesgue Measure	6
Paper-M 102: Topology	6
Paper-M 103: Algebra	6
Paper-M 104: Differential Equation	6
Paper-M105: Tensor and Mechanics	6
Semester II	
Paper- M201:Complex Analysis	6
Paper-M202:Functional Analysis	6
Paper-M 203: Hydrodynamics	6
Paper-M 204: Mathematical Methods	6
Paper-M205: Operation Research	6
Semester III	
Paper- M301: Computer Programming in C (Theory and Practical)	6
Paper-M302: Number Theory	6
Paper-M 303: Continuum Mechanics	6
Paper-M 304: Algebra II/Space Dynamics	6
Paper-M305: Special Theory of Relativity / Mathematical Logic	6
Semester IV	
Paper- M401: Graph Theory	6
Paper-M402: Numerical Analysis	6
Paper-M403: Functional Analysis II/ Fluid Dynamics	6
Paper-M404: Mathematical Statistics /Dynamical System.	6
Paper-M405: Fuzzy Sets and their Applications/ General Theory of Relativity and Cosmology.	6

Semester I

M101: Real Analysis and Lebesgue Measure

Unit 1: (Marks-20)

Uniform convergence at an interval. Cauchy's criterion. Test for uniform convergence. Properties of uniformly convergent sequences and series of functions. Uniform convergence and continuity. Integration, differentiation weirstrass approximation theorem (Statement only) and its application. Uniqueness theorem for power series. Abel's and Tauber's theorem. Fundamental properties.

Unit 2:(Marks-20)

Function of bounded variation, continuity, Differentiation, their continuity and monotonicity. Definition and Existence of R-S integral, properties of R-S integral integration and differentiation, fundamental theorem of calculus.

Unit 3:(Marks-20)

Lebesgue outer measure, Measurable sets and properties. Borel sets and their measurability characterization of measurable sets, Non-measurable sets, Measurable function, Properties, Operation of measurable function, sets of measure zero.

Unit 4:(Marks-20)

Lebesgue integral, Lebesgue integral of a bounded function, comparison of Riemann integral and Lebesgue integral. Integral of non negative measurable function, General Lebesgue integral. Convergence of Lebesgue integral, Bounded convergence théorem (statement only) Monotone Convergence theorem (statement only), Lebesgue Convergence theorem (statement only)..

Text Books:

1. Malik and Arora-Mathematical Analysis
2. L. Royden- Real analysis, Prentice Hall of India,

Reference Books:

1. W. Rudin, Principles of Mathematical Analysis, 3rd Edition, McGraw Hill
2. Real Analysis-Goldberg
3. Real Analysis-Dipak Chatterjee, Prentice Hall of India
4. Jain and Gupta-Lebesgue Measure and integration, Willey Eastern Ltd

M102 : Topology

Unit 1:(marks-20) Metric Space:

Convergence of sequences, completeness, Bair's theorem, continuous mappings, spaces of continuous function's Euclidean and unitary spaces.

Unit 2:(marks-10)Topological Space:

Continuity and homeomorphism, subspace, bases and sub bases. Weak topologies.

Unit 3:(marks-20) Compactness:

Compact spaces, product spaces, Tychonoffs theorem and locally compact spaces. Compactness for metric spaces, Ascoli's theorem.

Unit 4(marks-20)Separations:

TI-space and Hausdorff spaces, Completely regular spaces and normal spaces, Urysohn's lemma and Tietze extension theorem.

Unit 5:(marks-10)Connectedness:

Connected spaces, components of a space, totally disconnected spaces, locally connected spaces.

Books Recommended:

Simmons G.F. : Introduction to Topology and Modem Analysis, McGraw Hill

M103: Algebra

Unit l:(marks-20)

Direct product and Direct sums of Groups, Decomposable groups, Normal and Subnormal series of groups, Composition series, Jordan Holder theorem, solvable groups.

Unit 2;(marks-20)

Divisibility in Commutative rings, PID, UFD and their properties, Eisenstein's irreducibility criterion.

Unit 3:(marks-20)

Field theory-Extension fields, Algebraic and Transcendental, Splitting field, perfect fields, Finite fields (Moore's theorem etc.). Construction by ruler and compass, elements of Galois theory.

Unit 4:(marks-20)

Canonical forms, similarity of linear transformations, Invariant subspaces, Reduction to triangular forms, nilpotent transformations, index of nil potency, invariants of a nilpotent Transformation, Primary decomposition theorem, Jordan blocks and Jordan forms.

Text Books:

1. S.Singh and Zameruddin-Modern Algebra
2. Hoffman and Kunz-Linear Algebra

Reference Books:

1. I.N.Herstein-Topic in Algebra
2. C.Musili-Rings and Modules
3. D.S. Malik, J.N.Mordien, M.K.Sen-Fundamental of Abstract Algebra.
4. K.B.Dutta-Matrix and linear algebra.
5. Liner Algebra-S.Liptestuz. Schaum's outline series.

M104: Differential Equation.

Unit 1: (marks-20)

Solution of 2nd order differential equations with variable coefficients including method of variation of parameters. Statement only Existence theorem of 1st order equation, Statements of existence theorems for system of 1st order equations and for nth order differential equations, Wronskian.

Unit 2:(marks-20)

Method of series solution of 2nd order differential equations with particular reference to legendre. Bessel and Gauss. Simultaneous differential equations and total differential equations.

Unit 3:(marks-20)

Origin of partial differential equations of 1st order, LaGrange's method of solving 1st order linear partial differential equations. Particular solutions under various prescribed conditions. Linear homogeneous equations with more than two independent variables.

Unit 4:(marks-20)

Char pit's method of solving non-linear 1st order partial differential equations. Complete Integrals. Standard forms of non-linear 1s order partial differential equations.

Books Recommended:

1. Theory and problems of differential equations-Frank Ayres Jr. Schaum's Outline Series,McGraw Hill.
2. Advance Differential Equations-Raisinghannia.
3. Partial Diffrential Equation-Gupta Malik and Mittal Pragati Prakashan

M105 Tensors & Mechanics

Unit l:(marks-10)

Transformation of coordinates, summation convention, Kronecker delta, definition of tensors covariant, contra variant and mixed tensor, Cartesian tensors, rank of a tensor, symmetric and antisymmetric tensors, outer and inner product of tensors, contraction, quotient law. Riemannian space, metric tensor, fundamental tensors, associate tensors, magnitude of a vector, angle between two vectors Parametric curves.

Unit 2:(marks-10)

Christoffel's three-index symbols (or brackets) and properties, covariant differentiation of tensors, divergence and curl of a vector and gradient of a scalar.

Intrinsic derivatives, curvature of a curve, parallel displacement of vectors.

Unit 3:(marks- 10)

Forces in three dimension and general conditions of equilibrium, Poisot's central axis, wrench, cylindroids.

Unit 4(marks- 10)

Virtual works, bending moments, equilibrium of slightly elastic beams, general equations of a bent rod, equations of three moments, work done in bending a rod.

Unit 5:(marks-10)

Newton's laws and inertial frame of reference, general equations of motions, conservative force fields, general principle of conservation of energy, linear momentum and angular momentum.

Unit 6:(marks-1 0)

Motion in two dimensions, motion under a central force with particular reference to inverse square law of force, Kepler's laws of planetary motions, two body problem, motion in resisting medium and motion when the mass varies.

Unit 7:(marks-10)

Motion in three dimensions, velocity and acceleration in cylindrical and spherical polar coordinates, motion on cylindrical spherical and conical surfaces.

Unit 8:(marks- 10)

Revision of moments and general equations of motion of rigid body, motion in two dimensions under finite and impulsive forces, expression for K.E.; motion about a fixed axis.

Books recommended:

Tensors:

1. Agarwal D.C.: Tensor Calculus of Riemainan geometry
2. Ayers: Vectors and introduction to Tensor
3. Jeffreys and Jeffreys: Cartesian Tensor
4. Lass: Vector and Tensor Analysis
5. Sharma G.C. and Singh S.K.: A Text Book of Tensor and Riemannian Geometry.
6. Weatherburn: Riemannian Geometry

Statics:

1. Lamb: Statics(CUP)
2. Loney: Statics(CUP)
3. Ramsey: Statics(Cup)
4. Tyagi, Nand and Sharma: Statics, Krishna Prakashan mandir

Dyanamics:

1. Chorlton; Text Books of dynamics, Van Nostrand
2. Goldstein: Classical mechanics, Addison Wesley
3. Loney S.K.: Dynamics of a particle and of rigid bodies (CUP)
4. Ramsey: Dyanamics PartII
5. Singe and Griffith: Principles of mechanics, McGraw Hill
6. Spiegel M: Dyabanamics Part II

Semester 2

M201Complex Analysis

Unit I(Marks-20)

Analytic functions: The Cauchy Riemann equations, harmonic functions, elementary function's many valued functions.

Analytic functions as mappings: Isogonal and conformal Transormations. Bilinear transformations: geometrical inversion, coaxial circles, invariance of the cross-ratio. Fixed points of a bilinear transformation; some special bilinear transformations, e.g. real axis on itself, unit circle on itself, real axis on the unit circle etc.

Branch point and branch lines, concept of the Riemann Surface.

Unit 2: (Marks-20)

Integral along oriented curve, Cauchy's theorem, the Cauchy-Goursat theorem; Cauchy's integral and functions defined by integrals, the derivatives of a regular function; Morera's theorem, Cauchy's inequality, Liouville's theorem; Maximum modulus principle.

Unit 3: (Marks-20)

Taylor's and Laurent's theorem: Zeros and singularities, their classification, poles and zeros of meromorphic functions.

The argument theorem, "Rouche's theorem, location of roots of equations.

Unit 4: (Marks-20)

The residue theorem: evaluation of integrals by contour integration, special theorems used in evaluating integrals.

Books Recommended:

Phillips E.C.: Function of a Complex Variable, Oliver and Boyd

Shanti Narayan : Theory of Functions of a complex Variables, S. Chand and Co

Spiegel Murry R: Theory and Problems of Complex variables, Scheum's Outline Series TMH

M202 Functional Analysis

Unit 1: (Marks -20)

Banach Space: Definitions and some examples, Basic properties, continuous linear transformation, finite demensional normed linear spaces.

Unit 2:(Marks-20)

Hahn-Banach theorem, natural embedding of NLS*, open mapping theorem, closed graph theorem, Banach Steinhaus theorem, conjugate of an operator.

Unit 3:(Marks-20)

Hilbert Spaces: Definition and simple properties, orthogonal complement, orthogonal sets, conjugate space H*, adjoint of an operator, self adjoint of an operator, normal and unitary operator projection.

Unit 4:(Marks-20)

Finite-dimensional spectral Theory: Spectrum of an operator, spectral theorem.

Books recommended:

Simmons G.F.: Introduction to Topology and Modern Analysis, McGraw Hill

Books for reference:

Lahiri, BK.: Functional Analysis

Limaye, B.V.: Functionla Analysis

M203: Hydrodynamics

Unit 1 (Marks-20):

Kinematics of fluid motion: Path lines stream lines equations of continuity equation of motion and their integrals boundary conditions. Impulsive motions. Analysis of fluid motion and general theory of irrotational motion.

Unit 2 (Marks-20):

Motion in a plane: Use of Complex potential. Source. Sink doublet. Method of images. The Circle theorem. The theorem of Blasius. Motion past circular cylinder.

Unit 3:(Marks-20):

Motion in space: Motion past a sphere axisymmetric motion. Stoke's stream function and its use.

Unit 4(Marks-20):

Vortex motion : Properties of vortex filament motion due to rectilinear vortex and a system of vortices motion of a vortex filament due to the influence of others. Ranking vortex.

Text Books:

1. Continuum Mechanics -G.E.Mase. Schaum's outline series. McGraw Hill Book Company
2. A Treatise on Hydromechanics. Part II. W.H.Besant and A.S. Ramsay. CBS Publishers. Delhi
3. Text Book of Fluid Dynamics-Frank Chorlton. C.B.S Publishers. Delhi

Reference books:

1. Mathematical Theory of Continuum Mechanics-R. Chatterjee. Narosa Publishing House. New Delhi.
2. An Introduction to Fluid Mechanics-G.K.Batchelor. Foundation Books. New Delhi.
3. Hydrodynamics- M.D. Raisinghania S. Chand and Co. Limited.

M204: Mathematical Methods

Unit l: (Marks-20):

Lap lace "Transform with application to the solution of differential equations.

Unit 2: (Marks-20):

Fourier Transform: Fourier Integral Transform, Application of Fourier Transform to ordinary and partial differential equations of initial and boundary value problems.

Unit 3(Marks-20):

Integral equations:Solution of Linear Integral Equations, Fredholm's Integral Equations with separable kernels, Voltera's Integral Equations

Unit 4(Marks-20):

Method of successive Approximations, Fredholm's method, Voltera's method.

Books Recommended:

The Mathematics of Physics and Chemistry, by Margenue and Murphy.

Methods of Applied Mathematics by Francis B. Hilderbrand.

Fourier Transforms, by Ian N. Sneddon

Theory and Problems of Laplace transforms, by M.R. Spiegel.

M205: Operations Research

Unit-1 (Marks-10):

History and Development of Operations Research. Operation Research and its Scope. Necessity of Operation Research in Industry and Management. General idea of queuing problem-Markovian and non Markovian queues. Queuing theory and its operating characteristic queuing model-M/M/ M/M/K. General equations of the models.

Unit-2(Marks-10):

Simulation: Theory of simulation. Monte Carlo method application to the problems of replacement and maintenance inventory, queuing and financial problems.

Unit-3. (Marks-20):

Linear Programming: Simplex method. Theory of the simplex Method Duality and sensitivity Analysis. Other Algorithms for Linear Programming Dual Simplex Method. Integer programming-Branch and Bound technique. Concept of cutting plane. Gomory's all integer cutting plane method.

Applications to Industrial Problems:- Optimal product mix and activity levels. Petroleum refinery operations Blending problems. Economic interpretation of dual linear programming problems. Input-output analysis.

Unit-4: (Marks-10):

Transportation and Assignment Problems.

Unit-5: (Marks-10):

New York Analysis- Shortest Path Problem. Minimum Spanning Tree Problem. Maximum Flow Problem, Minimum Cost Flow Problem. Network simplex Method. Project Planning and Control with PERT-CPM

Unit 6: (Marks-20):

Nonlineear Programming : One and Multi-Unconstrained Optimization. Fuhn-Tucker Conditions for Constrained Optimization. Quardratic Programming. Separable Programming Convex Programming Non-convex Programming.

Text Books:

1. Kanti Swarup.P.K. Gupta and Manmohan: Operations Research. S.Chand and Co.
2. H.A.Taha.Operations Research-An introduction.Macmillan Publishing Co. Inc. New York.

Reference Books:

1. FS Hillier and GJLiberman. Introduction to Operations Research (Sixth Edition). McGraw Hill International Edition. Industrial Engineering Series. 1995 (This book comes with a CD containing tutuorial software)
2. P.K.Gupta and D.S. Hira: Operations Research-An Introduction S.Chand and Co.

Semester 3

M301Computer Programming in C

Unit 1(Marks-20):

An Overview of Programming: The basic model of computation, Algorithms, Flow charts, programming languages, compilation, linking and loading, efficiency and analysis of algorithms.

C Essentials: Character set, variables and identifiers, built in data types, operators and expressions, constants, type conversions, basic input/put operations, anatomy of a C program.

Unit 2 (Marks-20):

Control Flow: Conditional branching, The switch statement, looping, nested loops, the break and continue statements, the goto statement, infinite loops.

Unit 3(Marks-20):

Arrays, Pointers and Functions: Declaration, initialization, pointer arithmetic. Basics of functions, passing arguments, declaration and calls, return values.

SAMPLE PROGRAMS FOR PRACTICAL (MARKS-20)

To evaluate an arithmetic expression. To find GCD, Factorial. Fibonacci series. Prime number generation. Reversing digits of an integer. Finding square root of a number. To find the roots of a quadratic equation. To find the greatest and smallest of a finite set of numbers. To find the sum of different algebraic and trigonometric series. Addition, subtraction and multiplication of matrices.

Books Recommended:

Rajaraman V.: Computer Oriented Numerical methods, Prentice Hall of India, New Delhi

Balaguruswamy E.: ANSI C

Kernighan W. and Ritchie D.K.: The C programming Language, PHI.

M302: (Number theory)

Unit 1. (Marks-20)

Principle of mathematical induction, least common multiple, greatest common divisor. Euclidean algorithm, prime numbers, unique factorization theorem)

Unit 2: (Marks-20)

Operations of congruences, Residue sets mod m, Euler's theorem, order of a mod m, linear congruences, the theorems of Fermat and Wilson, The Chinese Reminder theorem, Polynomial congruences.

Unit 3: (Marks-20)

Primitive roots, Indices, quardratic residuce mod m, Euler's criterion, The Legendre sybmbol, The law of quadratic reciprocity. The Jacobi symbol.

Unit 4: (Marks-20)

Multiplicative Arithmetic functions, τ and σ functions, Mobius function, Euler's function, The inversion formula. Linear Diophantine equations, equations of the form $x^2+y^2=z^2$, related equations, Representation of a number by sum of two of four squares.

Books recommended:

Burton D.M.: Elementary Number Theory, Universal Book stall, New Delhi

M303: Continuum Mechanics

Unit: 1 (Marks-20)

The continuum concept. Homogeneity isotropy mass density. Cauchy's stress principle. Stress tensor. Equations of equilibrium. Stress quaric of Cauchy. Principal stresses. Stress invariants. Deviator and spherical stress tensors.

Unit 2 (Marks-20) Analysis of Strain:

Lagrangian and Eularian descriptions. Deformation tensors. Finite strain tensor. Small deformation theory. Linear strain tensors and physical interpretation. Stress ratio and finite strain interpretation strain quardric of Cauchy. Principal strains. Strain invariants. Spherical and Deviator strain components. Equations of Compatibility.

Unit 3 (Marks-20) Motion:

Material derivatives path lines and stream lines. Rate of deformation and Vortiety with their physical interpretation. Material derivatives of volume. Surface and line elements. Volume surface and line integrals. Fundamental laws of continuum Mechanics.

Unit 4 (Marks-20) Constitutive equations of Continuum Mechanics:

Linear elasticity. Generalized Hook's Law. Strain energy function. Elastic constants for isotropic homogeneous materials. Elaststatic and Elastodynamic problems.

Fluids: Viscous Stress tensor. Barotropic flow : Stokesian fluids. Newtonian fluids. Navier stokes equations. Irrotational flow. Perfect fluids. Bernoulis equation. Circulation.

Text Books:

1. Continuum Mechanics -G.E.Mase. Schaum's outline series. McGraw Hill Book Company
2. Mathematical Theory of Continuum Mechanics-R.Chatterjee. Narosa Publishing House. New Delhi.

M304 Algebra II

(Optional):

Unit 1: (Marks-20)

Posets and lattices, Modular, Distributive lattices, Direct product (sum) of an infinite family of groups. Structure theorems for finitely generated abelian groups

Unit 2 (Marks-20)

Sylow's theorem and its applications : Free abelian groups, free groups, free products of groups, representation of a group.

Unit 3(Marks-20)

Modules, submodules, Direct product and direct sum of modules, prime ideals in commutative rings, complete ring of quotients of a commutative rings.

Unit 4 (Marks-20)

Primitive rings, Radical, completely reducible module and rings, Artinian and Noetherian rings and modules.

Text/Reference Books:

1. Theory of groups-M.Hall
2. Lecutes of rings and modules-J Lambek
3. Modern Algebra- Singh and Zameeruddin, Vikas Publishing House
4. Lattices and Bollean Algebra-V.K. Sarma, Vikas Publishing House.
5. Basic Abstract Algebra- Bhattacharyya, Jain and Nagpaul, CUP, 1997
6. Infinite Abelian group- L Fuch, Academic Press

M304 Space Dynamics (Optional)

Unit-1 Basic formulae of a spherical triangle-The Two-body problem: (Marks-20)

The motion of the centre of mass. The relative motion. Kepler's equation. Solution by Hamilton Jacobi Theory. The Determination of Orbits: Laplace's Gauss Methods.

Unit-2: The Three Body problem: (Marks-20)

General Three Body Problem. Restricted Three Body Problem. Jacobi integral. Curves of zero velocity. Stationary solutions and their stability. The n-body problem: The motion of the centre of Mass. Calssical integrals.

Unit-3. Perturbation : (Marks-20)

Osculating orbit, perturbing forces. Secular and Periodie perturbations, Lagrange's planetary Equations on terms of perturbing forces and in terms of perturbed Hamiltonian. Motion of the moon-The perturbing forces. Perturbation of Keplerian elements of the moon by the sun.

Unit-4 Flight Mechanics: (Marks-20)

Rocket performance in a vacuum, vertically ascending paths. Gravity twin trajectories. Multi-stage rocket in a vacuum. Definitions pertinent to single stage rocket. Performance limitations of single stage rockets. Definitions pertinent to multi stage rockets. Analysis of multi-stage rockets neglecting gravity. Analysis of multi-stage rockets including gravity.

Unit-5 Rocket performance with aerodynamic forces. (Marks-20)

Short-range non-lifting missiles. Ascent of a sounding rocket. Some approximate performance of rocket powered aircraft.

Text Books:

1. Fundamentals of Celestial Mechanics. The Macmillan Company. 1962 J.M.A.Danby
2. Clestial Mechanics. The Macmillan Company. 1958- E. Finaly. Freundlich.
3. Orbital Dynamics of Space Vehicles. Prentice Hall INC. Engle Wood Cliff. New Jersey 1963-Ralph Deutsch

Reference Books:

1. An Introduction of Clestial Mechanics. Intersciences Publishers. INC 1960- Theodre E.Sterne.

2. Flight Mechanics Vol 1. Theory of flight paths.Addision Wiley Publishing Company INC. 1962-Angelo Miele.

M305 Special Theory of Relativity(Optional)

Unit 1: (Marks-20)

Inertial and non-inertial frames, Geometry of Newtonian mechanics, Galilean Transformations, Back-ground of the fundamental postulates of the special theory of relativity, Lorentz transformation. Relativistic concept of space and time and relativity of motion, Geometrical interpretation of Lorentz transformation as a rotation. Lorentz transformation as a group.

Unit 2: (Marks-20)

Relativistic addition law of velocities and its interpretation in terms of Robb's rapidity, Invariance of sped of light, consequences of Lorentz transformation eg (i) Lorentz Fitzgerald contraction (ii) Time dilation (iii) Simultaneity of events, Proper length and proper time, Application in problems. Transformation of acceleration.

Unit-3: (Marks-20)

Relativistic mechnics. Variation of mass with velocity, Transformation of mass, force and density Equivalence of mass and energy, Transformation of momentum and energy, Energy momentum vector. Applications in problems, Relativistic Lagrangian and Hamiltonian.

Unit-4: (Marks-20)

Minkowaki's space, Geometircal representation of simultaneity, Contraction and dilation, space like and time like intervals, position. Four vectors, Four velocity, Four forces and Four momentums, Relativistic equations of motion, Covariant four- dimensional formulation of the laws of mechanics.

Unit-5: (Marks-20)

Electrodynamics: Fundamentals of electrodynamics, Transformation of differential operators, D'Alembert operator, Drivation of Maxwell's equation, Electromagnetic potentials and Lorentz condition, Lorentz face, Lorentz transformations of space and time in four-vector form, Transformations of charge and current density, Invariance of Maxwell's equations, Transformation equations of electric field strength and magnetic field induction components, Invariance of E^2-H^2 and E.H.

Reference Books :

1. Introduction to Special Relativity, Wiley Eastern Lt.(1990) Robert Resnick
2. The Mathematical Theory of relativity, Cambridge University Press 1965 A S Eddington.
3. Relativistic Mechanics (Theory of Relativity) Pragati Prakashan, 2000-Satya Prasash

M305

Mathematical Logic (Optional)

Unit 1(Marks-20):

Informal statement calculus: Statements and connectives, truth functions and truth-tables, normal forms, adequate sets of connectives, arguments and validity.

Unit 2 (Marks-20):

Formal statement calculus: Formal definitions of Proof. Theorem and Deduction the formal theory L of statement calculus the deduction theorem and its converse.

Unit 3(Marks-20)

Adequacy theorem for 1: Valuation in L. tautology in L. the Soundness theorem. Extensions of L. consistency of an extension the adequacy theorem of L.

Unit 4(Marks-20)

Informal predicate Calculus: Symbolism of predicate calculus. First order language interpretation truth values of well-formed formulas satisfaction and truth. Formal Predicate Calculus: Predicate Calculus as a formal theory the adequacy theorem of K.

Unit 5(Marks-20):

Mathematical Systems: First order systems with equality the theory of groups first order arithmetic formal set theory consistency and models.

Books:

Text Book: Logic for Mathematics by A.G. Hamilton

Ref. Book: Introduction of Mathematical Logic by Elliot Mendelson

Semester 4

M401Graph theory

Unit 1: . (Marks-20)

Graphs, subgraphs, walk, paths, cycles and components, intersection of graphs, Degrees, Degree sequence. Threes, spanning tree, cycles, cocycles. Cycle space. Cocycle space, connectivity, cut vertices, cut edges, blocks., connectivity parameters, Menger's theorems.

Unit-2: (Marks-20)

Eulerian and Traversible graphs: Characterization theorems, characterization attempts for Hamiltonian graphs, two necessary and sufficient conditions of a graph to be Hamiltonian, Factorisations, Basic concepts, 1-factorizat5ion, 2- factorization, coverings, critical points, and lines

Unit-3: (Marks-20)

Planarity: Subdivision of graph, identification of vertices, plane and planar graph, outer planar graph, Euler's polyhedron formula, Kuratowski's theorems, Genus, thickness, coarseness and crossing number of a graph.

Unit 4: (Marks-20)

Algebraic graph theory: Adjacency matrix and spectrum of graphs, vertex, partition and the spectrum.

Text Books:

1. Harary: Graph Theory, NAROSA Publishing Co.

2. Algebraic Graph Theory

M402 Numerical Analysis

Unit 1: Interpolation formulae (Marks-20)

Newton's Forward Interpolation Formula, Newton's Backward Interpolation Formula, Newton's divided difference interpolation formula, Lagrange's interpolation formula, Gauss Forward Interpolation formula, Gauss Backward Interpolation formula, Stirling's formula, Bessel's formula.

Unit 2. Numerical Differentiation and Integration (Marks-20)

Numerical Differentiation and Integration, Simpson's rule, Weddle's central difference formula, quadrature formula, Gauss's quadrature formula, Euler's formula for summation and quadrature.

Unit 3. Solution of Algebraic and Transcendental Equations: (Marks-20)

Numerical Solutions of Algebraic and Transcendental Equations. Solutions by the method of iteration and the Newton-Raphson method, cases of repeated roots.

Unit 4. Linear Equations: (Marks-20)

Direct method for solving systems of linear equations (Gauss Elimination, LU decomposition, Cholesky decomposition), iterative methods (Jacobi, Gauss-Seidel, Relaxation methods)

Text Books:

Houscholder A.S.: Principles of Numbrical Analysis, McGraw Hill, New York.
Jain M.K.: Numerical Analysis for scientists and Engineers, S.Publishers.
Kung: Numerical Analysis, McGraw Hill Book Co.
Niyogi P. : Numerical Analysis and Algorithms, Tata McGraw Hill
Rajaraman V. : Computer Oriented Numerical Methods, Prentice Hall of India, New Delhi.

M403 Fluid Dynamics (Optional)

Unit 1(Marks-20)

Waves: Long wave and surface wave stationary wave. Energy of the waves. Waves between different media. Group velocity Dynamical significance of Group velocity. Surface tension and Capillary waves. Effect of Surface tension in water waves.

Unit 2 (Marks-20) :

Viscous fluid motion: Navier-Stokes equation of motion rate of change of vorticity and circulation rate of dissipation of energy. Diffusion of a viscous filament.

Unit 3 (Marks-20):

Exact solution of Navier Stokes Equation: Flow between plates. Flow through a pipe (circular elliptic). Suddenly accelerated plane wall. Flow near an Oscillating flat plate. Circular motion through cylinders.

Unit 4 (Marks-20):

Laminar Boundary Layer Theory: General outline of Boundary layer flow. Boundary layer thickness. Displacement thickness. Energy thickness. Flow along a flat plate at zero incidence. Similarity solution and Blasius solution for flow about a flat plate.

Karman's momentum integral equation. Energy integral equation. Pohlhausen solution of momentum integral equation.

Two dimensional Boundary layer equations for flow over a curved surface. Blasius solution for flow past a cylindrical surface phenomenon of separation.

Text Books:

1. Hydrodynamics- Horace Lamb. Cambridge University Press

2. Theoretical Hydrodynamics: I.M.Milne Thomson.McMillan Company. 3. Boundary Layer Theory: H. Schliehting Translated by J.Kertin. McGraw Hill Book 'Company Inc. New York

Reference Books:

1. Modern Development of Fluid Dynamics. Voll- S.Goldstein. Dover publication. New York

2 An Introduction to Fluid Dynamics, G.K.Batchelor. Functions

M403 Functional Analysis II (Optional)

Unit-1 (Marks-10)

Vector topologies: Examples First properties Mazur's and Eidelheit's separation theorems Metrizable vector topologies.

Unit-2 (Marks-15)

The Open Mapping Theorem: The closed graph Theorem and the uniform Boundedness Principle for F-spaces. Topologies induced by families of functions. Weak and Weak* topologies. Compactness. Adjooint operator Projection and complementation.

Unit-3 (Marks-15)

Convexity: The Hahn- Banach theorem for locally convex spaces. The Banach Alauglu Theorem for topological vector spaces. Krein-Milman theorem. Milman theorem.

Unit-4 (Marks-15)

Definition of Banach Algebra and Examples Singluar and Non singular elements. The Abstract index. The spectrum of an element. Gelfand Formula. Multiplicative. Linear Function. And the maximal ideal space. Gleason Kahane Zelazko Theorem.

Unit-5 (Marks-15)

The Gelfand Transforms. The spectral Mapping Theorem. Isomentric Gelfand Tranform. Maximal ideal spaces for Dise Algebra and the Algebra .

Unit-6 (Marks- 10)

(* algebras-Definition and Examples, Self Adjoint. Unitary normed positive and projection elements in (*-algebras, Commutative (*-algebras.(*-Homomorphisms. Representation of Commutative (*- algebras. Subalgebras and the spectrum. The spectrum theorem. The Continuous functional Calculus . Positive linear functionals and slates in (*- Algebras, The GNS Construction.

Text Books

1. Megginson Robert E-An introduction to Banach space theory. Springer verlag
2. W. Rudin-Functional Analysis Tata McGraw Hills.
3. E.E.Bonsall and J.Duncan-Complete Normed Algebras. Springer verlag Reference Book
4. Folland. Garald B-Real Analysis Modern Techniques and their applications (John Wiky)

M404Mathematical Statistics

Unit 1(Marks-16)

Probability: Mathematical and statistical definitions. Discrete Samplespace, Axiomatic approach, Theorems of Total and Compound probability, Repeated Trials, Baye's theorem. Random Variable and its distribution, Mathematical Expectaions, Expectation of sum and product of random variables, Expectation of functions of random variables. Distribution of more than one random variables. Tshebysheffs lemma. Weak law of large numbers. Theorems of Markoff and Khintchine, Bernouilli's and Poisson's theorems. Characteristic function. Probability generating unctions, Central limit theorem.

Unit 2 (Marks-16)

Binomial distribution, Posson, distribution, Normal distribution, Hypergemetric distribution, Multionomial distributions, Betta and Gamma distribution, Pearsonia system of vurves, derivation of the differential equations and its solutions yielding curves of types, ULM and IV. Bivariate Normal distribution. Regression and Correlation (including Multiple, partial and Interclass correlation)

Unit 3 (Marks-16)

Principle of least squares of curve fitting (including orthogonal polynomials).

Unit 4 (Marks-16)

Theory of sampling: Random and simple, random sampling, idea of sampling distribution, large sample test, Exact sampling distribution - and T,F,Z and X^2(with derivations) and associated tests of significance.

Unit 5 (Marks-16)

Estimation: Requirement of a good estimator, Method of maximum likelihood (including Cramer-Rao inequality)

Books Recommended:

1. An introduction to Probability theory and its Application by W.Feler
2. An Introduction to Mathematical Probability, by J.V. Uspensky
3. Correlation and Frequency curves, by Elderton
4. Modern Probability and its Application by Ferzen
5. Probability Theory by M. Leeve
6. Mathematical Methods of Statistics by H. Cramer
7. Linear Statistical Interference and its Application by C.R. Raw
8. The Advance Theory of Statistics by Kendell and Smart
9. Sampling Method by Cox and Cochrun
10. Sampling Survey of Murphy
11. Sampling Survey by F. Yats

M404 Dynamical Systems and Fractal Geometry

Unit 1: . (Marks-10)

Nonlinear Oscillators, Conservative system. Hamiltonian System, Various types of Oscillators in nonlinear system, Solutions of nonlinear differential equations.

Unit-2: (Marks-10)

Orbit of a map, fixed point, equilibrium point, periodic point, circular map, configuration space and phase space. Origin of bifurcation, Stability of a fixed point, equilibrium point, Concept of a limit cycle and torus.

Unit-3: (Marks-10)

Hyperbolicity, Quadratic map, Period Doubling phenomenon, Feigenbaum's Universal constant.

Unit 4: (Marks-10)

Turning point, Transcritical, Pitch Fork and Hopf Bifurcation.

Unit 5: (Marks-10)

Randomness of Orbits of a dynamical system, Chaos, Strange Attractors, Various roots to Chaos, Onset mechanism of turbulence.

Unit-6: (Marks-15)

Construction of the middle third Cantor set, Von Koch Curve, Sierpinski gasket, self similar fractals with different similarity ratio, Julia Set, measure and mass distribution, Housdorff measure, scaling property, effect of general transformations on Housdorff measure, Housdorff dimension and its properties, s-sets, calculation of Housdorff dimension and its properties, s-sets, calculation of Housdorff dimension in simple cases

Unit 7: (Marks-15)

Unit measurement of a set at scale d, box dimension, its equivalent versions, properties of box dimension, box dimension of middle third cantor set and other simple sets, some other definitions of dimension, upper estimate of box dimension, mass distribution principle, generalized cantor set and its dimension

Text Book:

1. Robert C. Hilborn: Non linear Dynamics and Chaos

2. D. K. Arrowsmith, Introduction to dynamical systems, Cambridge University Press, 1990.

3. Kenneth Falconer : Fractal Geometry, John Wiley and Sons, 1995

4. M. F. Barnsley : Fractals everywhere, A. P. 1988.

Reference Books:

1. R.L. Devany : An introduction to Chaotic Dynamical Systems, Addission-Wesley Publishing Co. Inc. 1989.

2. K. J. Falconer : The Geometry of Fractal Sets, Cambridge University Press, 1985

M405 General Theory of Relativity and Cosmology (Optional):

Unit 1 (Marks 20)

Geodesics, Derivation of the equation of geodesics, Geodesic co-ordinates, intrinsic derivatives, First Curvature, Parallel transport, parallel vectors. Related theorems of intrinsic derivatives and parallel displacement.

Unit 2 (Marks 15)

Riemann Christoffel Curvature tensors and their properties, Riciitensor, Bianchi identities, Einstein tensor Divergence of Einstein tensor, Condition of Flat Space, Riemann Curvature.

Unit-3 (Marks 15)

Theory of gravitation, principle of covariance and equivalence, geodesic principle, Simple consequences of the principle of equivalence (i) the equality of inertial and gravitational masses(ii) effect of gravitational potential on the rate of a clock, (iii) The clock paradox, the energy momentum tensor, Energy momentum tensor in case of a perfect fluid, conservation of energy and Momentum.

Unit-4 (Marks 15)

The gravitational fluid in empty space in presence of matter and energy. Newtonian equation of motion as an approximation of geodesic equations. Poission's equation as an approximation of Einstein field equation, Schwarzschid exterior solution and its isotropic form, planertry orbits and analogues of Kepler's laws in general relativity. Relation between M and m, Isotropic co-ordinates. The three crucial' tests (i) The advancee of perihelion (ii) Bending of light rays in a gravitational field (iii) Gravitational red-shift in spectral lines. Schawarzachid interior solutions., Boundary conditions.

Unit-5 (Marks 15)

Cosmology, Mach principle, Einstein modified field equations with cosmological term, Static cosmological models of Einstein and de-sitter, their derivations, properties and comparision with the actural universe. Huble's Law, cosmological principles, Wely's postulates. Non-static cosmological models. Derivation of Robertson": Walker metric, Redshift, Redshift versus distance relation Angular size versus red: shift relation and source counts in R.W space time.

Text Books

1. The Mathematical Theory of Relativity, Cambridge Univesity Press-1965-A.S. Eddington
2. A First course in general relativity, Cambridge University Press, 1990-B.F. Shutz
3. The Theory of Relativity-C.Moller
4. An Introduction to Riemannian Geometry and Tensor Calculus, - G.,E. Weatherburn. Cambridge University Press, 1950

M 405 Fuzzy Sets and their application (Optional)

Unit 1 (Marks-10)

Fuzzy sets: Basic Definitions. D-level sets. Convex fuzzy sets. Basic operations on Fuzzy sets. Types of Fuzzy sets. Cartesian products. Algebraic products Bounded sum and difference. T-conoroms.

Unit-2 (Marks-10)

Extcusion Principle: the Zadeh extension principle Image and inverse image of fuzzy sets. Fuzzy numbers. Elements of Fuzzy Arithmetic.

Unit-3 (Marks-10)

Fuzzy relations and Fuzzy Graphs: Fuzzy relations and fuzzy sets. Composition of Fuzzy relations. Min-max composition and its properties. Fuzzy equivalence relations. Fuzzy compatiability relations. Fuzzy relation equations. Fuzzy graphs, Similarity relation.

Unit-4 (Marks-10)

Possibility Theory: Fuzzy measures. Evidence theory. Necessity measure. Probability measure. Possiblity distribution. Possibility theory and fuzzy sets. Possibility theory versus probability theory.

Unit-5 (Marks-10)

Fuzzy Logic: An overview of classical logic. Multivalued logic. Fuzzy propositions. Fuzzy quantifiers. Linguistic variable and hedges. Inference from conditohnal fuzzy propositions, the compositional rule of inference. Appliction in Vivil, Mechanical and Industrial Engineering.

Unit-6 (Marks-10)

Approximate reasoning: An overview of fuzzy expert system. Fuzzy implications and their selection. Multiconditional approximate reasoning. The role of fuzzy relation equation.

Unit-7 (Marks-10)

Introduction to fuzzy control: Fuzzy controllers. Fuzzy rule base. Fuzzy inference engine. Fuzzification. Defuzzification and the various Defuzzification methods (the centre of area, the centre of maxima and the mean of maxima methods). Introduction of Fuzzy Neural Network, Autometa and Dynamical Systems:

Unit-8 (Marks-10)

Decision making in Fuzzy environment: Individual decision making. Multiperson decision makding. Multicriteria decision making. Multi stage decision making. Fuzzy ranking methods. Fuzzy linear programming. Application in Medicine and Economics.

Text Books:

1. G.J.Klir and B. Yuan-Fuzzy sets and Fuzzy Logic, Theory and Applications, Prentiee Hall of India, 1995
2. H.J.Zimmermann-Fuzzy set theory and its application, Allied Publishers Ltd. 1991

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M101

Institute of Distance and Open Learning

Gauhati University

M.A./M.Sc. in Mathematics

Semester 1

Paper I

Real Analysis and Lebesgue Measure

Contents:

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Real Analysis

UNIT 1

Sequences and series of functions

1. Pointwise and uniform convergence:

Definition 1.1: Suppose $\{f_n\}$, n= 1, 2, 3, …, is a sequence of real valued functions defined on a set E of real numbers. If for each $x \in E$, the sequence $\{f_n(x)\}$ of real numbers converges and $\lim_{n\to\infty} f_n(x) = f(x)$, $x \in E$, then we say that $\{f_n\}$ **converges pointwise to** f on E.

Definition 1. 2: A sequence $\{f_n\}$ of functions $\{f_n\}$, n = 1, 2, 3, …, is said to be **converge uniformly** to a function f on E, if for every $\varepsilon>0$, there is an integer N such that

$$n \geq N \text{ implies } |f_n(x) - f(x)| \leq \varepsilon, \quad \text{for all } x \in E.$$

Note: If a sequence $\{f_n\}$ converges uniformly to f on E, then it converges pointwise to f on E. However, the converse is not true as the solved example 3 shows.

Theorem 1.3:

(Cauchy's criterion for uniform convergence). The sequence $\{f_n\}$ of functions defined on E, **converges uniformly** on E if and only if every $\varepsilon>0$ there exists a positive integer N such that for all m, $n \geq N$ and for all $x \in E$.

$$|f_n(x) - f_m(x)| < \varepsilon.$$

Definition 1.4: A series Σf_n of functions (each defined on a set E) is said to **converge pointwise (resp. uniformly)** on E to a sum S(x) if the sequence of functions $\{s_n\}$ (sequence of partial sums) defined by

$$s_n(x) = \sum_{i=1}^{n} f_i(x) \text{ converges pointwise (resp. uniformly) to } S(x).$$

Theorem 1.4: (Cauchy's criterion of convergence of series) A series Σf_n converges uniformly on E if and only if given $\varepsilon>0$, there exists a positive integer N such that for all $n \geq N$, $p \geq 1$ and for all $x \in E$,

$$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| < \varepsilon.$$

Theorem 1.5: Suppose $\{f_n\}$ converges pointwise to f on E.

Let $\quad M_n = \sup_{x\in E} |f_n(x) - f(x)|, n \geq 1.$

Then $\{f_n\}$ **converges to f uniformly** if and only if $M_n \to 0$ as $n \to \infty$.

Theorem 1.6 (Weierstrass M-test): Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose

$$|f_n(x)| \leq M_n \ \forall\ x \in E, \ n = 1, 2, 3, \ldots$$

Then Σf_n converges uniformly on E if ΣM_n converges.

Note: The converse of the theorem 1.6 is not true.

Uniform convergence and continuity

Theorem 1.7 (a): If $\{f_n\}$ is a sequence of continuous functions on $E \subseteq R$, and if $\{f_n\}$ converges uniformly to f on E, then f is continuous.

(b) If a series Σf_n converges uniformly to f on $E \subseteq R$ and its term f_n are continuous at a point $x_0 \in E$, then the sum function f is continuous at x_0.

Theorem 1.8 (Dini's theorem on uniform convergence):

(a) If a sequence $\{f_n\}$ of continuous functions on a closed interval [a, b] is monotonic increasing, and converges pointwise to a continuous function f, then $\{f_n\}$ converges uniformly to f on [a, b].

(b) If the sum function of a series Σf_n, with non-negative continuous terms defined on [a, b] is continuous, then the series Σf_n converges uniformly on [a, b].

Uniform convergence and integration :

Theorem 1.9 (a) If a sequence $\{f_n\}$converges uniformly to f on [a, b], and each function f_n is integrable on [a, b] , then f is integrable on [a, b], and the sequence $\left\{\int_a^x f_n dt\right\}$ of indefinite integrals of f_n converges uniformly to $\int_a^x f dt$ on [a, b].

(b) If a series Σf_n converges uniformly to f on [a, b], and each term $f_n(x)$ is integrable, then f is integrable on [a, b] and the series $\sum\left(\int_a^x f dt\right)$ converges uniformly to $\int_a^x f dt$ on [a, b].

Uniform convergence and differentiation

Theorem 1.10 (a) Let $\{f_n\}$ be a sequence of differentiable functions on [a,b] such that it converges at least at one point $x_0 \in [a, b]$. If the sequence of derivatives $\{f_n'\}$converges uniformly to G on [a, b], then the given sequence $\{f_n\}$ converges uniformly on [a, b].

(b) Let Σf_n be a series of differentiable functions on [a, b] and such that it converges at least at one point $x_0 \in [a, b]$. If the series of differentials $\Sigma f_n'$ converges uniformly to G on [a, b], then the given series Σf_n converges uniformly on [a, b] to a function f, where $f'(x) = G(x)$.

2. Equicontinuous families of functions

Definition 2.1: Let $\{f_n\}$ be a sequence of functions defined on an interval [a,b]. We say that $\{f_n\}$ is **pointwise bounded** on [a, b] if the sequence $\{f_n(x)\}$ is bounded for every $x \in [a, b]$. In other words, $\{f_n\}$ is pointwise bounded, if there is a real valued function ϕ on [a, b] such that $|f_n(x)| \leq \phi(x)$, for all $x \in [a, b]$ and for all $n \geq 1$. The sequence $\{f_n\}$ is **uniformly bounded** on [a, b], if there is a positive number M such that $|f_n(x)| \leq M$, for all $x \in [a, b]$ and $n \geq 1$.

Definition 2.2: A family $\mathfrak{I}$ of functions defined on a set $E \subseteq R$ is said to be **equicontinuous** on E, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathfrak{I}$

$$|f(x) - f(y)| < \varepsilon \text{ whenever } x, y \in E \text{ with } |x - y| < \delta.$$

Note that every member of an equicontinuous family is uniformly continuous.

Theorem 2.3: If $\{f_n\}$ is a uniformly convergent sequence of continuous functions on an interval [a, b], then $\{f_n\}$ is equicontinuous on [a, b].

Theorem 2.4: If $\{f_n\}$ is pointwise bounded and equicontinuous on [a, b], then $\{f_n\}$ contains a uniformly convergent subsequence and $\{f_n\}$ is uniformly bounded on [a, b].

3. Power series

Definition 3.1: The series $$\sum_{n=0}^{\infty} a_n x^n, \qquad (1)$$

where a_n are real numbers dependent on n but not on x, is called a **power series.** Clearly, for any values of a_n, the series (1) is convergent at $x = 0$. Let S be the set of values of x for which the series (1) is convergent. The set S is called the **region of convergence** of the power series. The power series (1) is said to be

(i) nowhere convergent if $S = \{0\}$.

(ii) everywhere convergent if $S = R$.

(iii) absolutely convergent at x, if $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent.

Theorem 3.2: If $\overline{\lim} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, then the power series $\Sigma a_n x^n$ is convergent (in fact absolutely convergent) for $|x| < R$ and divergent for $|x| > R$.

Definition 3.3: In view of theorem 3.2, the region of convergence S of a power series $\Sigma a_n x^n$ is an interval. The end points of this interval are $-R$ and R (may or may not be inclusive), where R is 0, ∞ or $\frac{1}{\overline{\lim}|a_n|^{\frac{1}{n}}}$ according as $\overline{\lim}|a_n|^{\frac{1}{n}}$ is ∞, 0 or non-zero finite. R is called the **radius of convergence** of the power series.

Note 3.4: If $\lim\left|\frac{a_n}{a_n+1}\right|$ exists (finitely or infinitely), then $R = \lim\left|\frac{a_n}{a_n+1}\right|$, because then $\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim|a_n|^{\frac{1}{n}} = \overline{\lim}|a_n|^{\frac{1}{n}}$.

Theorem 3.5: Suppose that the series $\Sigma a_n x^n$ converges for $|x| < R$, then we have the following:

(a) For every $\varepsilon > 0$, the series $\Sigma a_n x^n$ is uniformly convergent on $[-R+\varepsilon, R-\varepsilon]$.

(b) If $f(x) = \Sigma a_n x^n$, $|x| < R$, ten f is differentiable (and therefore continuous) in $(-R,R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad |x| < R.$$

4. Fourier series

Definition 4.1: **A trigonometric polynomial** is a finite sum of functions

$$f(x) = a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx) \qquad \text{....(4.1)}$$

where x is a real number and $a_0, \ldots, a_N, b_0, \ldots, b_N$ are constants (here we allow them to be complex numbers).

Since $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, we can write (4.1) as

$$f(x) = \sum_{-N}^{N} c_n e^{inx}, \quad (x \text{ is real}) \qquad \text{....(4.2)}$$

We see that f(x) in (4.1), and therefore in (4.2), is periodic with period 2π.

Result 4. 2: If n is any integer, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 0, \text{if } n \neq 0 \\ 1, \text{ if } n = 0. \end{cases}$$

Therefore the constants c_n in (4.2) are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Definition 4.3: A **trigonometric series** is a series of functions

$$\sum_{-\infty}^{\infty} c_n e^{inx} \qquad \text{....(4.3)}$$

where x is real and c_n are complex numbers. The **nth partial sum** of (4.3) is the trigonometric polynomial

$$\sum_{-N}^{N} c_n e^{inx}$$

If f is an integrable function on [-π, π], then the trigonometric series $\sum_{-N}^{N} c_n e^{inx}$, where the numbers c_n are given by

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,\ n \in Z, \qquad \text{....(4.4)}$$

is called the **Fourier series** of f. The constants c_n are called the **Fourier coefficients** of f.

Definition 4.4: Let $\{\phi_n\}$ (n = 1, 2, ...) be a sequence of complex functions defined on [a, b]. We call that $\{\phi_n\}$ an **orthogonal system of functions** on [a,b], if

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}\,dx = 0 \text{ for all } n \neq m.$$

(here $\overline{\phi_m(x)}$ is the complex conjugate of $\phi_m(x)$.)

If in addition $\int_a^b |\phi_n(x)|^2 dx = 1$ for all n, then $\{\phi_n\}$is said to be **orthonormal**.

Definition 4.5: If $\{\phi_n\}$ is an orthonormal system of function defined on [a, b] and

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}\,dt,\ n \geq 1, \qquad \text{...(4.5)}$$

we write $$f(x) \sim \sum_1^{\infty} c_n\phi_n(x),\ x \in [a,b], \qquad \text{....(4.6)}$$

and call the series in (4.6) the **Fourier series of f relative to $\{\phi_n\}$**. The constants c_n in (4.5) is called the **nth Fourier coefficient of f relative to $\{\phi_n\}$**.

Note: Symbol '~' is used in (4.6) instead of equality, because nothing is said about the convergence of the series.

Theorem 4.6: Let $\{\phi_n\}$ be orthonormal on [a, b]. Let $s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$

be the nth partial sum of the Fourier series of f, and suppose $t_n(x) = \sum_{m=1}^{n} v_m \phi_m(x)$.

Then $$\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx,$$

and equality holds if and only if $v_m = c_m$ for all m.

Theorem 4.7: If $\{\phi_n\}$ is orthonormal on [a, b] and if

$$f(x) \sim \Sigma c_n \phi_n(x), \text{ then } \sum |c_n|^2 \le \int_a^b |f(x)|^2 dx. \text{ In particular, } \lim_{n\to\infty} c_n = 0.$$

5. Solved problems

Section 1:

1. Suppose $\{f_n\}$ is a sequence of monotonic functions on [a, b], and $\{f_n\}$ converges pointwise to a continuous function f on [a, b]. Prove that the convergence is uniform on [a, b].

Solution: First we assume that each f_n is monotonic increasing on [a, b]. Then f is monotonic increasing on [a, b]. Let $\varepsilon>0$ be given.

Since f is continuous on [a, b], f is uniformly continuous on [a, b] and so $\exists\, \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ for all } x, y \in [a, b] \text{ with } |x - y| < \delta. \quad ...(5.1)$$

Choose a partition $P = (x_0, x_1, \ldots, x_k)$ of [a, b] with $\mu(P) < \delta$.

Then by (5.1) $|f(x_i) - f(x_{i-1})| < \varepsilon$ for $i = 1, 2, \ldots, k$. ....(5.2)

Since $\lim_{n\to\infty} f_n(x_i) = f(x_i)$, $i = 1, 2, \ldots, k$, there exists a positive integer N such that

$$|f_n(x_i) - f(x_i)| < \varepsilon \text{ for all } n \ge N, i = 1, 2, \ldots, k. \quad ....(5.3)$$

Now, let $n \ge N$, $x \in [a, b]$. Let $x \in [x_{i-1}, x_i]$. Then

$$f_n(x) - f(x) \le f_n(x_i) - f(x_{i-1})$$

$= f_n(x_i) - f(x_i) + f(x_i) - f(x_{i-1})$

$< 2\varepsilon$, by (5.3) and (5.2).

Similarly, $f(x) - f_n(x) \le f(x_i) - f_n(x_{i-1})$

$= f(x_i) - f(x_{i-1}) + f(x_{i-1}) - f_n(x_{i-1})$

$< 2\varepsilon.$

Therefore $|f_n(x) - f(x)| < 2\varepsilon$ for all $n \ge N$, $x \in [a,b]$, and hence $\{f_n\}$ converges uniformly to f on [a, b].

The case is similar, if each f_n is monotonic decreasing. Now, if $\{f_n\}$contains both increasing and decreasing, then the sequence $\{f_n\}$ can be broken into two parts : one containing increasing and the other decreasing functions. If one of the two parts is finite, then the terms of this part can be deleted from $\{f_n\}$ without effecting the convergence and by the above cases $\{f_n\}$ converges uniformly to f.

If both the parts are infinite, then they are uniformly convergent subsequences of $\{f_n\}$ converging to the same limit of f. Consequently, $\{f_n\}$ converges uniformly to f.

2. Obtain a set of sufficient conditions for term by term differentiation of the series $\sum_{n=1}^{\infty} f_n(x)$ in the interval [a, b].

Solution: Consider the series obtained by term by term differentiation of the given series, viz. $\sum_{n=1}^{\infty} f_n'(x)$. We prove the following : If

(1) Σf_n converges at least pointwise on [a, b],

(2) $\Sigma f_n'$ converges uniformly to g on [a, b] and

(3) each f_n' is continuous on [a, b],

Then Σf_n converges uniformly on [a, b], and

$$f'(x) = g(x) \text{ i.e.} \frac{d}{dx}\left(\sum f_n(x)\right) = \sum f_n'(x).$$

Since the series $\Sigma f_n'$ converges uniformly on [a, b] and each f_n' is continuous on [a, b], therefore the sum function g is continuous on [a, b]. Thus $\int_a^x g(t)dt$ is differentiable and $\frac{d}{dx}\int_a^x g(t)\,dt = g(x)$.

Again, because each f_n' is continuous on [a, b] and $\Sigma f_n'$ converges uniformly to g on [a, b], we get $\sum \int_a^x f_n'(t)\,dt$ converges uniformly $\int_a^x g(t)\,dt$ on [a, b]. Since

$$\int_a^x f_n'(t)\,dt = \left[f_n(t)\right]_a^x = f_n(x) - f_n(a),$$

we get $\sum_{n=1}^{\infty}\left[f_n(x) - f_n(a)\right]$ converges uniformly on [a, b]. Because

$$\sum_{n=1}^{\infty}\left[f_n(x) - f_n(a)\right] = \sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(a) = f(x) - f(a),$$

Σf_n converges uniformly to f on [a, b].

Now, $\int_a^x g(t)dt = \sum_{n=1}^{\infty} f_n'(t)dt = f(x) - f(a),$

and therefore, $g(x) = \frac{d}{dx}\int_a^x g(t)dt = \frac{d}{dx}\left[f(x) - f(a)\right] = f'(x)$

i.e. $f'(x) = g(x)$

i.e. $\frac{d}{dx}\left(\sum f(x)\right) = \sum f'(x).$

3. Show that the sequence $\{f_n\}$, where $f_n(x) = x^n$ is uniformly convergent on [0, k], k<1 and only pointwise convergent on [0, 1].

Solution: Let $0 < k < 1$. Then, for $x \in [0, k]$

$$|f_n(x)| = |x^n| \le k^n.$$

Since $\lim_{n\to\infty} k^n = 0$ (for $0 < k < 1$), (in view of theorem 1.5) the sequence $\{f_n\}$converges uniformly to the zero function on [0, k]. Next, since each f_n is continuous on [0, 1], the limit function must be continuous on [0, 1], if $\{f_n\}$ converges uniformly on [0, 1]. However,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases},$$

which is discontinuous at x = 1. Thus, (by theorem 1.7(a)) $\{f_n\}$converges only pointwise on [0, 1].

4. Show that for $-1 < x < 1$, $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots = \frac{1}{1-x}$.

Solution : Consider the series

$$\log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \ldots \quad (5.4)$$

The nth partial sum of (5.4) is

$$S_n = \log\{(1-x)(1+x)(1+x^2)\cdots(1+x^{2^{n-2}})\} = \log(1-x^{2^{n-1}})$$

For $-1 < x < 1$, $\lim_{n\to\infty} S_n = 0$, and therefore the series (5.4) converge pointwise to the zero function on $(-1, 1)$.

Next, consider the series $\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots + \frac{2^n x^{2^n-1}}{1+x^{2^n}} + \cdots$ (5.5)

If $0 < k < 1$ then

$$\left|\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right| \le 2^n x^{2^n-1} \le 2^n k^{2^n-1}, \text{ for all } x \in [-k, k].$$

Now, $\sum_{n=1}^{\infty} 2^n k^{2^n-1}$ can be shown to be convergent. Therefore, by Weierstrass M-test, the series (5.5) converges uniformly to the zero function in $[-k, k]$. Thus the series (5.4) is pointwise convergent to the zero function, contains terms with continuous and the series of the derivatives of (5.4) is uniformly convergent on $[-k, k]$ for every $k \in [0, 1)$.

Therefore (5.4)can be differentiated term by term. We get therefore

$$\frac{-1}{1-x} + \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots = 0, \quad \text{for } x \in [-k, k],$$

i.e. $$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots = \frac{1}{1-x}, \quad (5.6)$$

for $-k \le x \le k$. Since k is arbitrary in $[0, 1)$, (5.6) is valid for all x with $-1 < x < 1$.

Section 2.

5. Let $f_n(x) = \frac{x^2}{x^2 + (1-nx)^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly bounded on $[0, 1]$. Does the sequence contain a uniformly convergent subsequence? Is the sequence $\{f_n\}$ equicontinuous?

Solution: For $x\in [0, 1]$, we have

$$|f_n(x)| = \frac{x^2}{x^2+(1-nx)^2} \le 1.$$

Therefore, $\{f_n\}$ is uniformly bounded on [0, 1]. (here n = 1).

We show that $\{f_n\}$ does not have any subsequence which is uniformly convergent on [0, 1]. For $x\in [0, 1]$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2}{x^2+(1-nx)^2} = 0.$$

Thus $\{f_n\}$ converges pointwise to f on [0, 1], where f(x) = 0, $x\in [0, 1]$. Let, if possible, $\{f_n\}$ have a subsequence $\{f_{n_k}\}$ which is uniformly convergent to f on [0, 1]. Let $0 < \varepsilon < 1$. then there exists a positive integer K such that

$$|f_{n_k}(x) - f(x)| < \varepsilon \quad \text{for all } x\in [0, 1] \text{ and } k \ge K. \qquad (5.7)$$

Now, for each k, $\frac{1}{n_k} \in [0,1]$ and $f_{n_k}\left(\frac{1}{n_k}\right) = 1$.

Thus $\left|f_{n_k}\left(\frac{1}{n_k}\right) - f\left(\frac{1}{n_k}\right)\right| = |1-0| = 1$ which is not less than ε for all k, contradicting (5.7). Hence the claim.

The sequence $\{f_n\}$ can not be equicontinuous, because, otherwise together with the fact that $\{f_n\}$ is uniformly (so pointwise) bounded, it will imply that $\{f_n\}$ contains a uniformly convergent subsequence.

6. Prove theorem 2.3

Solution: Let $\varepsilon>0$ be given. Since $\{f_n\}$ converges uniformly on [a,b], there exists an integer m such that

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{3} \quad \text{for all } n \ge m \text{ and } x\in [a, b]. \quad \text{....(5.8)}$$

Also since f_i, $1 \le i \le m$ are continuous on [a,b] they are uniformly continuous on [a, b] and therefore there exists $\delta > 0$ such that

$$|f_i(x) - f_i(y)| < \frac{\varepsilon}{3} \text{ whenever } |x-y| < \delta, \qquad \text{....(5.9)}$$

x, y $\in$ [a, b] and $1 \le i \le m$. For $n > m$, we have

$$|f_n(x)-f_n(y)| \le |f_n(x)-f_m(x)|+|f_m(x)-f_m(y)| + |f_m(x)-f_n(y)|$$

$$< \varepsilon,$$

whenever x, y $\in$ [a, b] with $|x-y| < \delta$. Thus for all n,

$$|f_n(x)-f_n(y)| < \varepsilon, \text{ whenever } x, y \in [a,b] \text{ with } |x-y| < \delta.$$

Hence $\{f_n\}$ is equicontinuous on [a, b].

Section 3

7. Prove that a power series $\Sigma a_n x^n$ converges uniformly on any closed interval contained in (–R, R) where R is the radius of convergence of the power series.

or

Prove theorem 3.5(a).

Solution: Let $\varepsilon > 0$ be given. For $|x| \le R - \varepsilon$, we have

$$|a_n x^n| \le |a_n (R-\varepsilon)^n|. \quad \text{....(5.10)}$$

Let $M_n = |a_n(R-\varepsilon)^n|$. Since $R - \varepsilon$ lies in the region (–R,R) of convergence of $\Sigma a_n x^n$, the series $\Sigma a_n(R-\varepsilon)^n$ converges absolutely, i.e. the series ΣM_n converges. Therefore, in view of (5.10), it follows by Weierstran M-test that the series $\Sigma a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

Now, if I is any closed interval contained in (–R,R), then we can find $\varepsilon>0$ such that $I \subseteq [-R+\varepsilon, R-\varepsilon]$. Since $\Sigma a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$, it converges uniformly on I.

8. Show that a power series can be differentiated term by term within the interval on convergence or Prove theorem 3.5 (b).

Solution: Let R be the radius of convergence of the power series $\Sigma a_n x^n$ and let $x_0 \in$ (–R, R), the interval of convergence of $\Sigma a_n x^n$. We choose $\varepsilon>0$ such that $x_0 \in [-R+\varepsilon, R-\varepsilon]$.

Let $f_n(x) = a_n x^n$. Then f_n is differentiable on $[-R+\varepsilon, R-\varepsilon]$ and $f_n'(x) = na_n x^n$. Now, consider the power series $\Sigma a_n x^n$. Since

$$\overline{\lim} |na_n|^{\frac{1}{n}} = \overline{\lim} |a_n|^{\frac{1}{n}} . \overline{\lim}\, n^{\frac{1}{n}}$$
$$= \overline{\lim} |a_n|^{\frac{1}{n}} = R,$$

$\Sigma na_n x^n$ has the same interval of convergence as of $\Sigma a_n x^n$. Therefore, (by theorem 3.5(a)) $\Sigma na_n x^n = \Sigma f_n'(x)$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$. Thus by theorem 1.10 (b), the sum function of $\Sigma a_n x^n$ is differentiable and

$$\frac{d}{dx}\left(\sum a_n x^n\right) = \sum f_n'(x) = \sum na_n x^n,$$

at every point in $[-R+\varepsilon, R-\varepsilon]$, and in particular

$$\frac{d}{dx}\left(\sum a_n x^n\right)_{x=x_0} = \sum na_n x_0^n.$$

Since x_0 is arbitrary in $(-R, R)$, $\Sigma a_n x^n$ can be differentiated term by term in $(-R, R)$.

Section 4:

9. Prove Theorem 4.6

Solution: Let $\int$ denote the integral over $[a, b]$, and Σ the sum from 1 to n. Now,

$$\int f\overline{t_n} = \int f \sum \overline{\gamma_m}\,\overline{\phi_m} = \sum \overline{\gamma_m} \int f\overline{\phi_m} = \sum \overline{\gamma_m} c_m.$$

Again. $\int |t_n|^2 = \int t_n \overline{t_n} = \int \left(\sum \gamma_m \phi_m\right)\left(\sum \overline{\gamma_k}\,\overline{\phi_k}\right) = \sum\sum \gamma_m \overline{\gamma_k} \int \phi_m \overline{\phi_k} = \sum |\gamma_m|^2$

because $\int \phi_m \overline{\phi}_k = \begin{cases} 0 & \text{if } m \neq k \\ 1 & \text{if } m = k. \end{cases}$

Therefore, $\int |f - t_n|^2 = \int (f - t_n)\left(\overline{f} - \overline{t_n}\right) = \int |f|^2 - \int f\overline{t_n} - \int \overline{f} t_n + \int |t_n|^2$

$$= \int |f|^2 - \sum c_m \overline{\gamma_m} - \sum \overline{c_m} \gamma_m + \sum |\gamma_m|^2$$
$$= \int |f|^2 + \sum \left\{(\gamma_m - c_m)\left(\overline{\gamma_m} - \overline{c_m}\right) - c_m \overline{c_m}\right\}$$

$$= \int |f|^2 + \sum |\gamma_m - c_m|^2 - \sum |c_m|^2,$$

which obviously take the minimum value if and only if $\gamma_m = c_m$. Thus

$$\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx,$$

and the equality holds if and only if $\gamma_m = c_m$ for all m.

10. With the notations as in theorem 4.6, prove that

$$\int_a^b |s_n - f|^2 dx = \int_a^b |f|^2 dx - \sum_{m=1}^{n} c_m^2.$$

Solution: We have

$$\int_a^b f\overline{s_n} = \int_a^b f\sum_{m=1}^{n}\overline{c_m}\,\overline{\phi_m} = \sum_{m=1}^{n}\overline{c_m}\int_a^b f\,\overline{\phi_m} = \sum_{m=1}^{n}\overline{c_m}c_m = \sum_{m=1}^{n}|c_m|^2, \text{ and}$$

$$\int_a^b |s_n|^2 = \int_a^b s_n\overline{s_n} = \int_a^b\left(\sum_{m=1}^{n}c_m\phi_m\right)\left(\sum_{k=1}^{n}\overline{c}_k\,\overline{\phi_k}\right) = \sum_{m=1}^{n}|c_m|^2.$$

Thus $$\int_a^b |f-s_n|^2 = \int_a^b (f-s_n)(\overline{f}-\overline{s_n}) = \int_a^b |f|^2 - \int_a^b f\overline{s_n} - \int_a^b \overline{f}s_n + \int_a^b |s_n|^2$$

$$= \int_a^b |f|^2 - \sum_{m=1}^{n}|c_m|^2 - \sum_{m=1}^{n}|c_m|^2 + \sum_{m=1}^{n}|c_m|^2$$

$$= \int_a^b |f|^2 - \sum_{m=1}^{n}|c_m|^2.$$

11. Prove theorem 4.7.

Solution: Let $s_n(x) = \sum_{m=1}^{n}c_m\phi_m(x)$. Then, as in example (10), we have

$$\int_a^b |f-s_n|^2dx = \int_a^b |f|^2dx - \sum_{m=1}^{n}|c_m|^2.$$

Since $\int_a^b |f-s_n|^2 \ge 0$, we get $$0 \le \sum_{m=1}^{\infty}|c_m|^2 \le \int_a^b |f|^2. \qquad ....(5.11)$$

As the $\int_a^b |f|^2$ is independent of n, letting $n\to\infty$ we get $\sum_{m=1}^{\infty}|c_m|^2 \le \int_a^b |f|^2$.

Since f is integrable, $\sum|c_m|^2$ is a convergent series and therefore $\lim_{n\to\infty}c_n = 0$.

6. Exercises:

1. Show that $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$ is uniformly convergent on $[\delta, \pi/2]$ for any $\delta>0$.

2. Prove theorem 1.5.

3. Examine uniform convergence of the series

$$\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\cdots$$

4. Show that the series

$$\sum_{k=1}^{\infty}[\, kxe^{-kx^2} - (k-1)xe^{-(k-1)x^2}\,]$$

converges pointwise but not uniformly on [0, 1].

5. Prove theorem 1.7. Use theorem 1.7 (a) to show that the sequence $\{f_n\}$, where $f_n(x) = x^n$, is not uniformly convergent on [0,1].

6. Prove theorem 2.4.

7. Suppose $\{f_n\}$ is a equicontinuous system of functions on [a, b] and $\{f_n\}$ converges pointwise on [a,b]. Prove that $\{f_n\}$converges uniformly on [a,b].

8. Show that the sequence $\{f_n\}$, where $f_n(x) = nxe^{-nx^2}$ is equicontinuous on [0,1].

9. Use theorem 3.4 (b) to show that

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)=\sum_{n=1}^{\infty}x^n \qquad \text{for } -1 < x < 1. \text{ Deduce that}$$

$$\log\frac{1}{1-x} = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

10. Define orthonormal and orthogonal family of functions.

Show that

(a) $\frac{1}{\sqrt{2\pi}}e^{inx}$, n= 1, 2, ...,

(b) $\frac{1}{\sqrt{2\pi}},\frac{\cos x}{\sqrt{\pi}},\frac{\sin x}{\sqrt{\pi}},\frac{\cos 2x}{\sqrt{\pi}},\frac{\sin 2x}{\sqrt{\pi}},\cdots$, are orthonormal families of functions on [-π, π].

For proofs of the theorems stated and for more exercise, see the following books:

1. W. Rudin, Principles of Mathematical Analysis.
2. S.C. Malik & S,Arora, Mathematical Analysis.

G.U. Questions

1996

2 (a) Let $\{f_n\}$ be a sequence of functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a,b]$, and let

$$M_n = \sup\{\,|f_n(x) - f(x)| : x \in [a,b]\}.$$

Prove that $\{f_n\}$ converges uniformly to f on [a,b] if and only if $\lim_{n\to\infty} M_n = 0$. 5

(b) Examine uniform convergence of the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \cdots \qquad \text{for } x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$ 5

(c) Show that the series $\sum \left[kxe^{-kx^2} + (k-1)xe^{-(k-1)x^2}\right]$

converges pointwise but not uniformly for $x \in [0,1]$. $2+4=6$

3. **(a)** If $\{f_n\}$ is a sequence of equicontinuous function on [a,b] and if $\{f_n\}$converges pointwise on [a,b], prove that $\{f_n\}$converges uniformly on [a,b]. 6

(b) Prove that a power series $\Sigma a_n x^n$ converges uniformly on any closed interval contained in (–R,R) where R is the radius of convergence of the power series. 5

(c) Define an **orthogonal family** of functions and give an example. What is the **Fourier series** of a function relative to a sequence of orthonormal functions defined on a closed interval? 2+1+2 = 5

1997

2. **(a)** Let D be a subset of R and a sequence of functions $\{f_n\}$ be uniformly convergent on D to a limit function f. Let $x_0 \in D$ and $\lim_{x\to x_0} f_n(x) = a_n$. Then prove that

(i) the sequence $\{a_n\}$ is convergent;

(ii) $\lim_{x\to x_0} f(x)$ exists and equals $\lim_{n\to\infty} a_n$. 3+2 = 5

(b) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is uniformly convergent on $\left[\delta, \frac{\pi}{2}\right]$ for any $\delta>0$. 5

(c) Show that for $x \in (-1,1)$

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\cdots=\frac{1}{1-x}.$$ 6

3. **(a)** For each $n \in N$, let

$$f_n(x) = nx^2, \quad 0 \le x \le \frac{1}{n}$$
$$= x, \quad \frac{1}{n} < x \le 1.$$

(i) Show that the sequence $\{f_n\}$ converges pointwise to a limit function f on [0, 1].

(ii) Calculate M_n where $M_n = \sup_{x \in [0,1]} |f_n(x) - f(x)|$.

Show that the sequence $\{f_n\}$ is uniformly continuous on [0, 1] 3 + 3 = 6

(b) Show that a (real) power series can be differentiated term by term within the interval of convergence. 5

(c) If $\{\phi_n\}$ is orthonormal on [a, b] and if $\sum c_n \phi_n(x)$

is the Fourier series of f relative to $\{\phi_n\}$, then prove that

$$\sum_{n=1}^{\infty} |c_n|^2 \le \int_a^b |f(x)|^2 dx$$

and deduce that $c_n \to o$ as $n \to \infty$. 4+1 = 5

1998

2. (a) Define **uniform convergence** of a sequence of functions defined on an interval of the real line. Using this definition establish a necessary and sufficient condition for uniform convergence of a sequence $f_n\}$ defined on $[a,b] \subset R$. 1+6 = 7

(b) Suppose $\{f_n\}$ is a function of monotonic functions on [a,b], and $\{f_n\}$ converges pointwise to a continuous function f on [a,b]. Prove that the convergence is uniform on [a,b]. 4

(c) If $\{f_n\}$ is a uniformly convergent sequence of continuous functions on [a,b], then prove that it is equicontinuous on [a,b]. 5

3. (a) Justifying all the steps to prove that

$$\int_0^1 \sum_{n=1}^{\infty} \frac{x^n}{n^2} dx = \sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}. \qquad 6$$

(b) If a series Σf_n of differentiable functions converges pointwise to f on [a,b], and each f_n' is continuous on [a,b], and the series $\Sigma f_n'$ converges uniformly to g on [a,b], then prove that $f'(x) = g(x)$. 5

(c) What is a trigonometric series? When a trigonometric series is said to be the **Fourier series** of a function f? How is this notion generalized to the notion of a Fourier series of a function **relative** to a sequence of orthonormal functions defined on a closed interval of R? 1+1+3 = 5

UNIT 2

The Riemann-Stieltjes Integral

1.Introduction :

Calculus deals with two geometric problems: finding the tangent line to a curve, and finding the area of a region under a curve. The first is studied by a limit process known as *differentiation* and the second by another limit process – *integration*. The formation of a independent theory of integration is due to the German mathematician, Georg Friedrich Bernhard Riemann, who gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence of the geometric concepts. The Riemann integral, which is studied in any undergraduate course of mathematics, has many refinements and generalizations, the most noteworthy being Lebesgue theory of integration. In this chapter, we discuss the extension due to a French mathematician Thomas Jan Stieltjes known as the *Riemann-Stieltjes integral*. The Riemann-Stieltjes integration is the process of integrating a bounded function f with respect to another bounded(monotonic) function α. In case $\alpha(x) = x$, this integral becomes Riemann integral of f.

2. Riemann-Stieltjes Integral

A *partition* P of [a,b] is a finite set of points, say

$$P=\{x_0, x_1,\ldots\ldots x_n\}, \quad (1)$$

such that $a=x_0<x_1<\ldots\ldots<x_n=b$. We shall use the symbol Δx_i to denote the ith subinterval $[x_{i-1}, x_i]$, of the partition as also its length $x_i - x_{i-1}$. The *mesh* or *norm* of P is defined by

$$\mu(P) = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$$

A partition P′ of [a,b] is *finer* than P(or a *refinement* of P) if $P \subseteq P'$. Clearly, in that case $\mu(P')\le\mu(P)$ If P_1 and P_2 are two partitions of [a, b] then by their *common refinement* we mean the refinement $P=P_1\cup P_2$ of both P_1 and P_2.

Definition 2.1 Let f and α be bounded real functions on [a,b] and α be monotonically increasing on [a,b], $b\ge a$. Let P be a partition of [a,b]. We write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}),\ i = 1, 2, \cdots, n.$$

Since α is monotonically increasing, $\Delta\alpha_i \ge 0$. We have

$$\sum_{i=1}^{n} \Delta\alpha_i = \alpha(b) - \alpha(a).$$

Let m_i and M_i be the infimum and the supremum of f on Δx_i. The *upper* and the *lower* sums of f corresponding to P are defined by

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i,$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

The *upper* and *lower integrals* of f with respect to α are defined as follows:

$$\overline{\int_a^b} f d\alpha = \inf U(P, f. \alpha), \qquad \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha),$$

where the infimum and the supremum are taken over all partitions P of [a,b]. We say that f is *Riemann-Stieltjes intergrable* (or simply *integrable*) with respect to α on [a,b] and write $f \in R(\alpha)$ if

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha.$$

In that case, the *Riemann-Steiltjes integral* of f with respect to α is defined to be

$$\int_a^b f d\alpha = \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha.$$

Theorem 2.2 (Riemnn's Condition of Integrability) *A function f is integrable with respect to* α *on* [a.b] *if and only if for every* $\in > 0$ *there exist a partition P of* [a,b] *such that* $U(P,f,\alpha) - L(P,f,\alpha) < \in$.

Theorem 2.3 *Suppose* $f \in R(\alpha)$ *on* [a,b], $m \le f \le M$, ϕ *is continuous on* [m,M] *and* $h(x) = \phi(f(x))$ *on* [a,b]. *Then* $h \in R(\alpha)$ *on* [a,b].

3. The Integral as a Limit of Sums

Definition 3.1 Corresponding to every partition P of [a,b] let us choose points $t_1, t_2, \ldots, t_n$ such that $x_{i-1} \le t_i \le x_i (i=1,2,\ldots,n)$, and let us consider the sum

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \qquad (2)$$

We say that $\lim_{\mu(P)\to 0} S(P, f, \alpha) = A$ (3)

if, for every $\in > 0$, there exists a $\delta > 0$ such that $\mu(P) < \delta$ implies $|S(P, f, \alpha) - A| < \varepsilon$ with all choices of t_i.

Theorem 3.2 *If* $\lim S(P,f,\alpha)$ *exists as* $\mu(P) \to 0$ *then*

$$f \in R(\alpha), \quad \text{and} \quad \lim_{\mu(P)\to 0} S(P.f.\alpha) = \int_a^b f d\alpha$$

Theorem 3.3 *If* f *is continuous on* [a,b], *then* $f \in R(\alpha)$ *on* [a,b]. *Moreover,* $\lim_{\mu(P)\to 0} S(P,f,\alpha)$ *exists and equals* $\int_a^b f d\alpha$.

Reduction to a Riemann integral

Theorem 3.4 *If* $f \in R$ (i.e. *if* f *is Riemann integrable*) *on* [a,b] *and* α *is monotonic increasing on* [a,b] *such that* $\alpha' \in R$ *on* [a,b], *then* $f \in R(\alpha)$, *and*

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$

Integration by parts

Theorem 3.5 *If* f *is continuous and monotonic increasing and* α *is monotonic increasing on* [a,b], *then*

$$\int_a^b f d\alpha = [f(x)\alpha(x)]_a^b - \int_a^b \alpha\, df,$$

i.e. $$\int_a^b f d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a)$$

4. Integration of vector valued functions

Definition 4.1 *Let* $f_1, f_2, \ldots, f_k$ *be real valued functions on* [a,b], *and for* $x \in [a,b]$ *let* $\bar{f}(x) = (f_1(x), f_2(x), \cdots, f_k(x))$. *Then* $\bar{f}$ *is a vector valued function of* [a,b] *into* R^k. *If* α *is monotonically increasing on* [a,b], *then we say* $\bar{f}$ *is Riemann-Stieltejes integrable with respect to* α *on* [a,b] (*and write* $\bar{f} \in R(\alpha)$) *if* $f_i \in R(\alpha)$ *on* [a.b] *for* i=1,2,...,k. *If this is the case, then we define*

$$\int_a^b \bar{f} d\alpha = \left(\int_a^b f_1 d\alpha, \int_a^b f_2 d\alpha, \cdots, \int_a^b f_k d\alpha\right)$$

If $\bar{f}$ is a vector-valued function of [a,b] into R^k, then by $|\bar{f}|$ we mean the real valued function defined by

$$|\bar{f}|(x) = \left(f_1^2(x) + f_2^2(x) + \cdots + f_k^2(x)\right)^{1/2}$$

Theorem 4.2 *If* $\bar{f}$ *maps into* R^k *and if* $\bar{f} \in R(\alpha)$ *for some monotonically increasing function* α *on* [a,b], *then* $|\bar{f}| \in R(\alpha)$, *and*

$$\left|\int_a^b \bar{f} d\alpha\right| \le \int_a^b |\bar{f}| d\alpha.$$

5 Functions of Bounded Variation

Definition 5.1 Let $\bar{f}$ be a vector valued function defined on [a,b]. Corresponding to a partition P of [a,b], we consider the sum $\sum_{i=1}^{n}|\bar{f}(x_i)-\bar{f}(x_{i-1})|$. If the set of these sums for different partitions of [a,b] is bounded above, then the function $\bar{f}$ is said to be of *bounded variation* on [a,b]. In that case we define

$$V(\bar{f};a,b)=\sup\sum_{i=1}^{n}|\bar{f}(x_i)-\bar{f}(x_{i-1})|,$$

where the supremum is taken over all partitions of [a,b], and we call $V(\bar{f};a,b)$ the *total variation* of $\bar{f}$ on [a,b].

Theorem 5.2 *Let* $\bar{f}=(f_1,f_2,\ldots,f_k)$ *be a vector valued function of* [a,b] *into* R^k. *Then* $\bar{f}$ *is of bounded variation on* [a,b] *if and only if each of the functions* f_i, $i=1,2,\ldots,k$, *is of bounded variation on* [a,b]. *For* $1\le i\le k$, *we have*

$$V(f_i;a,b)\le V(\bar{f};a,b)\le\sum_{i=1}^{k}V(f_i;a,b).$$

It is easy to see that if $\bar{f}$ is of bounded variation on [a,b], then $\bar{f}$ is of bounded variation on any subinterval [a,c] and [c,b], where a<c<b, and $V(\bar{f},a,b)=V(\bar{f},a,c)+V(\bar{f},c,b)$.

Definition 5.3 Let $\bar{f}$ be a function of bounded variation on [a,b]. For any $x\in[a,b]$ define $v_{\bar{f}}(x)=V(\bar{f},a,x)$ Then $v_{\bar{f}}$ is a monotonic increasing bounded real function on [a,b] and is called the *variation function* of $\bar{f}$.

Theorem 5.4 (Jordan Theorem) *If* f *is a real function of bounded variation on* [a,b], *then there exists monotonic increasing functions* p *and* q *on* [a,b] *such that, for* $a\le x\le b$, $f(x)=p(x)-q(x)$ *and* $v_f(x)=p(x)+q(x)$.

6. Mean Value Theorems

Theorem 6.1 [First Mean Value Theorem] *If* f *is continuous on* [a,b], *and* α *is monotonic increasing on* [a,b], *then there is* $\xi\in[a,b]$ *such that*

$$\int_a^b f\,d\alpha=f(\xi)\{\alpha(b)-\alpha(a)\}.$$

Definition 6.2 Let α be a real function on [a,b] of bounded variation. Then $\alpha=\beta-\gamma$, for some monotonically increasing functions β and γ on [a,b]. If $f\in R(\beta)$ and $f\in R(\gamma)$ on [a,b], then we say that $f\in R(\alpha)$ on [a,b] and define

$$\int_a^b f\,d\alpha=\int_a^b f\,d\beta-\int_a^b f\,d\gamma.$$

Theorem 6.3 [Second mean Value Theorem] *If* f *monotonic and* α *is continuous and of bounded variation on* [a,b] *then there exists* $\xi \in [a,b]$ *such that*

$$\int_a^b f\,d\alpha = f(a)[\alpha(\xi) - \alpha(a)] + f(b)[\alpha(b) - \alpha(\xi)]$$

Change of variable

Theorem 6.4 *If*

(i) f *is continuous on* [a,b], *and*

(ii) ϕ *is a continuous and strictly monotonic function on* $[\alpha,\beta]$, where $a=\phi(\alpha)$,

$b=\phi(\beta)$, then $\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(y))d\phi(y)$

Solved Problems

1. Prove Theorem 2.2.

Solution : First, let the given condition be satisfied. For every P we have

$$L(P, f, \alpha) \le \underline{\int_a^b} f\,d\alpha \le \overline{\int_a^b} f\,d\alpha \le U(P, f, \alpha)$$

Thus the given condition implies

$$0 \le \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon.$$

Hence, if the given condition is satisfied for every $\varepsilon > 0$, we have

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha,$$

that is, $f \in R(\alpha)$.

Conversely, suppose $f \in R(\alpha)$, and let $\varepsilon > 0$ be given. Then there exists partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) - \int_a^b f\,d\alpha < \frac{\varepsilon}{2},$$

$$\int_a^b f\,d\alpha - L(P_2, f, \alpha) < \frac{\varepsilon}{2}.$$

We choose P to be the common refinement of P_1 and P_2. Then by the fact that

$$L(P_1, f, \alpha) \le L(P, f, \alpha) \le U(P, f, \alpha) \le U(P_1, f, \alpha),$$

we have

$$U(P,f,\alpha)-L(P,f,\alpha)\le U(P_1,f,\alpha)-L(P_2,f,\alpha)<\varepsilon$$

This proves the result.

2. If $f \in R(\alpha)$, $g\in R(\alpha)$ on [a,b] then $f+g \in R(\alpha)$, $cf \in R(\alpha)$ for every constant c, and

$$\int_a^b (f + g)d\alpha = \int_a^b fd\alpha + \int_a^b gd\alpha,$$
$$\int_a^b cfd\alpha = c\int_a^b fd\alpha.$$

Solution: First, we prove the result for cf. We note that for any partition P of [a,b]

$L(P, cf, \alpha) = cL(P, f, \alpha)$ and $U(P, cf, \alpha) = cU(P, f, \alpha)$ if $c \ge 0$, and
$L(P, cf, \alpha) = cU(P, f, \alpha)$ and $U(P, cf, \alpha) = cL(P, f, \alpha)$ if $c < 0$.

Consequently, we have

$$\underline{\int_a^b} cfd\alpha = \sup cL(P,f,\alpha)= c\sup L(P,f,\alpha)= c\underline{\int_a^b} fd\alpha \quad \text{if } c \ge 0, \qquad \text{and}$$

$$\underline{\int_a^b} cfd\alpha = \sup cU(P,f,\alpha)= c\inf U(P,f,\alpha)= c\overline{\int_a^b} fd\alpha \quad \text{if } c<0.$$

Similarly,

$$\overline{\int_a^b} cfd\alpha = c\overline{\int_a^b} fd\alpha \quad \text{if } c\ge 0 \text{ and}$$
$$= c\underline{\int_a^b} fd\alpha \quad \text{if } c < 0$$

Since $f \in R(\alpha)$, both upper and lower integrals of f are equal to its integral. Consequently, replacing them by $\int_a^b fd\alpha$ in the above equations we see that the upper and lower integrals of cf are both equal to $c\int_a^b fd\alpha$. Hence, $cf \in R(\alpha)$ and

$$\int_a^b cfd\alpha = c\int_a^b fd\alpha.$$

Next, we prove the result for (f+g). Suppose h=f+g. Note that for any partition P of [a,b] we have

$$L(P, f, \alpha) + L(P, g, \alpha) \le L(P, h, \alpha) \le U(P, h, \alpha) \le U(P, f, \alpha) + U(P, g, \alpha) \quad (1)$$

Let $\varepsilon > 0$ be given . Since, f , g $\in R(\alpha)$,there exist partitions P_1 and P_2 such that

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\varepsilon}{2} \text{ and } U(P_2, g, \alpha) - L(P_2, g, \alpha) < \frac{\varepsilon}{2}.$$

These inequalities persist if P_1 and P_2 are replaced by their common refinement P.

Then (1) implies

$$U(P, h, \alpha) - L(P, h, \alpha) < \varepsilon$$

and hence $h = f + g \in R(\alpha)$. Now, with the same P we have

$$U(P, f, \alpha) < \int f d\alpha + \frac{\varepsilon}{2} \quad \text{and} \quad U(P, g, \alpha) < \int g d\alpha + \frac{\varepsilon}{2},$$

and hence by (1) we have

$$\int h d\alpha \le U(P, h, \alpha) < \int f d\alpha + \int g d\alpha + \varepsilon$$

Since $\in$ was arbitrary, we conclude that

$$\int h d\alpha \le \int f d\alpha + \int g d\alpha. \qquad (2)$$

Replacing f and g in (2) by $-f$ and $-g$ we get

$$-\int h d\alpha \le -\int f d\alpha - \int g d\alpha,$$

that is,

$$\int h d\alpha \ge \int f d\alpha + \int g d\alpha. \qquad (3)$$

Combining (2) and (3), we get the required result.

3. Prove Theorem 2.3.

Solution: Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on [m,M], there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in R(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b] such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \qquad (1)$$

Let m_i, M_i be the infimum and the supremum of f on Δx_i and m_i^*, M_i^* be the analogous numbers for h. We divide the numbers 1,2,…,n into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \le \in$.

For i $\in B$, $M_i^*-m_i^* \le 2K$, where $K = \sup|\phi(t)|$, $m \le t \le M$. By (1), we have

$$\delta \sum_{i\in B} \Delta\alpha_i \le \sum_{i\in B} (M_i - m_i)\Delta\alpha_i < \delta^2$$

so that $\sum_{i\in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P,h,\alpha) - L(P,h,\alpha) &= \sum_{i\in A}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i\in B}(M_i^* - m_i^*)\Delta\alpha_i \\ &\le \in [\alpha(b) - \alpha(a)] + 2K\delta \le \in [\alpha(b) - \alpha(a) + 2K] \end{aligned} \quad (2)$$

Since $\in$ is arbitrary, Riemann's Condition of Integrability implies that $h \in R(\alpha)$.

4. Prove theorem 3.2.

Solution : Let lim $S(P,f,\alpha)$ exists as $\mu(P)\to 0$ and equals A. Given $\varepsilon>0$ there exists $\delta>0$ such that $\mu(P)<\delta$ implies

$$A - \frac{\varepsilon}{2} < S(P,f,\alpha) < A + \frac{\varepsilon}{2}.$$

WE choose one such P. If we let the points t_i range over the intervals $[x_{i-1}, x_i]$ and take the lub and the glb of the numbers $S(P,f,\alpha)$ obtained in this way, then from the above inequality we get

$$A - \frac{\varepsilon}{2} \le L(P,f,\alpha) \le U(P,f,\alpha) < A + \frac{\varepsilon}{2}. \quad (1)$$

Thus

$$U(P,f,\alpha) - L(P,f,\alpha) \le \varepsilon,$$

and hence, by Riemann's condition of integrability, we have $f \in R(\alpha)$ on [a,b].

Moreover, because $L(P,f,\alpha) \le \int_a^b f d\alpha \le L(P,f,\alpha)$, from (1) we get

$$\int_a^b f d\alpha = A = \lim_{\mu(P)\to 0} S(P,f,\alpha)$$

5. Prove that a monotonic bounded function on a closed interval is of bounded variation. Show by an example that a continuous function may not be so.

Solution: Let f be bounded and monotonic increasing on [a,b] . For any partition $P=\{x_0, x_1, \ldots, x_n\}$ of [a,b] we have

$$\sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \sum_{i=1}^n \{f(x_i) - f(x_{i-1})\} = f(b) - f(a), \quad (1)$$

and, consequently, f is of bounded variation and V(f;a,b)=f(b)-f(a). Similarly, if f is bounded and monotonic decreasing then V(f;a,b)=f(a)-f(b) and the result follows.

Next, consider the function f defined on [a,b] by

$$f(x) = \begin{cases} x \sin x \dfrac{\pi}{x}, & \text{when } 0 < x \le 1 \\ 0, & \text{when } x = 0. \end{cases}$$

Then, f is a continuous function but not of bounded variation (verify).

6. Prove that the variation function $v_{\bar{f}}$ of a vector valued function $\bar{f}$ of bounded variation is continuous if and only if $\bar{f}$ is continuous.

Solution : First, Let $v_{\bar{f}}$ be continuous at a point $c \in [a,b]$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|v_{\bar{f}}(x) - v_{\bar{f}}(c)| < \varepsilon \text{ for all } x \in [a,b] \text{ with } |x - c| < \delta$$

Moreover, $|\bar{f}(x) - \bar{f}(c)| \le |v_{\bar{f}}(x) - v_{\bar{f}}(c)|$. Hence

$$|\bar{f}(x) - \bar{f}(c)| < \varepsilon \text{ for all } x \in [a,b] \text{ with } |x - c| < \delta,$$

which implies that $\bar{f}$ is continuous at c.

Conversely, let $\bar{f}$ be continuous at c, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\quad |\bar{f}(x) - \bar{f}(c)| < \dfrac{\varepsilon}{2}$ for all $x \in [a,b]$ with $|x - c| < \delta$, (1)

By definition of total variation, we can find a partition of [c,b] such that

$$\sum_{i=1}^{n} |\bar{f}(x_i) - \bar{f}(x_{i-1})| > V(\bar{f};c,b) - \frac{\varepsilon}{2}, \tag{2}$$

and such that $0 < x_1 - c < \delta$. (3)

Thus $|\bar{f}(x_1) - \bar{f}(c)| < \dfrac{\varepsilon}{2}$. Again (2) gives on using (3) ,

$$V(\bar{f};c,b) - \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \sum_{i=1}^{n} |\bar{f}(x_i) - \bar{f}(x_{i-1})| \le \frac{\varepsilon}{2} + V(\bar{f};x_1,b)$$

i.e. $V(\bar{f};c,b) - V(\bar{f};x_1,b) < \varepsilon$ or $|v_{\bar{f}}(x_1) - v_{\bar{f}}(c)| < \varepsilon$.

This shows that

$\lim_{x \to c+} v_{\bar{f}}(x) = v_{\bar{f}}(c)$. Similarly, it can be shown that $\lim_{x \to c-} v_{\bar{f}}(x) = v_{\bar{f}}(c)$.

Hence $v_{\bar{f}}$ is continuous at c.

The result follows because c is an arbitrary point of [a,b].

Exercise

1. If P^* is a refinement of P, then prove that

$$L(P^*,f,\alpha) \geq L(P,f,\alpha) \text{ and } U(P^*,f,\alpha) \leq U(P,f,\alpha).$$

2. Using Ex.1 prove that for any two partitions P_1, P_2 of [a,b]

$$L(P_1,f,\alpha) \leq U(P_2,f,\alpha).$$

3. Prove that $\underline{\int_a^b} fd\alpha \leq \overline{\int_a^b} fd\alpha$.

4. If $f \in R(\alpha)$, then there exists a number λ between the bounds of f such that

$$\int_a^b fd\alpha = \lambda\{\alpha(b) - \alpha(a)\}$$

5. If $f \in R(\alpha)$, $g \in R(\alpha)$ on [a,b], $a \leq b$ such that $f(x) \geq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b fd\alpha \geq \int_a^b gd\alpha.$$

6. If $f \in R(\alpha)$ on [a,b], and if $a<c<b$, then $f \in R(\alpha)$ on [a,c] and on [c,b], and

$$\int_a^c fd\alpha + \int_c^b fd\alpha = \int_a^b fd\alpha.$$

7. If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$, then $f \in R(\alpha_1+\alpha_2)$ and

$$\int_a^b fd(\alpha_1 + \alpha_2) = \int_a^b fd\alpha_1 + \int_a^b fd\alpha_2;$$

if $f \in R(\alpha)$ and c is a positive constant, then $f \in R(c\alpha)$ and $\int_a^b fd(c\alpha) = c\int_a^b fd\alpha$.

8. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on [a,b], then show that

(a) $fg \in R(\alpha)$;

(b) $|f| \in R(\alpha)$ and $\left|\int_a^b f d\alpha\right| \le \int_a^b |f| d\alpha$.

9. Suppose α increases on [a,b], $a \le x_0 \le b$, α is continuous at x_0, $f(x_0)=1$, and $f(x)=0$ if $x \ne x_0$. Prove that $f \in R(\alpha)$ and that $\int_a^b f d\alpha = 0$.

10. Suppose $f \ge 0$, f is continuous on [a,b], and $\int_a^b f d\alpha = 0$. Prove that $f(x)=0$ for all $x \in$ [a,b].

11. Show that $\int_0^3 x d(x - [x]) = \frac{3}{2}$, where [x] is the greatest integer not exceeding x.

12. If f is monotonic on [a,b], and if α is continuous on [a,b], then show that $f \in R(\alpha)$ on [a,b].

13. Show with an example that a function f may be integrable with respect to an increasing function α on an interval [a,b], although $\lim_{\mu(P)\to 0} S(P, f, \alpha)$ may not exist.

14. If f is continuous on [a,b] and α has a continuous derivative on [a,b], then

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx.$$

15. Prove that a function of bounded variation is always bounded. Give an example to disprove the converse of this result.

16. If a vector valued function $\bar{f}$ with domain [a,b] and a range of subset R^m is of bounded variation on [a,b] and $v_{\bar{f}}$ is continuous at $c \in$ [a,b], then prove that $\bar{f}$ is continuous at c.

Questions from G.U. Question Papers

1996

1 (a) Let f and α be bounded functions on [a,b] and α be monotonic increasing on [a,b]. If lim $S(P,f,\alpha)$ exists as $\mu(P) \to 0$, then prove that

$$f \in R(\alpha) \text{ and } \lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f d\alpha.$$

State a sufficient condition for existence of $\lim_{\mu(P)\to 0} S(P, f, \alpha) = \int_a^b f d\alpha$. 5+1=6

(b) If $f \in R(\alpha)$ on [a,b] and c is any constant, then prove that $cf \in R(\alpha)$ on [a,b]and

$$\int_a^b cf d\alpha = c\int_a^b f d\alpha. \qquad 4$$

(c) If a vector valued continuous function $\bar{f}$ with domain [a,b] and range a subset of R^m is of bounded variation on [a,b], then the variation $v_{\bar{f}}$ is continuous on [a,b]. 6

1997

1. (a) Suppose $f \in R(\alpha)$ on $[a,b] \subseteq \mathbf{R}$, $m \le f(x) \le \mathbf{M}$, ϕ is continuous on [m,**M**] and $h(x) = \phi(f(x))$ on [a,b]. Prove that $f \in R(\alpha)$ on [a,b]. 6

   (b) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ then prove that $f \in R(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \qquad 4$$

(c) Prove that a bounded monotonic function is of bounded variation, but a continuous function may not be so.

Remark. For proofs of the stated theorems and hints for the exercises the followinig two books are recommended.

1. Walter Rudin, *Principles of Mathematical Analysis.* 3rd Edition.
2. S.C.Malik and Savita Arora, *Mathematical Analysis*, 2nd Edition.

Integration in R^2 and R^3

In this chapter, we discuss the various forms of generalizations of Riemann integration of functions of one variable to the integrals of those functions whose domains are in R^2 and R^3. The classes of functions we will discuss will contain both scalar and vector-valued functions of two or three variables. We will introduce Green's theorem and its generalizations, viz. Stoke's theorem and Gauss' theorem.

1. Integration in R^2

1.1 Definition : A **plane curve** is a function C with domain a subset of R and range a subset of R^2. Generally, the range of a plane curve is also called a plane curve and usually described as the set of points (x, y) in R^2 for which

$$x = X(t),\ y = Y(t),\ a \le t \le b.$$

The curve is said ti be **closed** if $X(a) = X(b)$ and $Y(a) = Y(b)$, **smooth** if X' and Y' exists and do not vanish simultaneously, and **simple** if it does not have multiple points, i.e. if it does not pass through a point more than once.

1.2 **Line Integral:** Let $x = X(t)$, $y = Y(t)$, $a \le t \le b$ be a plane curve C anf f be a bounded function defined at every point on C. The Riemann-Stieltje's integrals

$$\int_C f(x,y)dx = \int_a^b f(X(t), Y(t))dX(t)$$

and $$\int_C f(x,y)dy = \int_a^b f(X(t), Y(t))dY(t),$$

if they exists are called the **line integrals** of f over C. For a vector-valued function $\overline{F} = (f,g)$ defined on C, the line integral of $\overline{F}$ over C is defined to be

$$\int_C \overline{F}\cdot d\overline{r} = \int_C (fdx + gdy).$$

1.3 **Definitions:** A **domain** is an open connected set of points, a **region** is a domain bounded by close curves together with the boundary points. A region is **regular with respect to y-axis** , if any line parellel to y-axis meets its boundary at atmost two points(i.e. if the boundary of the region is given by $x = a$, $x = b$, $y = \phi(x)$, $y = \psi(x)$, ϕ and ψ are continuous and $\phi \le \psi$). A region which is regular with respect to x-axis is similarly defined. A **piece-wise regular** region is one which can be expressed as a finite union of regular regions. The contour(boundary) of a region is said to be described in a **positive sense**, if the interior of the region lies to the left as one advances along the contour.

1.4 Double Integral: Let E be a region and f be a bounded function of two variables defined at each point of E. Consider any partition P of E. That is, P is a finite set $\{\Delta S_1, \Delta S_2, \ldots, \Delta S_n\}$ of small areas obtained by dividing the region E by finite number of curves. Let (ξ_i, η_i) be any point in ΔS_i, i = 1,2,...,n. Consider the sum

$$S(P,f) = \sum f(\xi_i, \eta_i)\Delta S_i.$$

If $\lim_{\mu(P)\to 0} S(P,f)$ exists, (where by $\mu(P)$ we mean the maximum of the areas ΔS_i), then we say that f is integrable over E and define the **double integral** of f over E to be

$$\iint_E f\,dx\,dy = \lim_{\mu(P)\to 0} S(P,f)$$

1.5 Reduction to Iterated Integral: If a double integral $\iint_E f\,dx\,dy$ exists for a function f defined on a closed regular domain E bounded by the curves

$$y = \phi(x),\ y = \psi(x);\ \ x = a,\ x = b$$

where ϕ, ψ are continuous, and $\phi(x) \le \psi(x)$, for all $x \in [a,b]$ and if the integral $\int_{\phi(x)}^{\psi(x)} f dy$ exists for each fixed point $x \in [a, b]$, then the iterated integral $\int_a^b dx \int_{\phi(x)}^{\psi(x)} f dy$ also exists, and

$$\iint_E f dx dy = \int_a^b dx \int_{\phi(x)}^{\psi(x)} f dy.$$

Note: If f is continuous in E, then f is integrable over E and so the double integral can be evaluated as iterated integral, if E is regular.

1.6 **Green's theorem in $\mathbf{R^2}$:** If a domain E, regular with respect to both the axes, is bounded by a contour C, and f and g are two single valued functions which along with their partial derivatives f_y and g_x are continuous on E, then

$$\iint_E \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx dy = \int_C (f dx + g dy)$$

Note: Green's theorem provides a formula connecting a line integral along a closed contour with an appropriate double integral over the domain bounded by that contour.

1.7 **Change of variables in a double integral :** Let E_1 be a region in uv-plane, bounded by a contour C_1. Suppose that the transformations $x=X(u,v)$, $y=Y(u,v)$ transforms the region E_1 to a region E in xy-plane, bounded by a contour C in such a way that the mapping gives ont-to-one correspondence between the points of E_1 and E and between the points of C_1 and C. If the functions X and Y posses continuous first order partial derivatives and if the Jacobian

$$J = \frac{\partial(X,Y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

does not change sign in E, then

$$\iint_E f\,dx\,dy = \iint_{E_1} f(X,Y)|J|\,du\,dv,$$

for any function f(x,y) for which the double integral $\iint_E f\,dx\,dy$ exists.

2 Integration in R^3

2.1 Line integrals:

In analogy to a plane curve, a curve C in space is a mapping from an interval [a,b] into R^3, and is given by

$$x = X(t),\ y = Y(t),\ z = Z(t),\quad a \le t \le b. \qquad \ldots(1)$$

By C we also mean the set of points (X(t), Y(t), Z(t)), $a \le t \le b$.

Let C be such a curve. Let a bounded vector-valued function $\overline{F}$=(f,g,h) be defined at every point of the curve C. Then the **line integral** (if it exists) of $\overline{F}$ along C is defined as

$$\int_C f\,dx + g\,dy + h\,dz = \int_C \overline{F}\cdot d\overline{r}. \qquad \ldots(2)$$

(Here $\overline{F} = \hat{i}f + \hat{j}g + \hat{k}h$ and $\overline{r} = \hat{i}x + \hat{j}y + \hat{k}z$).

If X, Y, Z posses continuous derivatives in [a,b] and if $\overline{F}$ = (f, g, h) is continuous at every point of C, then the line integral (2) exists and

$$\int_C f\,dx + g\,dy + h\,dz = \int_a^b [f(X,Y,Z)X' + g(X,Y,Z)Y' + h(X,Y,Z)Z']dt.$$

2. 2 Line integral with respect to arc length:

Let the curve C as in (1) be smooth, then it can be represented as $x = \theta(s)$, $y = \phi(s)$, $z = \psi(s)$, $0 \le s \le \ell$, where ℓ is the length of the curve as t varies from a to b. The line integral (2) then reduces to

$$\int_0^\ell \left\{ f(\theta,\phi,\psi)\frac{dx}{ds} + g(\theta,\phi,\psi)\frac{dy}{ds} + h(\theta,\phi,\psi)\frac{dz}{ds} \right\} ds$$

$$= \int_a^b \left(f\frac{dx}{ds} + g\frac{dy}{ds} + h\frac{dz}{ds} \right)\frac{ds}{dt}\,dt,$$

where $\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dX}{dt}\right)^2 + \left(\dfrac{dY}{dt}\right)^2 + \left(\dfrac{dZ}{dt}\right)^2}$.

2.3 Surface Integral:

A **surface** in R^3 is a vector-valued function with domain a subset of R^2 and range a subset of R^3. If E is a domain (or region) in R^2 and X, Y, Z are real valued functions defined on E, then

$$x = X(u, v),\ y = Y(u, v),\ z = Z(u, v),\ \ (u, v) \in E, \qquad \ldots(3)$$

and equivalently the range set

$$\{(X(u, v), Y(u, v), Z(u, v)) : (u, v) \in E\}$$

represents a surface.

If E is a region in xy-plane, then $z = \psi(x, y)$, $(x, y) \in E$, represents the surface $\{(x, y, \psi(x, y)) : (x, y) \in E\}$ in R^3. This surface has the property that any line parallel to z-axis cuts it atmost at one point and so said to be **regular** with respect to z-axis. In this case, E is the projection of the surface on xy-plane. Similarly, $x = \theta(y,z)$ and $y = \phi(z,x)$ with appropriate domains are surfaces which are said to be regular with respect to x and y axes respectively.

The surface (3) is said to be smooth if X, Y and Z posses continuous first order partial derivatives at each point of E and

$$\frac{\partial(Y,Z)}{\partial(u,v)}, \frac{\partial(Z,X)}{\partial(u,v)}, \frac{\partial(X,Y)}{\partial(u,v)}$$

do not all vanish simultaneously at any point.

2.4 Area of a surface: If a smooth surface is regular with respect to z-axis with E as its projection on xy-plane, then the area of the surface is

$$S = \iint_E \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

Similarly $S = \displaystyle\iint_{E_1} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\,dy\,dz$, and

$$S = \iint_{E_2} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dz\, dx,$$

if the surface is regular with respect to other axes (E_1 and E_2 are projections of the surface on yz and zx planes respectively).

2.5 Surface integral of a scalar function:

Let S be a smooth surface bounded by a contour C. Let f be a real valued function of three variables x, y, z defined at each point of the surface S. Then the surface integral of f over S is defined by

$$\iint_S f(x, y, z) dS. \qquad \ldots(4)$$

If the surface is regular with respect to z-axis and described by $z=\psi(x,y)$, $(x,y)\in E$, where E is the projection of S on xy-plane, then the surface integral (4) reduces to a double integral as follows :

$$\iint_S f(x, y, z) dS = \iint_E f(x, y, \psi(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy.$$

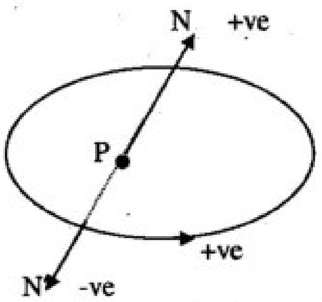
Similarly, if the surface is regular with respect to other axes (as described in the beginning of the section), then

$$\iint_S f(x, y, z) dS = \iint_{E_1} f(\theta(y, z), y, z) \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dydz$$

$$\iint_S f(x, y, z) dS = \iint_{E_2} f(x, \phi(z, x), z) \sqrt{1 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2}\, dzdx.$$

2.6 Orientation of a surface: A **two sided surface** is one which has two distinct sides in the sense that it is not possible to pass from one side to the other along a continuous path which lies on the surface and which does not cross any boundary curve of the surface. For example, the surface of a sphere is two sided, whereas that of the Möbius strip is not.

Let S be a smooth two-sided surface. Let NPN′ be the normal to the surface at a point P. The two opposite vectors PN and PN′ are obtained, one of which is called the positive normal and the other the negative normal to the surface. The side of the surface, which faces the positive normal at P is called the positive side and the other the negative side of the surface. Now the positive direction of the normals at all other points of the surface is fixed.



Let C be any contour on a surface S. The direction in which C is described is considered to be positive if the interior of C lies on the left side when a person walks in that direction with his heads towards the positive normal. Otherwise, C is considered to be negatively described.

2.7 Surface integral of a vector-valued function:

Let S be a smooth two sided surface and let $\overline{F} = (f, g, h)$ be a vector valued function defined at each point of the surface. If $\overline{n}$ denotes the unit vector along the normal at any point along the side of S under consideration, then the **surface integral** of $\overline{F}$ over S is defined to be the surface integral (of scalar function)

$$\iint_S \overline{F}\cdot\overline{n}\,dS = \iint_S (f\cos\alpha + g\cos\beta + h\cos\gamma)\,dS \qquad (\text{if} \quad \overline{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma.)$$

If the surface S is regular with respect to the three axes with D_1, D_2, D_3 as its projections on yz, zx and xy planes respectively, then

$$\iint_S \overline{F}\cdot\overline{n}\,dS = \iint_{D_1} f(x,y,z)\,dydz + \iint_{D_2} g(x,y,z)\,dzdx + \iint_{D_3} h(x,y,z)\,dxdy.$$

So, the surface integral is generally denoted by

$$\iint_S f\,dydz + g\,dzdx + h\,dxdy.$$

Note : (1) The surface integral of a vector-valued function taken over the opposite sides of a surface have opposite signs, i.e.

$$\iint_S \overline{F}\cdot\overline{n}\,dS = -\iint_{S'} \overline{F}\cdot\overline{n}\,dS, \quad \text{where S and S' are two sides of the surface.}$$

(2) If a smooth surface S is represented by

$$x = X(u, v),\ y = Y(u, v),\ z = Z(u, v),\ (u, v) \in D,$$

then the surface integral of $\overline{F} = (f, g, h)$ over S is given by

$$\iint_S \overline{F}\cdot\overline{n}\,dS = \iint_D \left[f\frac{\partial(Y,Z)}{\partial(u,v)} + g\frac{\partial(Z,X)}{\partial(u,v)} + h\frac{\partial(X,Y)}{\partial(u,v)} \right] dudv.$$

2.8 Stokes' Theorem (First generalization of Green's Theorem) :

If S is a smooth oriented surface bounded by a curve C oriented in the same sense, and f, g, h are three functions which along with their first partial derivatives are continuous in a three dimensional domain containing S, then

$$\int_C (fdx + gdy + hdz) = \iint \left[\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dydz + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) dzdx + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy \right].$$

In vectorial notation, if $\overline{F}$=(f, g, h) is the vector-valued function in the consideration, we have

$$\int_C \overline{F}\cdot d\overline{r} = \iint_S \text{curl}\,\overline{F}\cdot\overline{n}\,dS.$$

[Stokes' theorem connects a line integral to an appropriate surface integral.]

2.9 Volume integrals (triple integrals):

Let E be a three dimensional region (i.e. an open subset of R^3 together with its boundary) and f be a bounded function of three variables defined at each point of E. Consider a partition P of E, a finite set $\{\Delta E_1, \Delta E_2, \ldots, \Delta E_n\}$ of small volumes obtained by dividing the region E by finite number of surfaces. Let (x_i, y_i, z_i) be any point in ΔE_i, i=1, 2, …, n. Consider the sum

$$S(P,f) = \sum f(x_i, y_i, z_i)\Delta E_i.$$

If $\lim\limits_{\mu(P)\to 0} S(P,f)$ exists, (where by $\mu(P)$ we mean the maximum of the volumes ΔEi), then we say that f is integrable over E and the **triple integral (volume integral)** of f over E to be

$$\iiint_E f\,dv = \iiint_E f\,dxdydz = \lim_{\mu(P)\to 0} S(P,f)$$

In analogy to a regular region in R^2 , a three dimensional region E is **regular** with respect to z-axis if it is bounded by the surfaces –

$$z = \phi(x, y),\ z = \psi(x, y) \quad (\phi(x, y) \geq \psi(x, y)$$

and (on the sides) a lateral cylindrical surface. Let E be such a region with D as its projection on xy-plane. If f is a function defined on E such that the triple integral over E exists, and if for each point $(x, y) \in D$, the integral $\int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)dz$ exists, then

$$\iiint_E f(x,y,z)dxdydz = \iint_D \left[\int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)dz \right] dxdy.$$

A sufficient condition for existence of the triple integral of f over a region E bounded by pairwise smooth surfaces is that f is continuous in E.

2.10 Change of variables in triple integrals:

Let the functions $x = X(u, v, w)$, $y = Y(u, v, w)$, $z = Z(u, v, w)$

transform a domain E in cartesian co-ordinates x, y, z onto a domain E′ in the new co-ordinates u, v, w in one-to-one manner. Let X, Y, Z have continuous first order partial derivatives, and let f be a function defined on E. If the triple integral of f over E exists, then

$$\iiint_E f\, dxdydz = \iiint_{E'} F(u,v,w)|J|\, dudvdw,$$

where $F(u, v, w) = f(X(u, v, w), Y(u, v, w), Z(u, v, w))$ and J is the Jacobian

$$J = \frac{\partial(X,Y,Z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} & \frac{\partial Y}{\partial w} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} & \frac{\partial Z}{\partial w} \end{vmatrix}.$$

For Cylindrical polar co-ordinates :

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

Jacobian $J = r$.

For spherical polar co-ordinates :

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta,$$

Jacobian $J = r^2\sin\theta$.

2.11 Gauss' Theorem:

If a three dimensional regular (or piecewise regular) domain E is bounded by a smooth (or piecewise smooth) oriented surface S and f, g, h are real-valued functions which along with their partial derivatives f_x, g_y, h_z are continuous at each point of E and S, then

$$\iiint_E (f_x + g_y + h_z)dxdydz = \iint_S f\, dydz + g\, dzdx + h\, dxdy.$$

In vectorial notation, if $\overline{F} = (f, g, h)$, then

$$\iiint_E \operatorname{div} \overline{F}\, dv = \iint_S \overline{F} \cdot \overline{n}\, dS.$$

The Gauss Theorem is also called the **divergence theorem** or the **Green's theorem in space.**

3 Solved Examples:

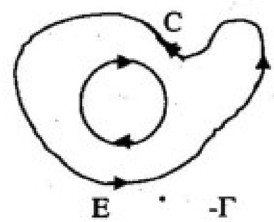
3.1 Prove that the line integral

$$\int_C \frac{x\,dy - y\,dx}{x^2 + y^2}$$

taken over any closed contour R^2 with the origin inside it and described in the positive direction is 2π.

Solution:

Consider a circular path Γ with centre as origin, described positively and lying completely inside C. Let E be the region lying between C and Γ. So E is a piecewise regular region bounded by C-Γ. By Greens' theorem



$$\int_{C-\Gamma} \frac{xdy - ydx}{x^2 + y^2} = \iint_E \left\{ \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) \right\} dxdy$$

$$= \iint_E \left\{ \frac{2}{x^2 + y^2} - \frac{2(x^2 + y^2)}{(x^2 + y^2)^2} \right\} dxdy$$

$$= \iint_E 0\, dxdy = 0.$$

i.e. $$\int_C \frac{xdy - ydx}{x^2 + y^2} - \int_\Gamma \frac{xdy - ydx}{x^2 + y^2} = 0$$

Therefore,

$$I = \int_C \frac{xdy - ydx}{x^2 + y^2} = \int_\Gamma \frac{xdy - ydx}{x^2 + y^2}.$$

On Γ, $x = a\cos\theta$, $y = a\sin\theta$ and therefore

$$I = \int_0^{2\pi} \frac{a^2\cos^2\theta + a^2\sin^2\theta}{a^2} d\theta$$

$$\int_0^{2\pi} d\theta = 2\pi.$$

3.2 Evaluate the double integral

$$\iint_E \sqrt{x^2 + y^2}\, dydx$$

over the region enclosed by triangle y = 0, y = x, x = 1, changing it to polar co-ordinates.

Solution: In polar co-ordinates the triangle has sides θ = 0, θ = π/4 and rcosθ = 1 and so it is given by 0 ≤ θ ≤ π/4, 0 ≤ r ≤ secθ. Thus the required integral

$$I = \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sec\theta} r \cdot r\, dr \quad , \qquad (\text{as } |J| = r)$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sec\theta} r^2 d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3}\sec^3\theta\, d\theta = \frac{1}{3}\int_0^1 \sqrt{1+t^2}\, dt = \frac{1}{6}\left[\sqrt{2} + \log\left(1\sqrt{2}\right)\right]$$

3.3 Show that

$$\int_\Gamma (x + y)\, dS = \sqrt{2}a^2,$$

where Γ is the quarter circle

$$x^2 + y^2 + z^2 = a^2,\ y = x$$

lying in the first octant.

Solution: The curve Γ is the part for which x ≥ 0, y ≥ 0, z ≥ 0, of the intersection of the surfaces.

$$2x^2 + z^2 = a^2 \text{ i.e. } \frac{x^2}{\left(\frac{a}{\sqrt{2}}\right)^2} + \frac{z^2}{a^2} = 1 \text{ and } y = x.$$

In parametric form, the curve is given by

$$x = \frac{a}{\sqrt{2}}\cos\phi \qquad y = \frac{a}{\sqrt{2}}\cos\phi \qquad z = a\sin\phi,$$

where $0 \le \phi \le \pi/2$.

Therefore,

$$\frac{dS}{d\phi}=\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2+\left(\frac{dz}{d\phi}\right)^2}=a.$$

We now have

$$\int_\Gamma (x+y)dS=\int_0^{\frac{\pi}{2}} 2\frac{a}{\sqrt{2}}\cos\phi\, a\, d\phi$$

$$=\sqrt{2}a^2\int_0^{\frac{\pi}{2}}\cos\phi\, d\phi=\sqrt{2}a^2.$$

3.4 Find the area of the surface of the paraboloid

$$\frac{x^2}{a}+\frac{y^2}{b}=2z$$

inside the cylinders $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k.$

Solution: The surface is given by

$z=\frac{x^2}{2a}+\frac{y^2}{2b},$ and so it is regular with respect to z-axis. We have

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2}.$$

The projection of the surface on xy-plane is the region E bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=k.$

Thus the required area

$$S=\iint_E\sqrt{1+\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2}\,dxdy\ =ab\iint_{E'}\sqrt{1+X^2+Y^2}\,dXdY$$

(where $X=\frac{x}{a}, Y=\frac{y}{b}, |J|=ab$ E is the region in the XY-plane bounded by the circle $X^2+Y^2=k$)

$$= ab\int_0^{2\pi} d\theta \int_0^{\sqrt{k}} \sqrt{1+r^2}\, r dr \text{ (transforming to polar co-ordinates)}$$

$$= 2\pi ab\int_0^{\sqrt{k}} r\sqrt{1+r^2}\, dr = \frac{2}{3}\pi ab\left[(1+k)^{\frac{3}{2}} - 1\right].$$

3.5 State and prove Stokes theorem.

Solution : For the statement, see 2.8.

Proof : Let the surface S be represented by

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \in D,$$

where D is an oriented surface in uv-plane with boundary Γ represented by

$$u = u(t),\ v = v(t),\quad a \le t \le b.$$

Now $\quad I = \int_C (f\, dx + g\, dy + h\, dz)$

$$= \int_a^b \left[f\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right)dt + g\left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right)dt + h\left(\frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}\right)dt \right]$$

$$= \int_\Gamma \left[\left(f\frac{\partial x}{\partial u} + g\frac{\partial y}{\partial u} + h\frac{\partial z}{\partial u}\right)du + \left(f\frac{\partial x}{\partial v} + g\frac{\partial y}{\partial v} + h\frac{\partial z}{\partial v}\right)dv \right]$$

$$= \iint_D \left[\frac{\partial}{\partial u}\left(f\frac{\partial x}{\partial v} + g\frac{\partial y}{\partial v} + h\frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(f\frac{\partial x}{\partial u} + g\frac{\partial y}{\partial u} + h\frac{\partial z}{\partial u}\right) \right] dudv$$

(using Greens' theorem)

Now, $\quad \frac{\partial}{\partial u}\left(f\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(f\frac{\partial x}{\partial u}\right)$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + f\frac{\partial^2 x}{\partial u \partial v} - \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} - f\frac{\partial^2 x}{\partial v \partial u}$$

$$= \frac{\partial f}{\partial z}\left[\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right] - \frac{\partial f}{\partial y}\left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right] = \frac{\partial f}{\partial z}\frac{\partial (z, x)}{\partial (u, v)} - \frac{\partial f}{\partial y}\frac{\partial (x, y)}{\partial (u, v)}.$$

Similarly,

$$\frac{\partial}{\partial u}\left(g\frac{\partial y}{\partial v}\right)-\frac{\partial}{\partial v}\left(g\frac{\partial y}{\partial u}\right)=\frac{\partial g}{\partial x}\frac{\partial(x,y)}{\partial(u,v)}-\frac{\partial g}{\partial z}\frac{\partial(y,z)}{\partial(u,v)},$$

$$\frac{\partial}{\partial u}\left(h\frac{\partial z}{\partial v}\right)-\frac{\partial}{\partial v}\left(h\frac{\partial z}{\partial u}\right)=\frac{\partial h}{\partial y}\frac{\partial(y,z)}{\partial(u,v)}-\frac{\partial h}{\partial x}\frac{\partial(z,x)}{\partial(u,v)}.$$

Thus, $$I=\iint_D\left[\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)\frac{\partial(y,z)}{\partial(u,v)}+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)\frac{\partial(z,x)}{\partial(u,v)}+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)\frac{\partial(x,y)}{\partial(u,v)}\right]dudv$$

$$=\iint_S\left(\frac{\partial h}{\partial y}-\frac{\partial g}{\partial z}\right)dydz+\left(\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}\right)dzdx+\left(\frac{\partial g}{\partial x}-\frac{\partial f}{\partial y}\right)dxdy$$

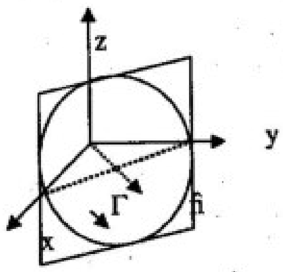
In vectorial notation, $\int_C \overline{F}\cdot d\overline{r}=\iint_S \text{curl}\,\overline{F}.\hat{n}\,dS.$

3.6 Evaluate $\int_\Gamma ydx+zdy+xdz$, where Γ is the curve

$$x^2+y^2+z^2-2ax-2ay=0,\ x+y=2a$$

and it begins at the point (2a, 0, 0) and goes at first below the z-plane.

Solution: We use Stokes' theorem and transform the line integral to a surface integral. Let S be the portion of the plane x + y = 2a bounded by Γ, which is a circle of radius $\sqrt{2}a$.



The given integral is the line integral $I=\int_\Gamma \overline{F}\cdot d\overline{r}$, where $\overline{F}=(y,z,x)$. By Stokes' theorem

$$I=\iint_S \text{curl}\,\overline{F}\cdot\hat{n}\,dS.$$

Now, curl $\overline{F}=(-1,-1,-1)$

and $\hat{n}=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$

Therefore,

$$\text{curl}\,\overline{F}\cdot\hat{n} = -\frac{2}{\sqrt{2}} = -\sqrt{2}.$$ On S, y = 2a – x. Thus,

$$I = \iint_S \left(-\sqrt{2}\right)dS = \left(-\sqrt{2}\right)\iint_S dS$$

$$= \left(-\sqrt{2}\right)\times \text{area of } S = -\sqrt{2}\times\pi\left(\sqrt{2}a\right)^2 = -2\sqrt{2}\pi a^2.$$

3.7 Evaluate $$\iint_S (y-z)dydz + (z-x)dzdx + (x-y)dxdy$$

where S is the portion of the surface $x^2 + y^2 - 2ax + az = 0,\ z \geq 0$.

Solution: Putting $f = \frac{y^2+z^2}{2}, g = \frac{z^2+x^2}{2}, h = \frac{x^2+y^2}{2}$,

we have

$$\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = y - z,\ \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} = z - x \ \text{ and } \ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = x - y.$$

Thus by Stokes' theorem

$$\iint_S (y-z)dydz + (z-x)dzdx + (x-y)dxdy$$

$$= \frac{1}{2}\int_C \left(y^2+z^2\right)dx + \left(z^2+x^2\right)dy + \left(x^2+y^2\right)dz,$$

where C is the boundary of S and is given by

$$(x-a)^2 + y^2 = a^2,\ z = 0.$$

On putting $x = a + a\cos\theta = a(1+\cos\theta),\ y = a\sin\theta$,

the line integral becomes

$$\frac{1}{2}\int_{-\pi}^{\pi}\left[a^2\sin^2\theta\left(-a\sin\theta\right) + a^2\left(1+\cos\right)^2\ a\cos\theta\right]d\theta$$

$$= \frac{1}{2}a^3\int_{-\pi}^{\pi}\left(-\sin^3\theta + \cos\theta + 2\cos^2\theta + \cos^3\theta\right)d\theta$$

$$= \frac{1}{2}a^3\int_{-\pi}^{\pi}2\cos^2\theta\, d\theta \quad \text{[the other integrals are zero]}$$

$$= 4a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta = 4a^3 \frac{1}{2}\frac{\pi}{2} = \pi a^3.$$

3.8 State and prove Gauss Theorem (or Greens' theorem in space or divergence theorem).

Solution: For the statement see 2.11.

Proof: Let us first consider the domain E to be regular with respect to z-axis, bounded above and below by surfaces S_1 and S_2 determined by

$$z = \psi(x, y) \text{ and } z = \phi(x, y), \ (\psi \geq \phi)$$

and by a lateral cylindrical surface S_3 (which may reduce to the common curve of S_1 and S_2) with generators parallel to the z-axis. Then $S=S_1 \cup S_2 \cup S_3$. Let D be the common projection of S_1 and S_2 on the xy-plane. Clearly, the orientation of D as projections of S_1 and S_2 are opposite. Thus,

$$\iiint_E h_z dxdydz = \iint_D \left[\int_{\phi(x,y)}^{\psi(x,y)} h_z dz \right] dxdy$$

$$= \iint_D [h(x, y, \psi) - h(x, y, \phi)] dxdy$$

$$= \iint_D h(x, y, \psi) dxdy + \iint_{-D} h(x, y, \phi) dxdy$$

$$= \iint_{S_1} h(x, y, z) dxdy + \iint_{S_2} h(x, y, z) dxdy$$

$$= \iint_S h(x, y, z) dxdy \qquad \left(\text{since } \iint_{S_3} h dxdy = 0.\right)$$

Thus, $\iiint_E h_z dxdydz = \iint_S h \, dxdy.$ ....(1)

If E is piecewise regular with respect to z-axis, then it can be divided into regular regions by surfaces. The triple integral over E is sum of the triple integrals over the subregions. Moreover, sum of the surface integrals over the oriented boundaries of these regions is the surface integral over S, since each new surface introduced will be counted twice with opposite orientations as boundaries of different subregions. Thus, (1) is also valid, when E is piecewise regular with respect to z-axis.

Since E is regular (or piecewise regular) with respect to all the axes, we get

$$\iiint_E f_x dxdydz = \iint_S f \, dydz \qquad ....(2)$$

and $$\iiint_E g_y dxdydz = \iint_S g \, dzdx \qquad ....(3)$$

Adding (1), (2) and (3) we get the required result.

3.9 Evaluate $\iint_S (4xy\,dydz - y^2\,dzdx + yz\,dxdy)$

where S is the outer surface of the cube bounded by x=0,y=0,z=0,x=1,y=1,z=1.

Solution: The cube E bounded by the oriented surface S is regular with respect to the axes. If $f = 4xy$, $g = -y^2$, $h = yz$, then f, g, h and their partial derivatives $f_x = 4y$, $g_y = -2y$, $h_z = y$ are continuous at each point of E and S.

Thus by Gauss theorem,

$$\iint_S f\,dydz + g\,dzdx + h\,dxdy = \iiint_E (f_x + g_y + h_z)dxdydz$$

$$= \iiint_E 3y\ dxdydz = 3\int_0^1 dx \int_0^1 y\,dy \int_0^1 dz = \frac{3}{2}.$$

3.10 Evaluate

$$\iint_S xz^2\,dydz + (x^2y - z^3)dzdx + (2xy + y^2z)dxdy$$

where S is the outer side of the entire surface of the hemispherical region bounded by

$$z = \sqrt{a^2 - x^2 - y^2} \text{ and } z = 0.$$

Solution: Let E be the upper half of the spherical region, which is bounded by S. Then E is regular with respect to axes.

Let $f = xz^2$, $g = x^2y - z^3$, $h = 2xy + y^2z$.

Then f, g, h and their partial derivatives $f_x = z^2$, $g_y = x^2$, $h_z = y^2$ are continuous in E and S.

Thus by Gauss theorem

$$\iint_E f\,dydz + g\,dzdx + h\,dxdy = \iiint_E (f_x + g_y + h_z)dxdydz$$

$$= \iiint_E (x^2 + y^2 + z^2)\,dxdydz$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{a} r^2(r^2 \sin\theta)dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \int_0^{a} r^4\,dr$$

$$= (2\pi)(2)\left(\frac{a^5}{5}\right) = \frac{4}{5}a^5.$$

3.11 **Evaluate**

$$\iiint_E (x+y+z)^n \, xyz \, dxdydz$$

where $E = \{(x, y, z) \in R^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$.

Solution: E is a regular region bounded by the surface S which is the tetrahedron with sides

$$x = 0, y = 0, z = 0, x + y + z = 1.$$

Thus the given integral is

$$I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyz(x+y+z)^n dz.$$

(since the region is described by $0 \leq x \leq 1$, $0 \leq y \leq 1-x$, $0 \leq z \leq 1-x-y$)

Now, we put $x + y + z = u$, $x + y = vu$ and $x = uvw$,

i.e. $x = uvw$, $y = uv(1-w)$, $z = u(1-v)$.

It may be seen that when x, y, z are positive and $x + y + z \leq 1$, then each of u, v, w lie between 0 and 1 and conversely. So, the region E is fully described when $0 \leq u \leq 1$, $0 \leq v \leq 1$, $0 \leq w \leq 1$. Moreover,

$$|J| = \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| = \left|-u^2 v\right| = u^2 v \text{ (on simplification)}$$

Therefore, $$I = \int_0^1 u^{n+5} du \int_0^1 v^3(1-v)dv \int_0^1 w(1-w)dw$$

$$= \frac{1}{n+6} \cdot \left(\frac{1}{4} - \frac{1}{5}\right) \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{120(n+6)}.$$

4 Exercises

(a) Integration on R^2

4.1 Compute $\int_\Gamma \frac{x\,dy - y\,dx}{x^2 + 4y^2}$, where Γ is the circle $x^2 + y^2 = 1$ in the positive direction on xy-plane.

4.2 Evaluate

$\iint_R \sqrt{(4a^2 - x^2 - y^2)}\,dxdy$ where R is the upper half of the circle $x^2 + y^2 - 2ax = 0$ in xy-plane.

4.3 Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right)dxdy$, where E is the region bounded by the co-ordinate axes and $x + y = 1$ in the first quadrant.

(b) **Integration on R^3**

4.4 Find the line integral $\int_C (y^2+z^2)dx + (z^2+x^2)dy + (x^2+y^2)dz$

where the curve C is the part for which $z \ge 0$ of the intersection of the surfaces

$x^2 + y^2 + z^2 = 2ax$, $x^2 + y^2 = 2bx$, $a > b > 0$, the curve begins at the origin and runs at first in the positive octant.

4.5 Find the line integral

$\int_C (y+z)dx + (z+x)dy + (x+y)dz$ where C is the circle $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$.

4.6 Show that $\int_\Gamma yz\,dx + zx\,dy + xy\,dz = 0$

where Γ is the arc of the curve $x = b\cot t$, $y = b\sin t$, $z = at/2\pi$, from the point it intersects $z = 0$ to the point it intersects $z = a$.

4.7 Show, using Stokes' theorem, that

$$\int_C x^2y^3\,dx + dy + z\,dz = -\frac{\pi a^6}{8} \quad \text{where C is the circle } x^2 + y^2 = a^2,\ z = 0.$$

4.8 Find the area of that part of the surface of the cylinder $x^2 + y^2 = a^2$ which is cut out by the cylinder $x^2 + z^2 = a^2$.

4.9 Show that the surface area of the sphere $x^2 + y^2 + z^2 = 1$ that lies inside the cylinder $2x^2(x^2 + y^2) = 3(x^2 - y^2)$ is

$$2\pi - 4\sqrt{2}\left\{\sqrt{3}\log\left(\sqrt{3}+\sqrt{2}\right) - 2\log\left(1+\sqrt{2}\right)\right\}$$

4.10 Evaluate $\iint_S (x\,dydz + dzdx + xz^2\,dxdy)$

where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

4.11 Evaluate $\iint_S x\,dS$, where S is the entire surface of the solid bounded by the cylinder

$x^2 + y^2 = 1$ and the planes $z = 0$, $z = x + 2$.

4.12 Compute $\iint_S (xz\,dxdy + xy\,dydz + yz\,dzdx)$ where S is the outer side of the pyramid

formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.

4.13 Evaluate $\iint_S (x\cos\alpha + y\cos\beta + z^2\cos\gamma)dS$, where S is the closed surface of the region bounded by the cone $x^2 + y^2 = z^2$ and the plane $z = 1$, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the outward drawn normal of S.

4.14 Evaluate the surface integral $\iint_S y^2 z\, dxdy + xz\, dydz + x^2 y\, dzdx$, where S is the outer surface of the region situated in the first octant and formed by the paraboloid of revolution $z = x^2 + y^2$, cylinder $x^2 + y^2 = 1$ and the co-ordinate planes.

4.15 Evaluate the surface integral $\iint_S x^2\, dydz + y^2\, dzdx + z^2\, dxdy$ taken over the surface of the ellipsiod $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

4.16 Compute the integral $\iiint_E xyz\, dxdydz$ over the domain bounded by $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

4.17 Compute the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the surface of the paraboloid $x^2 + y^2 = 3z$.

[the volume of the region E is given by $\iiint_E dxdydz$.]

4.18 Evaluate $\iiint_E z^2\, dxdydz$, taken over the region common to the surfaces

$x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = ax$.

Suggested Book

S. C. Mallik and S. Arora, *Mathematical Analysis,* 2nd ed. Wiley Eastern Ltd, 1991.

G.U. Questions:

1996

4. (a) State and Prove Stoke's theorem $\quad 2 + 7 = 9$

(b) Evaluate

$$\iint_S xz^2\, dydz + (x^2 y - z^3)dzdx + (2xy + y^2 z)dxdy$$

where S is the outer side of the entire surface of the hemispherical region bounded by

$z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$. 7

1997

4. (a) State and prove Gauss' divergence theorem. $2+7=9$

(b) Evaluate $\int_{\Gamma} y\,dx + z\,dy + x\,dz$, where Γ is the curve $x^2 + y^2 + z^2 - 2ax - 2ay = 0$, $x + y = 2a$, and it begins at the point (2a, 0, 0) and goes at first below the z-plane. 7

1998

4. (a)State and prove Stoke's theorem. $2+7=9$

(b)Evaluate $\iiint_{\Lambda} (x+y+z)^n \, dxdydz$

where $\Lambda = \{(x, y, z) \in R^3 : x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\}$. 7

□□□

UNIT 3 and 4

Lebesgue Measure and Integration

Riemann's definition of integral, which was put forward in 1854, had many limitations. The functions considered in this definition are necessarily bounded. Though, this drawback could be removed using the notion of improper integrals, the pointwise limit of a bounded sequence of Riemann integrable functions may not be integrable in this sense. At the beginning of nineteenth century, the French mathematician Lebesgue introduced a new concept of integral, removing most of the shortcomings of Riemann integral. In this chapter, we discuss the basics of Lebesgue measure and integration.

1. Lebesgue Outer measure

Definition 1.1. Let E be any subset of the real line. We define $m^*(E)$, the **Lebesgue Outer measure** (or briefly the **Outer measure**) of E as follows:

$$m^*(E) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n)\right\},$$

where the infimum is taken over all countable collections $\{I_n\}$of open intervals such that $E \subset \bigcup_{n=1}^{\infty} I_n$ (i.e. $\{I_n\}$is a covering of E by open intervals).

Here $\ell(I)$ denotes the length of the interval I, defined as follows: If I is bounded with end points $a,b \in R(a \leq b)$, then $\ell(I) = b-a$. If I is unbounded, then $\ell(I) = \infty$.

Note 1. 2 Let $E \subseteq R$. From the definition of outer measure it follows that given $\varepsilon \geq 0$, there exists a countable collection $\{I_n\}$of open intervals such that

$$\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(E) + \varepsilon$$

Note that $m^*(E)$ may be ∞ as well

All the sets we consider in the sequel without specifications, are subsets of the real line.

Theorem 1.3.(Properties of outer measure)

(a) $m^*(E) \geq 0$

(b) $m^*(\phi)=0$

(c) Outer measure is **monotonic increasing,** i.e. if $E \subseteq F$, then $m^*(E) \leq m^*(F)$.

(d) Outer measure is **translation invariant**, i.e. for any real number x, $m^*(E+x) = m^*(E)$.

where $E+x=\{y+x : y \in E\}$.

(e) If I is any interval, then $m^*(I)= \ell(I)$

(f) Outer measure is **countably subadditive,** i.e. if $E \subseteq \bigcup_{n=1}^{\infty} E_n$, then $m^*(E) \leq \sum_{n=1}^{\infty} m^*(E_n)$.

(g) If E is countable, then $m^*(E)=0$.

2. Lebesgue measurable sets

Definition 2.1. A set E is said to be **Lebesgue measurable**(or briefly **measurable**), if for each set $A \subseteq R$ we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \qquad \ldots(2.1)$$

(Here, by E^c we mean the complement R-E of E in R.)

If E is measurable, then we define m(E), the **Lebesgue measure** of E to be the outer measure of E, i.e. $m(E)=m^*(E)$.

Note 2. 2 By Theorem 1.3(f), for any set E and A, we have

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$$

Thus, **a set E is measurable if and only if for every set** $A \subseteq R$ we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \qquad \ldots(2.2)$$

Theorem 2.3.

(a) ϕ and R are measurable.

(b) If E is measurable, then E^c is measurable.

(c) If $m^*(E)=0$, then E is measurable.

(d) If E is measurable and x is any real number, then E+x is measurable and

$$m(E+x) = m(E).$$

(e) If $E_1, E_2, \ldots, E_n$ are measurable sets, then $\bigcup_{i=1}^{n} E_i$ is measurable. Further, if E_i are disjoint, then

$$m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Theorem 2.4. If $\{E_n\}$ is a disjoint sequence of measurable sets and $E = \bigcup_{n=1}^{\infty} E_n$, then for any set A

$$m^*(A\cap E)=\sum_{n=1}^{\infty} m^*(A\cap E_n).$$

Theorem 2.5. If $\{E_n\}$is a sequence of measurable sets, then $E=\bigcup_{n=1}^{\infty} E_n$ is measurable. Further, if E_i are disjoint, then $m(E)= \sum_{n=1}^{\infty} m(E_n)$.

Theorem 2.6. Every interval is measurable and its measure is its length.

Theorem 2.7. Every open (closed) set is measurable.

Theorem 2.8. Every conutable set is measurable with measure zero.

3. Measurable functions.

Definition 3.1. Let f be an extended real valued function defined on a measurable set E. Then f is said to be **measurable** if for every real number α

$$E(f>\alpha)=\{x\in E: f(x)>\alpha\} \text{ is a measurable set.}$$

[An extended real valued function on E is one which takes values in $R\cup\{\pm\infty\}$]

Examples 3.2. Every constant function on a measurable set is measurable.

More generally, every continuous function defined on a measurable set is measurable.

Theorem 3.3. For a function f defined on a measurable set E, the following are equivalent.

(i) f is measurable on E.

(ii) $E(f\geq\alpha)=\{x\in E: f(x)\geq\alpha\}$is measurable for every $\alpha\in R$.

(iii) $E(f<\alpha)=\{x\in E: f(x)<\alpha\}$is measurable for every $\alpha\in R$.

(iv) $E(f\leq\alpha)=\{x\in E: f(x)\leq\alpha\}$is measurable for every $\alpha\in R$.

Theorem 3.4. If f is measurable on E, then $E(f=\alpha)=\{x\in E: f(x)=\alpha\}$ is measurable for every extended real number α.

Definition 3.5. A property P is said to hold **almost everywhere** (a.e.) if the measure of the set

$$\{x: P \text{ is not true for } x\}$$

is zero. For example, if f and g are extended real valued functions defined on a measurable set E, then

(i) $f = 0$ a.e. means $m(\{x\in E: f(x)\neq 0\})=0$

(ii) $f = g$ a.e. means $m(\{x\in E: f(x)\neq g(x)\})=0$

(iii) $f \geq g$ a.e. means $m(\{x\in E: f(x) < g(x)\})=0$

Theorem 3.6. A bounded function f defined on a closed interval is Riemann integrable if and only if f is continuous a.e.(i.e. if and only if the set of points of discontinuity of f has Lebesgue measure zero)

Theorem 3.7. Let f and g be extended real valued functions defined on a measurable set E. If $f = g$ a.e. and f is measurable, then g is measurable.

Theorem 3.8. Let $\{f_n\}$be a sequence of measurable functions (with same domain of definition). Then

(i) $\max\{f_1, f_2, \ldots, f_n\}$ (ii) $\min\{f_1, f_2, \ldots, f_n\}$ (iii) $\sup f_n$ (iv) $\inf f_n$ (v) $\overline{\lim} f_n$

(vi) $\underline{\lim} f_n$ are all measurable.

Theorem 3.9. Let E be a measurable set of finite measure, and $\{f_n\}$ is a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x)\to f(x)$. Then given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A\subseteq E$ with $m(A)<\delta$ and an integer N such that for all $x\notin A$, and for all $n\geq N$, $|f_n(x)-f(x)|<\varepsilon$

Definition 3.10. A real valued function f is said to be a **simple function**, if there exists measurable sets E_i of finite measure and real numbers $\alpha_i, 1\leq i \leq n$, such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{E_i},$$

where χ_E is the characteristic function of E in R defined by

$$\chi_E(x) = 1 \quad \text{if } x\in E$$
$$= 0 \quad \text{if } x\in R-E.$$

Definition 3.11. Let [a,b] be an interval and P: $a=x_0< x_1< \ldots < x_n=b$ a partition of [a,b]. For any real numbers $\alpha_i, 1\leq i \leq n$, the function $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, where E_i are disjoint intervals with end points x_{i-1} and x_i, $1\leq i \leq n$, is called a **step function** on [a,b].

4. Lebesgue Integral

Definition 4.1. The Lebesgue integral is defined for different classes of functions in several steps as follows:

(1) **For simple functions:** The **Lebesgue integral for a simple function** $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$ is

defined by $$\int f d\mu = \sum_{i=1}^{n} \alpha_i m(E_i) \qquad \ldots(4.1)$$

In fact, the sum in (4.1) is independent of the representation of f. If E is any measurable set, then the Lebesgue integral of f over E is defined to be

$$\int_E f d\mu = \int f \cdot \chi_E d\mu.$$

(2) **For bounded functions defined on a measurable set of finite measure:** Let f be defined and bounded on E, where E is measurable and $m(E)<\infty$.

Then f is said to be **Lebesgue integrable** if

$$\inf_{f \le \psi} \int_E \psi d\mu = \sup_{f \ge \phi} \int_E \phi d\mu \qquad \ldots(4.2)$$

where the infimum and the supremum are taken over all simple functions ψ and ϕ, respectively, with $\phi \le f \le \psi$.

[A necessary and sufficient condition for (4.2) to hold is that f is measurable, see example.]

In that case, the **Lebesgue integral** of f is defined as $\int f d\mu = \sup_{\phi \le f} \int \phi d\mu$, where φ is simple.

(3) For non-negative measurable functions:

Let f be an extended real valued measurable function defined on a measurable set E such that $f(x) \ge 0, \ \forall x \in E$. The Lebesgue integral of f is defined by

$\int f d\mu = \sup_{h \le f} \int h d\mu,$ Where h is any bounded measurable function defined on E such that measure of $\{x : h(x) \ne 0\}$ is finite. In this case, f is said to be **Lebesgue integrable** if $\int f d\mu < \infty$.

(4) General Lebesgue Integral : Let f be any extended real valued function. Then the **positive part** f^+ and the **negative part** f^- of f are defined by

$$f^+(x) = \max\{f(x), 0\}$$

$$f^-(x) = \max\{-f(x), 0\}.$$

Then f^+ and f^- are nonnegative functions such that $f = f^+ - f^-$.

Now, let f be a measurable function. Then, by Theorem 3.8(i), f^+ and f^- are measurable functions. The function f is said to be **Lebesgue integrable** if f^+ and f^- are

Lebesgue integrable as nonnegative measurable functions (i.e. as in the sense of (3)). The **Lebesgue integral** of f in this case is defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu .$$

Note : (1) A function f is Lebesgue integrable (or briefly integrable) if and only if $\int f d\mu$ is defined in one of the senses of Definition 4.1 and $\int f d\mu < \infty$ (i.e. the integral is finite).

(2) The notation $f \in \mathcal{L}(\mu)$ or $f \in \mathcal{L}(m)$ is used to write that f is Lebesgue integrable.

(3) If f is an integrable function, and E is a measurable set contained in the domain of definition of f, then the function $f.\chi_E$ is integrable. The **integral of** f **over** E is defined to be $\int_E f d\mu = \int f \cdot \chi_E d\mu$.

Theorem 4.2. (Bounded Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that $|f_n(x)| \le M$ for all n and x. If $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in E$, then f is integrable over E and

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$$

Theorem 4.3.(Fatou's Lemma) If $\{f_n\}$ is a sequence of nonnegative measurable functions and $f_n(x) \to x$ a.e. on a set E, then

$$\int_E f d\mu \le \underline{\lim} \int_E f_n d\mu$$

Theorem 4.4. (Monotone Convergence Theorem) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim_{n\to\infty} f_n$ a.e. Then $\int f d\mu = \lim_{n\to\infty} \int f_n d\mu$

Theorem 4.5.(Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E and suppose that there is an integrable function g such that $|f_n(x)| \le g(x)$ for all $x \in E$. If $\lim_{n\to\infty} f_n = f$ a.e. in E, then f is integrable over E and

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu .$$

5. Solved Examples

5.1. Prove theorem 1.3(e).

Solution :(i) First, let I be an open interval and let $\alpha = m^*(I)$. Since I covers itself, therefore, from definition we get $\alpha \le \ell(I)$. If possible let $\alpha < \ell(I)$, say $\alpha = \ell(I) - \varepsilon$, where ε is a positive real. Now, we can find a conutable collection $\{I_n\}$of open intervals such that $I \subseteq \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} \ell(I_n) \leq \alpha + \varepsilon/2 = \ell(I) - \varepsilon/2,$$

(see Note 1.2). Let $I=(a,b)$ and for $0< \delta < b-a$, let $I(\delta)=[a+\delta, b-\delta]$. Then $I(\delta) \subseteq I \subseteq \bigcup_{n=1}^{\infty} I_n$. Since $I(\delta)$ is compact, there is a +ve integer k such that $I(\delta) \subseteq \bigcup_{n=1}^{k} I_n$.

Clearly,

$$b-a-2\delta = \ell(I(\delta)) \leq \sum_{n=1}^{k} \ell(I_n) \leq \sum_{n=1}^{\infty} \ell(I_n) \leq \ell(I) - \varepsilon/2.$$

Since δ is arbitrary, we get

$$\ell(I) = (b-a) \leq \ell(I) - \varepsilon/2,$$

which is absurd.

Hence $\alpha = \ell(I)$ i.e. $m^*(I) = \ell(I)$.

(ii) Next, let I be any bounded interval. Then given $\varepsilon>0$, we can find open intervals J_1 and J_2 such that $J_1 \subset I \subset J_2$ and that

$$\ell(I) < \ell(J_1) + \varepsilon \text{ and } \ell(J_2) < \ell(I) + \varepsilon.$$

Then $l(I) - \varepsilon < l(J_1) = m^*(J_1)$ (by (i))

$\leq m^*(I) \leq m^*(J_2)$ (by Theorem 1.3(c))

$= \ell(J_2)$ (by (i))

$\leq \ell(I) + \varepsilon$

Letting $\varepsilon \to 0$, we get $m^*(I) = \ell(I)$.

(iii) Finally, let I be any unbounded interval. Then given any real number M, we can find a bounded interval J such that $J \subset I$ and $\ell(J)=M$. Thus

$$m^*(I) \geq m^*(J) = \ell(J) = M$$

Hence $m^*(I) = \infty = \ell(J)$

5.2. Prove Theorem 1.3(g).

Solution: Let E be a countable subset of the real line and let $E=\{x_1,x_2,x_3,\dots\}$. Let $\varepsilon>0$ be arbitrary. Consider the open intervals

$$I_n=\left(x_n-\frac{\varepsilon}{2^n},\, x_n+\frac{\varepsilon}{2^n}\right), n=1, 2, \dots.$$

Then $x_n\in I_n$ and therefore $E\subseteq\bigcup_{n=1}^{\infty} I_n$. By definition of $m^*(E)$, we have

$$0\le m^*(E)\le\sum_{n=1}^{\infty}\ell(I_n)=\sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n-1}}=\varepsilon\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}=2\varepsilon \quad ,$$

i.e. $0\le m^*(E)\le 2\varepsilon$. Since $\varepsilon>0$ is arbitrary, we must have $m^*(E)=0$.

5.3. Let E and F be (Lebesgue) measurable sets. Prove that $E\cup F$ and $E\cap F$ are (Lebesgue) measurable.

Solution : Let A be any subset of the real line. Since E is measurable, we have

$$m^*(A)=m^*(A\cap E)+m^*(A\cap E^c) \qquad (1)$$

Again since F is measurable, we have

$$m^*(A\cap E)=m^*(A\cap E\cap F)+m^*(A\cap E\cap F^c)$$

and $\quad m^*(A\cap E^c)=m^*(A\cap E^c\cap F)+m^*(A\cap E^c\cap F^c)$.

Putting these in (1), we have

$$m^*(A)=m^*(A\cap E\cap F)+m^*(A\cap E\cap F^c)+m^*(A\cap E^c\cap F)+m^*(A\cap E^c\cap F^c) \qquad \dots(2)$$

Since (2) is true for any $A\subseteq R$, we replace A by $A\cap(E\cup F)$ to get

$$m^*(A\cap(E\cup F))=m^*(A\cap E\cap F)+m^*(A\cap E\cap F^c)+m^*(A\cap E^c\cap F), \qquad \dots(3)$$

(because $A\cap(E\cup F)\cap(E\cap F)=A\cap E\cap F$, $A\cap(E\cup F)\cap(E\cap F^c)=A\cap E\cap F^c$, $A\cap(E\cup F)\cap(E^c\cap F)=A\cap E^c\cap F$, $A\cap(E\cup F)\cap(E^c\cap F^c)=\phi$, and because $m^*(\phi)=0$).

Moreover, $A\cap E^c\cap F^c= A\cap(E\cap F)^c$. Hence from (2), we get

$$m^*(A)=m^*(A\cap(E\cup F))+m^*(A\cap(E\cup F)^c).$$

This shows that $E\cup F$ is measurable.

Since E and F are measurable, we have E^c and F^c are measurable (by Theorem 2.3(b)). By above result, we have $E^c \cup F^c$ is measurable. Hence, again by Theorem 2.3(b), $E \cap F = (E^c \cup F^c)^c$ is measurable.

5.4. Show that every interval is measurable and its measure is its length.

Solution : Let I be any interval. Let A be any subset of the real line. We show that

$$m^*(A) \geq m^*(A \cap I) + m^*(A \cap I^c).$$

Clearly, the result is true, if $m^*(A) = \infty$. So, let $m^*(A) < \infty$.

Let $\varepsilon > 0$ be given. Then there exists a countable collection $\{I_n\}$ of open intervals such that

$$\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon .$$

If $I'_n = I_n \cap I$ and $I''_n = I_n \cap I^c$, then I'_n and I''_n are intervals (or empty) and

$$\ell(I_n) = \ell(I'_n) + \ell(I''_n) = m^*(I'_n) + m^*(I''_n).$$

Further, $A \cap I \subseteq \left(\sum_{n=1}^{\infty} I_n \right) \cap I = \bigcup_{n=1}^{\infty} (I_n \cap I) = \bigcup_{n=1}^{\infty} I'_n$,

And similarly $A \cap I^c \subseteq \bigcup_{n=1}^{\infty} I''_n$. Therefore, we get using countable subadditivity of outer measure

$$\begin{aligned} m^*(A \cap I) + m^*(A \cap I^c) &\leq m^*(\cup I'_n) + m^*(\cup I''_n) \\ &\leq \Sigma m^*(I'_n) + \Sigma m^*(I''_n) \\ &= \Sigma(m^*(I'_n) + m^*(I''_n)) \\ &= \Sigma\, \ell(I_n) \\ &\leq m^*(A) + \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$m^*(A \cap I) + m^*(A \cap I^c) \leq m^*(A),$$

and therefore I is measurable. Moreover, $m(I) = m^*(I) = \ell(I)$.

5.5. Prove that every countable set is measurable with measure zero. Does there exist a uncountable set which is measurable?

Solution : Let E be any countable subset of R. To prove that E is measurable, we first show that $m^*(E) = 0$. (Do as in Example 5.2).

Now, let A subset of R. We show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

Since $A \cap E \subseteq E$ we have $0 \leq m^*(A \cap E) \leq m^*(E) = 0$ i.e. $m^*(A \cap E) = 0$. Again, $A \cap E^c \subseteq A$ implies that $m^*(A \cap E^c) \leq m^*(A)$. Therefore

$$m^*(A \cap E) + m^*(A \cap E^c) \leq 0 + m^*(A) = m^*(A).$$

Hence E is measurable. Moreover, $m(E) = m^*(E) = 0$.

There are many uncountable measurable sets. For example, if $a, b \in R$ with $a < b$, then $E = [a,b]$, being an interval, is measurable, which in uncountable.

5.6. Prove that a constant function is measurable.

Solution : Let E be a measurable set and c a real number. Let f be the function defined on E by $f(x) = c, \forall\ x \in E$.

To show that f is measurable.

Let α be any real number. Then we have

$$E(f > \alpha) = \{x \in E : f(x) > \alpha\} = E, \text{ if } \alpha < c$$

$$= \phi, \text{ if } \alpha \geq c.$$

Since both E and ϕ are measurable, the set $E(f > \alpha)$ is measurable and hence f is a measurable function.

5.7. Prove that every continuous function is measurable.

Solution : Let E be a measurable set and $f : E \to R$ be a continuous function. To show that f is measurable.

Let α be any real number. Then

$$E(f > \alpha) = \{x \in E : f(x) > \alpha\} = f^{-1}(\alpha, \infty).$$

Now (α, ∞) is an open subset of R and f is continuous. Thus, $f^{-1}(\alpha, \infty)$ is an open subset of E(in its subspace topology, induced from the usual topology of R). Therefore,

$$E(f > \alpha) = E \cap G,$$

where G is an open subset of R. Since every open subset is measurable, G is measurable. Now, E being measurable $E(f > \alpha) = E \cap G$ is measurable and hence f is a measurable function.

5.8. If f and g are measurable functions defined on the same domain, prove that f+g and fg are measurable.

Solution : If f and g are measurable functions defined on E. To show that f+g is measurable, we are to show that, for an arbitrary $\alpha \in R$, the set $E(f+g>\alpha)=\{x \in E: f(x)+g(x)>\alpha\}$ is measurable.

Let $\{r_i : i \in N\}$ be an enumeration of the rational numbers. It can be seen that $E(f+g>\alpha)=\bigcup_{i=1}^{\infty}[E(f>r_i) \cap E(g>\alpha-r_i)]$.

Since f and g are measurable, $E(f>r_i)$ and $E(g>\alpha-r_i)$ are measurable, and therefore their intersection is measurable, for each i. Thus $E(f+g>\alpha)$ is countable union of measurable sets and therefore is measurable. Hence f+g is a measurable function.

To prove that fg is measurable, we first show that the following: if f is measurable on E and $c \in R$, then cf is measurable.

If c=0, then cf is the constant function zero, and therefore is measurable. Let $c \neq 0$, For any real α we have

$$E(cf>\alpha)=E\left(f>\frac{\alpha}{c}\right) \quad \text{if } c>0$$

$$=E\left(f<\frac{\alpha}{c}\right) \quad \text{if } c<0.$$

Since f is measurable, both $E\left(f>\frac{\alpha}{c}\right)$ and $E\left(f<\frac{\alpha}{c}\right)$ are measurable. Therefore $E(cf>\alpha)$ is measurable and hence cf is measurable function.

Next, we show that if f is measurable on E, then f^2 is measurable on E. For any real α, we have

$$E(f^2>\alpha)=E, \text{ if } \alpha<0$$

$$=E(f>\sqrt{\alpha}) \cup E(f<-\sqrt{\alpha}), \text{ if } \alpha \geq 0,$$

and therefore $E(f^2>\alpha)$ is measurable. Hence f^2 is measurable on E.

Finally, we have

$$fg = \frac{1}{4}\{(f+g)^2-(f-g)^2\},$$

and if f and g are measurable on E, then by the above three results, fg is measurable on E.

5.9. Prove Theorem 3.7.

Solution : Let $C=\{x\in E:f(x)\neq g(x)\}$. Because f = g a.e. on E, we have m(E)=0. Now, let α be any real and let

$$A=\{x\in E:f(x)>\alpha\}$$

$$B=\{x\in E:g(x)>\alpha\}.$$

As f is measurable, A is a measurable set. To show that B is measurable. We have $A-B\subseteq E$ and $B-A\subseteq E$.

Since m(E)=0, we have

$$0\leq m^*(A-B)\leq m^*(E)=m(E)=0$$

i.e. $m^*(A-B)=0$. This implies that A-B is measurable. Similarly, B-A is measurable. Now, we have

$$A\cap B=A-(A-B)=A\cap(A-B)^c$$

and therefore measurable. Finally,

$$B=(A\cap B)\cup(B-A)$$

and therefore measurable. This completes the proof.

5.10. Prove that a bounded function defined on a measurable set of finite measure is integrable if and only if is measurable.

Or

Let f be defined and bounded on a measurable set E with m(E) finite. In order that

$$\inf_{\psi\geq f}\int_E \psi(x)d\mu=\sup_{\phi\leq f}\int_E \phi(x)d\mu$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f is integrable.

Solution : First, let f be measurable. Since f is bounded, there exists a +ve number M such that $|f(x)|<M$ for all $x\in E$. For any positive integer n, consider the sets

$$E_k=\{x\in E:\frac{k-1}{n}M<f(x)\leq\frac{k}{n}M\},\ -n\leq k\leq n.$$

Clearly, E_k is measurable for each k and $E_k\cap E_{k'}=\phi$ for $k\neq k'$.

Also $E=\cup\{E_k:-n\leq k\leq n\}$.

Thus we get

$$\sum_{k=-n}^{n} m(E_k) = m(E).$$

Now consider the functions ψ_n and ϕ_n defined by

$$\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_k}(x)$$

$$\phi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} (k-1)\chi_{E_k}(x)$$

Then ψ_n and ϕ_n are simple functions such that $\phi_n \le f \le \psi_n$.

Thus we get

$$\inf_{\psi\ge f}\int_E \psi d\mu \le \int_E \psi_n d\mu = \frac{M}{n}\sum_{k=-n}^{n} k m(E_k),$$

and $$\sup_{\phi\le f}\int_E \phi d\mu \ge \int_E \phi_n d\mu = \frac{M}{n}\sum_{k=-n}^{n} (k-1) m(E_k).$$

Thus

$$0 \le \inf_{\psi\ge f}\int_E \psi d\mu - \sup_{\phi\le f}\int_E \phi d\mu \le \frac{M}{n}\sum_{k=-n}^{n} m(E_k) = \frac{M}{n} m(E).$$

Since n is arbitrary, letting $n\to\infty$ we get

$$\inf_{\psi\ge f}\int_E \psi d\mu = \sup_{\phi\le f}\int_E \phi d\mu$$

Thus f is integrable, i.e. the condition is sufficient.

Conversely, let the function f be integrable, i.e. let

$$\inf_{\psi\ge f}\int_E \psi d\mu = \sup_{\phi\le f}\int_E \phi d\mu$$

Then given any positive integer n, there exist simple functions ϕ_n and ψ_n such that $\phi_n \le f \le \psi_n$ and

$$\int_E \psi_n d\mu - \int_E \phi_n d\mu < \frac{1}{n}.$$

Consider the functions $\psi^*=\inf \psi_n$, $\phi^*=\sup \phi_n$. As ϕ_n and ψ_n are measurable (because they are simple), ψ^* and ϕ^* are measurable. Moreover, $\phi^*\leq f \leq \psi^*$. We show that $\phi^*= f = \psi^*$ a.e. For this, consider the set

$$\Delta = \{x\in E: \phi^*(x)\neq\psi^*(x)\}$$

$$= \{x\in E: \phi^*(x) < \psi^*(x)\}$$

$$= \{x\in E: \psi^*(x) - \phi^*(x)>0\}$$

Then $\Delta = \bigcup_{k=1}^{\infty}\Delta_k$, where

$$\Delta_k=\{x\in E: \psi^*(x) - \phi^*(x)>\frac{1}{k}\}$$

$$\subseteq\{x\in E: \psi_n(x) - \phi_n(x)>\frac{1}{k}\}=\Delta_k^{(n)}, \text{ say.}$$

Now,
$$\int_E \psi_n d\mu - \int_E \phi_n d\mu < \frac{1}{n}$$

$$\Rightarrow \int_E (\psi_n - \phi_n) d\mu < \frac{1}{n}$$

$$\Rightarrow m(\Delta_k^{(n)})\cdot\frac{1}{k} \leq \int_E (\psi_n - \phi_n) d\mu < \frac{1}{n}$$

$$\Rightarrow m(\Delta_k^{(n)}) < \frac{k}{n}$$

$$\Rightarrow m(\Delta_k) < \frac{k}{n}.$$

As Δ_k is independent of n, we get $m(\Delta_k)=0$ for each k. Consequently, $m(\Delta)=0$. Thus $\phi^*=\psi^*$ a.e. and therefore $\phi^*= f =\psi^*$ a.e. Since ϕ^* and ψ^* are measurable, we get f is measurable.

5.11. Prove that every Riemann integrable function is Lebesgue integrable. Show that the converse is not true.

Solution : Let f be a bounded function defined on a closed interval [a,b]. For a partition P of [a,b], the lower sum $L(P,f)=\sum_{i=1}^{n} m_i\Delta x_i$ (with usual notations) in fact is given by $L(P,f)=\int_E \phi d\mu$, where ϕ is the step function $\phi=\sum m_i\chi_{E_i}$, $E_i=\Delta x_i$ and $E=[a,b]$. Clearly, $\phi \leq f$. We have

$$\underline{\int_a^b} fdx = \sup_P L(P,f) \le \sup_{\phi\le f} \int_E \phi d\mu \ ,$$

where the supremum is taken over all step functions $\phi \le f$. Since every step function is a simple function, we get

$$\underline{\int_a^b} fdx \le \sup_{\phi\le f} \int_E \phi d\mu \ ,$$

where ϕ is simple. Similarly,

$$\overline{\int_a^b} fdx \ge \inf_{\psi\ge f} \int_E \phi d\mu \ , \ (\psi \text{ is simple}).$$

We therefore have

$$\underline{\int_a^b} fdx \le \sup_{\phi\le f} \int_E \phi d\mu \le \inf_{\psi\ge f} \int_E \psi d\mu \le \overline{\int_a^b} fdx$$

If f is Riemann integrable, then $\underline{\int_a^b} fdx = \overline{\int_a^b} fdx$, and therefore

$$\sup_{\phi\le f} \int_E \phi d\mu = \inf_{\psi\ge f} \int_E \psi d\mu$$

and f is Lebesgue integrable.

5.12. If f and g are bounded measurable functions defined on a set E of finite measure and a, b$\in$R, then show that

$$\int_E (af + bg) = a\int_E f + b\int_E g \, .$$

Solution : Clearly af, bg, and af+bg are bounded measurable functions on E. First, we show that $\int_E af = a\int_E f$.

If $a = 0$, then clearly $\int_E af = 0 = a\int_E f$. Let $a \ne 0$. We note that a function ϕ is simple if and only if $a\phi$ is so.

Case I. $a>0$.

For any simple function ψ, we have $\psi \ge f$ if and only if $a\psi \ge af$. Thus

$$\int_E af = \inf_{\psi \ge f} \int_E a\psi = a \inf_{\psi \ge f} \int_E \psi = a\int_E f .$$

Case II. $a<0$.

For any simple function ϕ, we have $\phi \le f$ if and only if $a\phi \ge af$. Thus

$$\int_E af = \inf_{a\phi \ge af} \int_E a\phi = \inf_{\phi \le f} \int_E a\phi = a \sup_{\phi \le f} \int_E \phi = a\int_E f .$$

Therefore, for any a, $\int_E af = a\int_E f$

Next, we show that for any bounded measurable functions f and g on E,

$$\int_E f+g = \int_E f + \int_E g$$

If ψ_1 and ψ_2 are simple functions such that $\psi_1 \ge f$ and $\psi_2 \ge g$, then $\psi_1+\psi_2$ is a simple function such that $\psi_1+\psi_2 \ge f+g$. Thus

$$\int_E f+g = \inf_{\psi \ge f+g} \int_E \psi \le \inf_{\substack{\psi_1 \ge f \\ \psi_2 \ge g}} \int_E \psi_1 + \psi_2 = \inf_{\psi_1 \ge f} \int_E \psi_1 + \inf_{\psi_2 \ge g} \int_E \psi_2 = \int_E f + \int_E g .$$

$$\text{i.e. } \int_E f+g \le \int_E f + \int_E g$$

Similarly we have (with ϕ, ϕ_1, ϕ_2 simple functions)

$$\int_E f+g = \sup_{\phi \le f+g} \int_E \phi \ge \sup_{\substack{\phi_1 \le f \\ \phi_2 \le g}} \int_E \phi_1 + \phi_2 = \sup_{\phi_1 \le f} \int_E \phi_1 + \sup_{\phi_2 \le g} \int_E \phi_2 = \int_E f + \int_E g ,$$

i.e. $\int_E f+g \ge \int_E f + \int_E g$. Hence $\int_E f+g = \int_E f + \int_E g$

Finally, using the above results

$$\int_E (af+bg) = \int_E af + \int_E bg = a\int_E f + b\int_E g .$$

5.13. If f is bounded measurable function defined on a measurable set E of finite measure, then show that

$$\int_E f = \int_{E_1} f + \int_{E_2} f ,$$

where $E=E_1 \cup E_2$, $E_1 \cap E_2 = \phi$.

Solution : As $E=E_1 \cup E_2$ and $E_1 \cap E_2 = \phi$, we have $\chi_E = \chi_{E_1} + \chi_{E_2}$. Therefore,

$$\int_E f = \int f \cdot \chi_E = \int f(\chi_{E_1} + \chi_{E_2})$$

$$= \int (f \cdot \chi_{E_1} + f \cdot \chi_{E_2})$$

$$= \int f \cdot \chi_{E_1} + \int f \cdot \chi_{E_2}$$

$$= \int_{E_1} f + \int_{E_2} f.$$

[Here we have used the result that for bounded measurable functions f and g defined on E of finite measure $\int f + g = \int f + \int g$.]

5.14. If f is a bounded measurable function defined on a measurable set E of finite measure, then show that

$$\left|\int_E f\right| \le \int_E |f|.$$

Solution : First, we show the following. If f and g both are bounded measurable on E and if $f \le g$, then $\int_E f \le \int_E g$.

For any simple function ϕ with $\phi \le f$, we have $\phi \le g$.

Therefore,

$$\int_E f = \sup_{\phi \le f} \int_E \phi \le \sup_{\phi \le g} \int_E \phi = \int_E g,$$

$$\text{i.e. } \int_E f \le \int_E g.$$

Now, if f is bounded measurable on E, then so is $|f|$. Moreover, $-|f| \le f \le |f|$ on E. Therefore by the above result, we have

$$\int_E -|f| \le \int_E f \le \int_E |f|$$

$$\text{i.e.} -\int_E |f| \le \int_E f \le \int_E |f|$$

$$\text{i.e. } \left|\int_E f\right| \le \int_E |f|.$$

5.15. Let f be integrable over E and c any real number. Show that cf is integrable over E and $\int_E cf = c\int_E f$.

Solution : First, we note (see example 5.12) that if f is bounded measurable, then cf is bounded measurable and $\int_E cf = c\int_E f$.

Case I: f is nonnegative integrable and c>0.

Let h denote an arbitrary bounded measurable function which vanish outside a set of finite measure. Since c>0, we have h≤f if and only if ch≤cf. Therefore,

$$\int_E cf = \sup_{h\le cf}\int_E h = \sup_{h\le f}\int_E ch$$
$$= \sup_{h\le f} c\int_E h = c\sup_{h\le f}\int_E h = c\int_E f$$

Moreover, $\int_E cf < \infty$, because $\int_E f < \infty$. Therefore cf is integrable over E.

General Case: f is general integrable function and c any real number.

If c = 0, then cf = 0 and so integrable over E and $\int_E cf = 0 = c\int_E f$. Let c≠0.

Let f^+ and f^- be the positive and the negative parts of f respectively. We note that the positive and the negative parts of are given by

$$(cf)^+ = cf^+ \text{ and } (cf)^- = cf^- \text{ if } c>0$$

and $$(cf)^+ = (-c)f^- \text{ and } (cf)^- = (-c)f^+ \text{ if } c<0.$$

Since f is integrable over E, by definition f^+ and f^- are integrable over E. By case I, $(cf)^+$ and $(cf)^-$ are integrable over E and therefore cf is integrable over E. Moreover, if c>0,

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^- = \int_E cf^+ - \int_E cf^-$$
$$= c\int_E f^+ - c\int_E f^- = c\left(\int_E f^+ - \int_E f^-\right) = c\int_E f$$

and if c<0,

$$\int_E cf = \int_E (cf)^+ - \int_E (cf)^- = \int_E (-c)f^- - \int_E (-c)f^+$$
$$= (-c)\left(\int_E f^- - \int_E f^+\right) = c\left(\int_E f^+ - \int_E f^-\right) = c\int_E f$$

This completes the proof.

5.16. If f is a measurable function and E is a set of measure zero, then show that f is integrable over E and $\int_E f = 0$.

Solution : First, let f be bounded on E. Let $|f(x)| \le M \quad \forall x \in E$. Let $\phi = -M\chi_E$ and $\psi = M\chi_E$ Then ϕ and ψ are simple functions such that $\phi \le f.\chi_E \le \psi$, and therefore

$$0 = -M.m(E) = \int_E \phi \le \int_E f \le \int_E \psi = M.m(E) = 0 \quad \text{and we get } \int_E f = 0.$$

Next, let f be nonnegative on E. Then

$$\int_E f = \sup_{h \le f} \int_E h \text{, where h is bounded on E.}$$

$$= 0,$$

because $\int_E h = 0$ for any such h. In particular $\int_E f < \infty$ and so f is integrable over E.

Finally, let f be any measurable function. Let f^+ and f^- be the positive and negative parts of f respectively. Then, by the above, f^+ and f^- are integrable over E and $\int_E f^+ = \int_E f^- = 0$. Thus f is integrable over E and $\int_E f = \int_E f^+ - \int_E f^- = 0$.

6. **Exercises**

6.1. Prove Theorem 1.3(c).

6.2. If E is measurable, then show that E^c is measurable.

6.3. Show that every open set is measurable. Use 6.2 to show that every closed set is measurable.

6.4. Prove Theorem 3.4.

6.5. If f and g are measurable functions, then show that max{f,g}and min{f,g} are measurable. Use it to show that the positive and the negative parts of a measurable function is measurable.

6.6. Prove that every simple function is measurable.

6.7. If f and g are bounded measurable functions defined on a measurable set E of finite measure such that $f \le g$ a.e., then show that $\int_E f \le \int_E g$.

6.8. If f and g are nonnegative measurable functions defined on a measurable set E, then show that

$$\int_E f+g=\int_E f+\int_E g.$$

6.9. If f and g are integrable, then show that f+g is integrable and $\int f+g=\int f+\int g$.

6.10. If $f=0$ a.e. then show that f is integrable and $\int f=0$.

6.11. If f is integrable and $f=g$ a.e. then show that g is integrable and $\int g=\int f$

6.12. If f is integrable on a measurable set E, then show that |f| is integrable on E and $\left|\int_E f\right|\le\int_E |f|$.

G.U. Questions

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5.(a). Define a measurable set and prove that every interval is measurable. **(6)**

(b). If f and g are two measurable functions on $E\subset R$, then show that f+g and fg are also measurable on E. **(6)**

(c). If f is Lebesgue integrable on a measurable set E, then prove that |f| is Lebesgue integrable on E and $\left|\int_E f\,d\mu\right|\le\int_E |f|\,d\mu$. **(6)**

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5.(a) Prove that the union of two measurable sets is measurable. **(6)**

(b) If f = g almost everywhere and if f is a measurable function, then show that g is also measurable.

(c) If f and g are bounded measurable functions defined on a measurable set E of finite measure, then prove that

$\int_E \alpha f+\beta g=\alpha\int_E f+\beta\int_E g$ where α and β are any two real numbers. **(6)**

5(a) when is a function defined on a measurable set with values on the extended real line said to be **measurable** ? Give an example (with justification) of such a function. 1+3=4

(b) If f and g are two measurable functions on a set $E \subset R$, then show that f + g and fg are also measurable on E. 2+2=4

(c) Prove that every bounded measurable function on a measurable set $E \subset R$(with finite measure) is Lebesgue integrable on E. 5

(d) Prove that every function which is Riemann integrable on an interval is also Lebesgue integrable on that interval. 5.

♦♦♦

M101 Real Analysis and Lebesgue Measure

MA/MSc in Mathematics

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