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Semester 1

Paper II

Topology

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Paper 102

TOPOLOGY

UNIT 1

INTRODUCTION

The word 'Topology' is derived from two Greek words 'topos' meaning 'surface' and 'logos' meaning 'discourse' or 'study' . Topology thus literally means the study of surfaces . It is also often described as 'rubber sheet geometry' . For an excellent and exciting history of the development of topology , interested readers may consult the book - ' The Genesis of point set Topology , by Jerome . K. Manheim.'

Topology is one of the fundamental pillars of modern mathematics , and its outstanding characteristic is enormous applications in different branches of mathematical science. This work is an exposition of the fundamental ideas and results of 'General Topology' prescribed for the previous year students of the Mathematics Department under Gauhati University. It has been assumed , necessarily , that the reader has some prior familiarity with the basic notions of the theory of metric spaces and topological spaces . Problems after each chapter have been designed so as to stimulate the reader in such a way that they will encounter exciting challenges to solve them . Suggestions for improvement of the work are always welcome , and the author is highly pleased to answer to any question raised by the reader.

The main reference books are :

1) George F. Simmons, Introduction to Topology and Modern Analysis, McGraw-Hill Book Company, 1963.

2) J.R.Munkres, Topology, A First Course, Prentice- Hall, New Delhi, 1978.

3) M.A.Armstrong , Basic Topology, Springer, 1993.

PREREQUISITES

Much of the material in this chapter is assumed to be known. The purpose of this chapter is three fold: first, to give a quick review of the basic ideas of metric spaces required in the course; second, to set down some of the conventions and terminology that will be adhered to throughout this work; and the third, to give the student a chance to fill in on some concepts he may be unfamiliar or not-so-familiar with.

Since this chapter is not in the prescribed course, proofs of many results are not provided. I recall with due emphasis that the only way to learn mathematics, is to do mathematics. That tenet is the foundation of do -it-yourself, Socratic or Texas method, the method in which the teacher plays the role of an omniscient but largely uncommunicative referee between the learner and the facts.

1.1 Metric and Metric Space :

Let X be a non empty set of elements x,y,z,......... . A metric (or a distance function) is a mapping d of X×X into R (the set of reals) satisfying the following conditions :

$d(x,y) \geq 0$ (non-negativeness)

$d(x,y) = 0 \Leftrightarrow x = y$ (identity)

$d(x,y) = d(y,x)$ (symmetry)

$d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

A metric space consists of two aspects; a non-empty set X and a metric d on X. The elements of X are called the **points** of the metric space (X,d) . Whenever it can be done without causing confusion, we denote the metric space (X,d) by the symbol X which is used for the underlying set of points.

One can enjoy the beauty of metric spaces if he himself tries to work out the following examples of metric spaces, some of which are even useful in studying complex analysis and functional analysis.

Example 1. Let X be an arbitrary non-empty set, and define d by

$$d(x,y) = 0 \quad \text{if } x = y$$
$$= 1 \quad \text{if } x \neq y$$

This definition yields d as a metric known as the discrete metric on X

Example 2. Let C be the set of all complex numbers , show that the mapping

$d : C \times C \to R$ defined by

$d(z_1, z_2) = |z_1 - z_2|$ is a metric on R

The following two inequalities are of some use in working out some of the following examples.

Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two n-tuples of real or complex numbers .Then ,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{\frac{1}{2}}, \quad \text{(Cauchy's inequality)}$$

$$\text{and}\left(\sum_{i=1}^{n}|x_i + y_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n}|x_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}|y_i|^2\right)^{\frac{1}{2}}, \quad \text{(Minkowski's ineqality)}$$

Example 3. Let $X = R^n$ denote the set of all ordered n-tuples of real numbers for a fixed $n \in N$. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Define the mapping

$$d : R^n \times R^n \rightarrow R \text{ by}$$

$$d(x,y) = \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{\frac{1}{2}}$$

Prove that d is a metric on R^n. This is called the Euclidean metric.

Example 4. Let [0,1] denote the set of all real -valued continuous functions defined in the closed interval [0,1]. For $f,g \in C\,[0,1]$ define a map ρ by setting

$$\rho(f,g) = \sup_{0\le x\le 1}|\, f(x) - g(x)\,|, \qquad x \in [0,1]$$

Then show that ρ is a metric on C [0,1]. This space is called the space of continuous functions with the **Chebyshev metric**.

Example 5. Let c denote the set of all convergent sequence of real numbers ,

If $x = \{x_i\}$ and $y = \{y_i\}$ are elements of c, we introduce the distance between the sequences x and y by setting

$$\rho(x, y) = \sup_i |x_i - y_i|$$

Then (c,ρ) is a metric space.

Example 6. Let ℓ_p $(p\ge 1)$ be the set of sequences $x = \{x_i\}$ of real numbers subject to the condition that comment . Show that ℓ_p is a metric space with a metric defined by

$$d(x, y) = \left(\sum_{i=1}^{\infty}|x_i - y_i|^p\right)^{\frac{1}{p}}, \quad x, y \in \ell p$$

1.2 Open and Closed sets in a Metric Space.

Let (X,d) be a metric space . If x_0 is a point of X and r is a positive real number , the open sphere denoted by $S_r(x_0)$ or $S(x_0,r)$, with center x_0 and radius r is the subset of X defined by

$$S_r(x_0) = \{\ x \in X : d(x,x_0) < r\}.$$

An open sphere is also called an open ball denoted by $B_r(x_0)$ or $B(x_0,r)$. An open sphere is always non-empty, for it contains its center.

Similarly a closed sphere (or a closed ball) is denoted and defined by

$$S_r[x_0] = \{ x \in X : d(x,x_0) \leq r \}$$

A subset G of the metric space X is called an **open set** if, given any point x in G, there exists a positive real number r such that $S_r(x_0) \subseteq G$, that is if each point of G is the center of some open sphere contained in G. Moreover, a subset F of X is said to be a **closed set** if its complement F^c, is an open set.

We are now in a position to highlight below some of the fundamental properties of open sets and closed sets.

Theorem 1.03 In a metric space (X,d), prove the following.

i) The empty set Φ and the whole space X are open sets.

ii) Every open sphere is an open set, but the converse is not necessarily true.

iii) The union of an arbitrary collection of open sets is open.

iv) The intersection of finite number of open sets is open, but in case of infinite intersection the result may not be true.

v) A subset of X is open if and only if it is the union of a family of open spheres.

vi) If X is discrete, every set is open.

Theorem 1.04 In a metric space (X,d), the following are true.

The empty set Φ and the whole space X are closed sets.

Every closed sphere is a closed set, but the converse is not necessarily true.

The union of finite number of closed sets is closed, but in case of infinite union

the result may not be true.

i) The intersection of an arbitrary collection of closed sets is closed.

Every finite subset is always closed, that is, the complement of a finite set is

always open.

If X is discrete, every set is closed.

The following concepts are very useful in many branches of mathematics in addition to our present work. Throughout the following, X denotes a metric space with a metric d.

Theorem 1.05. Limit Points, Derived Sets

A point $x \in X$ is called **a limit point** (or **an accumulation point**) of a subset A of X , if every open set G containing x contains a point of A distinct from x.

Thus x is a limit point of A iff every open set G containing x intersects A in a point different from x , that is , if $(G \setminus \{x\}) \cap A \neq \Phi$ for all open sets G containing x.

The derived set of A , denoted by D(A) , is the set of all its limit points. The basic properties of derived sets are the following:

Theorem 1.06. Let (X,d) be a metric space and let A and B be the subsets of X . Then prove the following:

i) $D(\Phi) = \Phi$

ii) $A \subseteq B \Rightarrow D(A) \subseteq D(B)$.

iii) $x \in D(A) \Rightarrow x \in D(A - \{x\})$;

iv) $D(A \cup B) = D(A) \cup D(B)$;

v) $D(A \cap B) \subseteq D(A) \cap D(B)$;

[show with a counter example that the equality may not hold in general]

vi) D(A) is always a closed set;

vii) A is closed if and only if $D(A) \subseteq A$;

viii) $A \cup D(A)$ is a closed set;

ix) the derived set of a finite set is the null set;

x) A point $x \in X$ is a limit point of A if and only if every open sphere $S_r(x)$ contains infinitely many points of A.

§ 1.07. Closure Points and the Closure

A point $x \in X$ is called a closure point of a subset A of X , if every open set G containing x contains a point of A . The set of all closure points of A is called the closure of A, and it is denoted by $\overline{A}$ or cl(A). Equivalently $\overline{A}$ is a closed superset of A which is contained in every closed superset of A , that is, $\overline{A}$ is the smallest closed superset of A; or equivalently $\overline{A}$, equals the intersection of all closed supersets of A.

The following are some illuminating properties of closures.

i) $\overline{\Phi} = \Phi$;

ii) $\overline{\overline{A}} = \overline{A}$;

iii) $A \subseteq \overline{A}$;

iv) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$;

v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

vi) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$; (show with a counter example that the equality does not hold in general)

vii) $\overline{A} = A \cup D(A)$;

viii) A is closed $\Leftrightarrow$ $\overline{A} = A$;

ix) $\overline{A \times B} = \overline{A} \times \overline{B}$;

x) The closure of a finite set is the set itself.

§ 1.08. Interior points and the Interior.

Let (X,d) be an arbitrary metric space, and let A be a subset of X. A point in A is called an interior point of A if it is the center of some open sphere contained in A; and the interior of A, denoted by A^0 or Int(A) or A^i, is the set of all its interior points. Symbolically

$$A^0 = \{\, x \mid x \in A \text{ and } S_r(x) \subseteq A \text{ for some } r \}$$

Equivalently, A^0 is the largest open set contained in A; or A^0 is the union of all open sets contained in A. Its basic Properties are demonstrated by the following theorem.

Theorem 1.10 Let A and B be subsets of X. Then

i) $\Phi^0 = \Phi$;

ii) $A \subseteq B \Rightarrow A^0 \subseteq B^0$;

iii) $(A \cap B)^0 = A^0 \cap B^0$;

iv) $A^0 \cup B^0 \subseteq (A \cup B)^0$; (Exhibit a counter example to show that the equality may not hold in general)

v) A is open if and only if $A^0 = A$;

vi) $A^{00} = A^0$;

vii) $A^0 \subseteq A \subseteq \overline{A}$;

viii) $\overline{A}^c = (A^c)^0$

§ 1.09 Boundary Points and the Boundary:

A point in X is called a boundary point of A if each open sphere centered on the point intersects both A and A^c ; and the boundary of A, denoted by b(A) or A^b , is the set of all its boundary points . This concept possesses the following properties.

Theorem 1.12 Prove

i) b(A) is a closed set ;

ii) $b(A) = \overline{A} \cap \overline{A^c} = \overline{A} - A^0$;

iii) A is closed $\Leftrightarrow$ it contains its boundary ;

§ 1.10 Everywhere dense and nowhere dense sets : Separable spaces.

A subset A of X is said to be **dense** (or **everywhere dense**) if $\overline{A} = X$. For example , the set Q of rationals is dense in R with respect to the usual metric . The following theorem states the fundamental properties of a dense set.

Theorem 1.14 Let X be a metric space and A a subset of X . Then

A is dense .

$\Leftrightarrow$ the only closed superset of A is X.

$\Leftrightarrow$ the only open set disjoint from A is Φ.

$\Leftrightarrow$ A intersects every non-empty open set.

$\Leftrightarrow$ A intersects every open sphere.

A subset A of a metric space is said to be nowhere dense if its closure has empty interior i.e. $\overline{A}^{0} = \Phi$. For example , the set N of natural numbers is nowhere dense in R . The basic merit of a nowhere dense set can be obtained from the following theorem.

§ 1.11 Let X be a metric space and A a subset of X . Then

A is nowhere dense

$\Leftrightarrow$ $\overline{A}$ does not contain any not - empty open set

$\Leftrightarrow$ each non - empty open set has a non-empty open subset disjoint from $\overline{A}$

$\Leftrightarrow$ each non-empty open set has a non-empty open subset disjoint from A.

$\Leftrightarrow$ each non-empty open set contains an open sphere disjoint from A.

A set A is said to be **perfect** if A is dense in itself and closed , i.e. if $A = D(A)$. A metric space X is said to be separable if X contains a countable dense subset , i.e. if there exists a countable subset A of X such that $\overline{A} = X$. If we consider the metric space (R,d) where d denotes the usual metric on R , then Q is a countable dense subset of R and so (R,d) is a separable space.

In the same spirit , we should emphasis that the Cantor set is of particular importance in the study of many intrinsic properties of metric and topological spaces.

The Cantor Set : To construct the Cantor set, we proceed as follows:

Let $F_0 = [0,1]$. Divide F_0 into three equal parts and remove the middle third open interval $I_{1,1} = (1/3,2/3)$. This leaves two disjoint closed intervals $J_{1,1} = [0, 1/3]$ and $J_{1,2} = [2/3,1]$ each having length 1/3. Let $F_1 = J_{1,1} \cup J_{1,2}$. This completes the first stage of our construction .We now divide the two closed intervals $J_{1,1}$ and $J_{1,2}$ into three equal parts and remove their middle third open intervals $I_{2,1} = (1/9,2/9)$ and $I_{2,2} = (7/9, 8/9)$. This leaves four (i.e. 2^2) disjoint closed intervals $J_{2,1} = [0,1/9]$, $J_{2,2} = [2/9,1/3]$, $J_{2,3} = [2/3,7/9]$ and $J_{2,4} = [8/9,1]$, each having length $1/3^2$. Let $F_2 = J_{2,1} \cup J_{2,2} \cup J_{2,3} \cup J_{2,4}$, and $G_2 = I_{2,1} \cup I_{2,2}$. In general at the nth stage we remove $2^n - 1$ open intervals $I_{n,1}$, $I_{n,2}$......, $I_{n,2^n-1}$ and are left with 2^n closed intervals $J_{n,1}$, $J_{n,2}$......, $J_{n,2^n}$, each having length $1/3^n$. Let

$$\text{Let } F_n = \bigcup_{k=1}^{2^n} J_{n,k} \quad \text{and } G_n = \bigcup_{k=1}^{2^n-1} I_{n,k} \quad (n \in N)$$

Then Cantor's ternary set F is defined by

$$F = \bigcap_{n=1}^{\infty} F_n = [0,1] \cap \left(\bigcup_{n=1}^{\infty} G_n \right) .$$

F is a closed set and consists of those points in the closed unit interval [0,1] which "ultimately remain" after the removal of all the open intervals (1/3,2/3),(1/9,2/9),(7/9,8/9), ... Clearly , F contains the points : 0,1,1/3,2/3,1/9,2/9,7/9,8/9, ...

Problem Set 1

Ex1: Prove that each of the following sets is a metric space with a metric indicated in each set:

i) Let m be the set of bounded number sequences $x = \{x_1, x_2, ...\}$ with a metric defined by $d(x,y) = \sup | x_i - y_i |$, where $x = \{ x_i \}$, $y = \{ y_i \}$ belong to m.

ii) Let M[0,1] denote the set of all bounded functions x(t) of a real variable t , defined on the segment [0,1] . Introduce the metric by setting

$$d(x,y) = \sup_{t}\{\,|\,x(t) - y(t)\,|\,\},\ t \in [0,1].$$

iii). The set $\tilde{M}[0,1]$ of bounded measurable functions with a suitable metric to be defined by you.

Ex 2: Let X be a metric space , let x be a point of X , and let r be a positive real number . Give an example to show that the closure of $S_r(x)$ is not necessarily equal to $S_r[x]$.

Ex 3 : Let X be a metric space , and let A be a subset of X . If x is a limit point of A , show that each open sphere centered on x contains an infinite number of distinct points of A . Use this result to show that a finite subset of X is closed.

Ex 4: Show that a subset of a metric space is bounded $\Leftrightarrow$ it is non-empty and is contained in some closed sphere.

Ex 5: Describe the interior of the Cantor set.

Ex 6 : Describe the boundary of each of the following subsets of the real line : the integers , the rationals, $[0,1]$,$(0,1)$. Do the same for each of the following of the complex plane :

$$\{z : |z| < 1\}\ ;\ \{z : |z| \le 1\}\ ;\ \{z : I(z) > 0\}.$$

CONVERGENCE , COMPLETENESS AND BAIRE'S THEOREM:

One of our main aims in considering metric spaces is to study convergent sequences in a context more general than that of classical analysis. The fruits of this study are many , and among them is the added insight gained into ordinary convergence as it is used in analysis. Moreover , this chapter highlights two important theorems , viz. Cantor's Intersection Theorem and Baire's Category Theorem which crop up from time to time as an indispensable tool.

§ 2.01 : The Cauchy Sequence or the Fundamental Sequence

Let X be a metric space with a metric d , and let $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of points in X . We say that $\{x_n\}$ is a **Cauchy sequence** or a **fundamental sequence** if for each $\varepsilon > 0$, there exists a positive integer n_0 such that $m, n \geq n_0 \Rightarrow d(x_m,x_n) < \varepsilon$. For example , the sequence $\{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ is a Cauchy sequence in $X = (0,1]$ with the usual metric.

§2.02 : Convergent sequence :

We say that the sequence $\{x_n\}$ is **convergent** if there exists a point x in X such that either

i) for each $\varepsilon > 0$, there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow d(x_n,x_0) < \varepsilon$; or equivalently

ii) for each open sphere $S_\varepsilon(x)$ centered on x , there exists a positive integer n_0 such that x_n is in $S_\varepsilon(x)$ for all $n \geq n_0$.

The point x is called the limit of the convergent sequence $\{x_n\}$, and we usually symbolize this by writing $x_n \to x$.

§ 2.03 Complete metric space :

A metric space (X,d) is said to be complete if every Cauchy sequence in it is convergent . For example the Euclidean space (R^n,d) with the usual metric is a complete metric space .

The following theorem is illuminating , but can be proved easily.

§ 2.04 . Let (X,d) be a metric space . Then

i) a Cauchy sequence is not necessarily convergent;

ii) a convergent sequence is Cauchy , and has a unique limit;

iii) a Cauchy sequence is convergent $\Leftrightarrow$ it has a convergent sub-sequence ;

iv) If a convergent sequence in a metric space has infinitely many distinct points , then its limit is a limit point of the set of points of the sequence ;

v) if $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x,y)$;

vi) if y is a fixed point in X, $x_n \to x$, in X then $d(x_n, y) \to d(x,y)$.

Problem 2.05 : Let (X,d) be a complete metric space , and let Y be a subset of X. Prove that Y is complete if and only if Y is closed. Use the result to examine if each of the following sets in R is complete.

The set of natural numbers;

The set of irrationals ;

$Y = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$.

Solution : First part

We assume first that Y is complete as a subspace of X , and we will show that it is closed .Y will be closed if we can show that it contains all its limit points.

Let y be a limit point of Y .

$\Rightarrow$ for each positive integer n , the open sphere $S_{1/n}(y)$ contains a point y_n in Y ,

$\Rightarrow y_n \in S_{1/n}(y)$

$\Rightarrow d(y, y_n) < 1/n$

$\Rightarrow y_n \to y$ in X

$\Rightarrow \{y_n\}$ is a Cauchy sequence in X.

$\Rightarrow \{y_n\}$is a Cauchy sequence in Y , because $Y \subseteq X$ and $y_n \in Y, \forall n$.

$\Rightarrow \{y_n\}$ is convergent in Y , because Y is complete.

$\Rightarrow y_n \to y$ in Y.

$\Rightarrow y \in Y$.

Thus , Y contains all its limit points and so Y is closed.

Conversely , we assume that Y is closed. We need to show that it is complete.

Let $\{y_n\}$ be a Cauchy sequence in Y.

$\Rightarrow \{y_n\}$ is a Cauchy sequence in X.

$\Rightarrow y_n \to x$ in X , because X is complete .

We now want to show that x is in Y . If $\{y_n\}$ has only finitely many distinct points then $\{y_n\}$ must be of the form $\{y_1, y_2, \ldots, y_n, x, x, \ldots\}$ where x is infinitely repeated . So $x \in Y$. On the other hand, if $\{y_n\}$ has infinitely many distinct points, then by theorem 2.04 (iv), x is a limit point of the set of points of the sequence. Since Y contains this set, and moreover, since $A \subseteq B \Rightarrow D(A) \subseteq D(B)$, x is a limit point of Y. Because Y is closed, it follows that $x \in Y$. This leads Y to be complete.

Second part

i) N = {1,2,3, ...}

Since $D(N) = \Phi$, $\overline{N} = N$, hence by the above result N is complete.

ii) Q^c = the set of irrationals.

We know that $\overline{Q^c} = 3$, and so $\overline{Q^c}$ is not closed. Therefore, by the above theorem Q^c is not complete.

iii) A = {1,1/2,1/3, ... , 1/n, ...}.

We know, 0 is the only limit point of A (prove!), and so D(A) = {0}.

Thus $\overline{A} = A \cup D(A) = A \cup \{0\} \neq A$.

Hence, A is not closed and so by the above result A is not complete.

A sequence $\{A_n\}$ of subsets of a metric space is called a decreasing sequence or a nested sequence if

$A_1 \supseteq A_2 \supseteq A_3 \supseteq$ ....

The Cantor's Intersection Theorem gives conditions under which the intersection of such a sequence is non-empty.

Problem 2.06 Prove Cantor's Intersection Theorem : Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point. Here $\delta(F_n) = \sup \{d(x,y): x,y \in F_n\}$.

Give an example to show that the set F in the above theorem may be empty if the hypothesis $\delta(F_n) \to 0$ is dropped.

Solution: Let X be a complete metric space. For each n, we choose $x_n \in F_n$. Since $\delta(F_n) \to 0$, for every $\varepsilon > 0$, there exists a positive integer m_0 such that $\delta(F_{m_0}) < \varepsilon$. Again, since $\langle F_n \rangle$ is decreasing we have

$$n, m \geq m_0 \Rightarrow F_n, F_m \subset F_{m_0}$$

$$\Rightarrow x_n, x_m \in F_{m_0} \Rightarrow d(x_n, x_m) < \epsilon$$

$$\Rightarrow \{x_n\} \text{ is a Cauchy sequence} . \Rightarrow x_n \to x_0 \text{ for some } x_0 \in X.$$

We assert that $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

To prove this , let $m_0 \in N$ be arbitrary . Then

$n > m_0 \Rightarrow x_n \in F_{m_0}$, because $x_n \in F_n$ and $n > m_0 \Rightarrow F_n \subset F_{m_0}$.

Now if $\{x_n\}$ has only finitely many distinct points then x_0 is that point infinitely repeated, and is, therefore in F_m comment . If $\{x_n\}$has infinitely many distinct points , then x_0 is a limit point of the set of points of the sequence . This means that x_0 is a limit point of the subset $\{x_n : n \geq m_0\}$ of the set of points of the sequence . Since the set $\{x_n : n \geq m_0\}$ is contained in F_{m_0} , because F_{m_0} is closed.

To show uniqueness , let there be another point $x_0^* \in \bigcap_{n=1}^{\infty} F_n$.

Then $\quad d(x_0, x_0^*) \leq \delta(F_n)$ for every n.

Therefore , $d(x_0, x_0^*) = 0$, since $\delta(F_n) \to 0$ as $n \to \infty$ hence $x_0 = x_0^*$ and so $\bigcap_{n=1}^{\infty} F_n = \{x_0\}$.

Second Part : Consider the real line with the usual metric .

Let $\quad F_1 = \{ 1,2,3,\ldots\ldots , n,\ldots\ldots\}$,

$F_2 = \{ 2,3,4,\ldots\ldots , n,\ldots\ldots\}$,

$F_3 = \{ 3,4,5,\ldots\ldots , n,\ldots\ldots\}$,

....

$F_n = \{ n, n+1, n+2, \ldots\ldots\}$,

.... F_n's are closed,

Then $F_1 \supset F_2 \supset F_3 \supset \ldots\ldots\ldots \supset F_n \supset \ldots\ldots$

and $\delta(F_n) \to \infty$ as $n \to \infty$.

But $\bigcap_{n=1}^{\infty} F_n = \Phi$.

Definition 2.07 : Let X be a metric space . A set $A \subset X$ is said to be of the first category if A is the union of a countable family of nowhere dense sets. If A is not of the first category , it is said to be of the second category.

Problem 2.08: Show that the set of rational numbers is of first category.

Solution : Since the set Q of rational numbers is countable , it can be represented as

$$Q = \{x_1, x_2, \ldots, x_n, \ldots\}.$$

Write, $A_1 = \{x_1\}$, $A_2 = \{x_2\}$, ….. , $A_n = \{x_n\}$, …..

Now for each $n \in N$, $\overline{A}_n^{\,0} = \Phi$, and so A_n's are nowhere dense sets. Now , $Q = \{A_n\}$ and is therefore , of the first category.

The following theorem , known as the Baire Category theorem , is important in the sense that a complete metric space can't be covered by any sequence of nowhere dense sets.

Theorem 2.09 : Every complete metric space is of the second category as a subset of itself.

Proof : Let (X,d) be a complete metric space . We need to show that X is of the second category.

Suppose , if possible , X is of first category , so that X is the union of a sequence $\{A_n\}$ of nowhere dense sets . Since X is open and A_1 is nowhere dense , there is an open sphere S_1 of radius less than 1 which is disjoint from A_1 . Let F_1 be the concentric closed sphere whose radius is one half of that of S_1 , and consider its interior . Since A_2 is nowhere dense , int(F_1) contains an open sphere S_2 of radius less than ½ which is disjoint from A_2 .Let F_2 be the concentric closed sphere whose radius is one half of that of S_2 , and consider its interior. Since A_3 is nowhere dense , int(F_2) contains an open sphere S_3 of radius less than ¼ which is disjoint from A_3 . Let F_3 be the concentric closed sphere whose radius is one half of that of S_3 . Continuing in this way , we get a decreasing sequence $\{F_n\}$ of non-empty closed subsets of X such that $\delta(F_n) \to 0$. Since X is complete , Cantor's Intersection theorem guarantees that there exists a point x in X which is in all the F_n's. This point is clearly in all the S_n's , and therefore, (since S_n is disjoint from A_n), it is not in any of the A_n's . This is a contradiction . Consequently , X must be of the second category.

Problem Set 2

Ex 1 prove that

a) if a complete metric space is the union of a sequence of its subsets , Then the closure of at least one set in the sequence must have non-empty interior .

b) If $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space X, then there exists a point in X which is not in any of the A_n's.

Ex 2 Is the set of irrational numbers a set of second category?

Ex 3 Show that the Cantor set is nowhere dense.

Ex 4 Show that a closed set is nowhere dense ⇔ its complement is everywhere dense.

Ex 5 Explain in details why the set of complex numbers is of second category.

CONTINUOUS MAPPINGS

INTRODUCTION :

The central theme of analysis is that of a continuous function . The continuity of a function f at a point x in a metric space is a local property , and the continuity at a point is of interest primarily in analysis. In topics dealing with more primitive mathematical structure such as topological structures , one is interested only in functions which are continuous at every point of the domain of definition. This chapter is primarily concerned with some intrinsic results of continuous mappings and uniformly continuous mappings .

§ 3.01 Definition : Let (X,d_1) and (Y,d_2) be two metric spaces and $f : (X,d_1) \to (Y,d_2)$, a mapping of X into Y . f is said to be continuous at a point x_0 in X if either of the following equivalent conditions is satisfied :

i) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon$;

ii) for each open sphere $S_\varepsilon(f(x_0))$ centered on $f(x_0)$ there exists an open sphere $S_\delta(x_0)$ centered on x_0 such that

$$f(S_\delta(x_0)) \subseteq S_\varepsilon(f(x_0)).$$

If f is continuous at every point in the space X , then we say , f is continuous (on the whole space) . In general, δ depends both on x_0 and ε . However, if δ works uniformly over the entire space X in the sense that it does not depend on x_0 , f is said to be uniformly continuous on X . The formal definition is :

§ 3.02 Definition : A mapping f of X into Y is said to be uniformly continuous if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x_1, x_2) < \delta \Rightarrow d_2(f(x_1), f(x_2)) < \varepsilon$. It is clear that any uniformly continuous mapping is automatically continuous.

3.3 Problem : Explain the notions : 'continuity' and 'uniform continuity' in metric spaces . Examine these with each of the following functions :

i) $f: R \to R$, $f(x) = 5x$, $x \in R$,

ii) $g: R \to R$, $g(x) = x^2 + 1$, $x \in R$,

iii) $h: (0,1) \to R$, $h(x) = 1/x$, $x \in (0,1)$,

iv) $\Phi: (0,1) \to R$, $\Phi(x) = 1/(2-x)$, $x \in (0,1)$,

v) $\Psi : (0,1) \to R$, $\Psi(x) = \sin(1/x)$, $x \in (0,1)$.

Solution (i): Let x_0 be chosen arbitrarily, and after choice we keep it fixed. Let ε be any arbitrary positive quantity. We consider the real line R with the usual metric.

Now $|f(x)-f(x_0)|$, $x \in R$(Domain)

$= | 5x - 5x_0 |$

$= 5| x - x_0 |$

$< \delta$ whenever $0 < \delta$ and $\delta > | x - x_0 |$

$\leq \varepsilon$ where, $0 < \delta \leq \frac{\varepsilon}{5}$ ....(1)

Thus $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Hence f is continuous at x_0. Since x_0 was arbitrary, it follows that f is continuous at every point of R.

(1) reveals that δ depends only on ε, but not on x_0 and therefore, f is uniformly continuous on its domain 3.

(ii) Here, $|g(x) - g(x_0)| = | x^2 + 1 - x_0^2 - 1| = | x - x_0| | x + x_0| = | x - x_0|| x - x_0 + 2x_0| \leq$ $| x - x_0| (| x - x_0| + 2 | x_0 |)$

$< \delta (\delta + 2 | x_0 |)$, whenever $0 < \delta$ and $\delta > | x - x_0|$

$\leq \varepsilon$, where, $0 < \delta \leq - |x_0| + (x_0^2 + \varepsilon)^{1/2}$ .... (2)

Thus, $| x - x_0 | < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$.

Hence g is continuous at x_0. Since x_0 was arbitrary, it follows that g is continuous at every point of R (domain space).

From the relation (2), it is evident that g depends both on x_0 and ε, and therefore, g is not uniformly continuous.

We recall that a subset M of a metric space (X , d) is compact if every sequence in M has a sub-sequence convergent to a point of M.

Note:

i) If the domain space is a bounded closed interval [a,b], then g is uniformly continuous because of the fact that [a,b] is a compact set.

ii) In analysis, there are other standard methods in order to examine the uniform continuity of the functions given in the problem.

iii) the rest of the given mappings are left as exercise for the students.

We provide some fundamental theorems below, the first one expresses continuity at a point in terms of sequences which converges to the point.

Theorem 3.04 Let X and Y be metric spaces and f a mapping of X into Y. Prove that f is continuous at x_0 if and only if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$.

Proof : We first assume that f is continuous at x_0 . If $\{x_n\}$ is a sequence in X such that $x_n \to x_0$, we must show that $f(x_n) \to f(x_0)$. Let $S_\varepsilon f(x_0)$ be an arbitrary open sphere centered on $f(x_0)$ and with radius $\varepsilon > 0$. By our assumption , there exists an open sphere $S_\delta(x_0)$ centered on x_0 such that $f(S_\delta(x_0)) \subseteq S_\varepsilon f(x_0)$. Since $x_n \to x_0$, there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow x_n \in S_\delta(x_0) \Rightarrow f(x_n) \in f(S_\delta(x_0))$

$$\Rightarrow f(x_n) \in S_\varepsilon f(x_0) \quad (\text{because } f(S_\delta(x_0)) \subseteq S_\varepsilon f(x_0)).$$

Thus , for $\varepsilon > 0$, $\exists$ a positive integer n_0 such that $n \geq n_0$

$$\Rightarrow f(x_n) \in S_\varepsilon f(x_0) \Rightarrow f(x_n) \to f(x_0).$$

Conversely , let us assume that $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$. We are to show that f is continuous at x_0 . Suppose , if possible , f is not continuous at x_0 . Then there exists an open sphere $S_\varepsilon f(x_0)$ with the property that the image under f of each open sphere centered on x_0 is not contained in it . Consider the sequence of open spheres $S_1(x_0)$, $S_{1/2}(x_0)$, ... $S_{1/n}(x_0)$, ... Form a sequence $\{x_n\}$ such that $x_n \in S_{1/n}(x_0)$ and $f(x_n) \notin S_\varepsilon f(x_0)$. This yields that $x_n \to x_0$ but $f(x_n)$ does not converge to $f(x_0)$ which is a contradiction to our assumption . So our desired result must be true.

Corollary 3.05 : Let X and Y be metric spaces and f a mapping of X into Y . Then f is continuous if and only if $x_n \to x \Rightarrow f(x_n) \to f(x)$ for any x in X.

The proof of the following theorem can be obtained from any standard book on metric spaces, and hence they are left for the readers.

Theorem 3.06 : Let X and Y be metric spaces and f a mapping of X into Y. Prove

i) f is continuous $\Leftrightarrow$ $f^{-}(G)$ is open in X whenever G is open in Y.

ii) f is continuous $\Leftrightarrow$ $f^{-}(F)$ is closed in X whenever F is closed in Y.

iii) f is continuous $\Leftrightarrow$ $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.

Problem 3.07 : Let X and Y be metric spaces and A a non-empty subset of X . If f and g are continuous mappings of X into Y such that f(x) = g(x) for every x in A , show that f(x) = g(x) for every x in $\overline{A}$.

Proof : We first show that if $x \in \overline{A}$, then there is a sequence $\{x_n\}$ in A such that $x_n \to x$. Since $x \in \overline{A}$, for each $n \in N$, the open sphere $S_{1/n}(x)$ contains a point $x_n \in A$. Thus , $x_n \in S_{1/n}(x)$, n = 1,2,3, ...

$$\Rightarrow d(x_n, x) < 1/n \to 0 \text{ as } n \to \infty \Rightarrow x_n \to x.$$

Now, $f(x_n) = g(x_n)$, $\forall$ n = 1,2,3, ... because x_n's are in A .

$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n)$$

$\Rightarrow f(\lim_{n\to\infty}(x_n)) = g(\lim_{n\to\infty}(x_n))$, because f and g are continuous . $\Rightarrow f(x) = g(x)$, $\forall\, x \in \bar{A}$.

Problem 3.08 : Prove that the image of a Cauchy sequence under a uniformly continuous mapping is again a Cauchy sequence.

Proof : Let $f : (X , d_1) \to (Y , d_2)$ be a uniformly continuous mapping . Let $\{x_n\}$ be a Cauchy sequence in X . We show that $\{f(x_n)\}$ is a Cauchy sequence in Y . Let $\varepsilon > 0$ be given . Now , f is given to be uniformly continuous , and so for $\varepsilon > 0$, $\exists$ $\delta > 0$ such that $d_1(x, x^1) < \delta \Rightarrow d_2(f(x) , f(x^1)) < \varepsilon$, $x, x^1 \in X$ .... (I) Since $\{x_n\}$ is a Cauchy sequence , for $\delta > 0$ $\exists$ a positive number n_0 such that

$$m,n \geq n_0 \Rightarrow d_1(x_n , x_m) < \delta \Rightarrow d_2(f(x_n) , f(x_m)) < \varepsilon , \quad [\text{ applying (I)}]$$

The above result ensures that

$\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y .

Problem 3.09 : Let f be a continuous real function defined on R which satisfies the function $f(x + y)= f(x) + f(y)$. Show that this function must have the form $f(x)=mx$, for some real number m.

Solution : It is given that $f(x + y)=f(x) + f(y)$, $\forall\, x, y \in R$ (1)

Put $x = 0 = y$ in (1)

Then $f(0) = f(0) + f(0) \Rightarrow f(0) + 0= f(0) + f(0) \Rightarrow f(0) = 0$ (by the left cancellation law in the additive group of real numbers) (2)

Next , Put $y = - x$ in (1)

Then , $f(0) = f(x - x) = f(x +(-x)) = f(x) + f(- x)$

$\Rightarrow 0 = f(x) + f(- x)$, (by involving the result (2))

$\Rightarrow f(-x) = - f(x)$ .... (3)

Case (I) : Let x be a positive integer

Then $f(x) = f(\underbrace{1 + 1 + 1 + ... + 1}_{x\text{times}}) = \underbrace{f(1) + f(1) + ... + f(1)}_{x\text{times}}$, [thanks to (1)]

$= x\, f(1) = m\, x$, where $m = f(1)$

So , the result is true if x is a positive integer.

Case (II) Let x be a negative integer . Put $x = -y$ where y is a positive integer.

Now, $f(x)= f(-y) = - f(y)$ [by appealing to (3)]

$= -yf(1)$ as in the case (1)

$= mx$ where $m=f(1)$ Hence the result is true if x is a negative integer.

Case (III) Let x be a rational number.

Put

$$x = \frac{p}{q}, \quad q > 0 \text{ is a positive integer}$$
$$\Rightarrow p = qx$$
$$\Rightarrow f(p) = f(qx)$$
$$\Rightarrow mp = f(\underbrace{x + x + \cdots x}_{q \text{ times}}) \quad \text{left hand side result is due to case (I) and case(II)}$$
$$\Rightarrow mp = qf(x), \quad \text{[owing to (1)]}$$
$$\Rightarrow f(x) = mx.$$

Hence the result is true if x is rational.

Case IV Let x be an irrational . Then there exists a sequence $\{x_n\}$ of rationals such that $x_n \to x$.

Now, by the case (III),

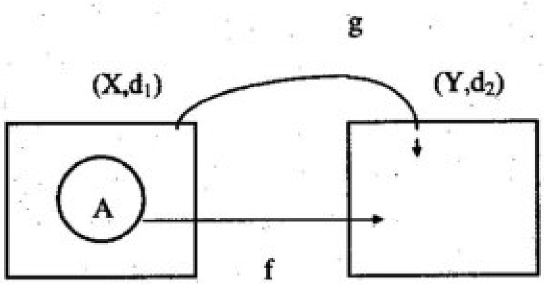
$$f(x_n) = mx_n, \quad \text{for all } n \in N$$
$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (mx_n)$$
$$\Rightarrow f\left(\lim_{n\to\infty} x_n\right) = mx \quad \text{because f is continuous.}$$
$$\Rightarrow f(x) = mx.$$

Thus, the proof is complete for all reals of x.

Uniformly continuous mappings – as opposed to those which are merely continuous — are of particular significance in analysis. The following theorem expresses a property of these mappings which is often useful .

Theorem 3.10: Let X be a metric space , let Y be a complete metric space, and let A be dense subspace of X . If f is uniformly continuous mapping of A into Y , then f can be extended uniquely to a uniformly continuous mapping g of X into Y.

Proof :



$$f(x)=g(x), \quad \text{for all} \quad x \in A$$

If A=X, the conclusion is obvious, because by putting f=g we get the required result. We therefore assume that $A \neq X$. We begin by showing how to define the mapping g . If x is a point in A, we define g(x) to be f(x). Now let x be a point in X—A . Since A is dense, x is the limit of a convergent sequence $\{a_n\}$ in A. Since a_n is a Cauchy sequence and f is uniformly continuous , $\{f(a_n)\}$ is a Cauchy sequence in Y , by problem 3.08. Since Y is complete, there exists a point in Y – we call this point g(x) – such that $f(a_n) \to g(x)$. We must make sure that g(x) depends only on x, and not on the sequence $\{a_n\}$. To show this, let $\{b_n\}$ be another sequence in A such that $b_n \to x$. Then $d_1(a_n,b_n) \to d_1(x,x)=0$, and by the uniform continuity of f, $d_2(f(a_n),f(b_n)) \to 0$.

Thus,

$$\lim_{n\to\infty} d_2(f(a_n), f(b_n)) = 0$$
$$\Rightarrow d_2\left(\lim_{n\to\infty} (f(a_n), f(b_n))\right) = 0$$
$$\Rightarrow d_2\left(\left(g(x), \lim_{n\to\infty} f(b_n)\right)\right) = 0$$
$$\Rightarrow \lim_{n\to\infty} f(b_n) = g(x), \qquad \text{because } d_2 \text{ is a metric.}$$

We next show that g is uniformly continuous. Let $\varepsilon > 0$ be given, and use the uniform continuity of f to find $\delta > 0$ such that for a and a' in A.

We have $d_1(a,a') < \delta \Rightarrow d_2(f(a),f(a')) < \varepsilon$.

Let x and x′ be any points in X such that $d_1(x,x') < \delta$. It suffices to show that $d_2(g(x),g(x')) < \varepsilon$. Let $\{a_n\}$ and $\{a_n'\}$be sequence in A such that $a_n \to x$ and $a_n' \to x'$. By the triangle inequality, we see that $d_1(a_n,a_n') \leq d_1(a_n,x)+ d_1(x,x')+ d_1(x',a_n')$. This inequality, together with the facts that $d_1(a_n,x) \to 0$, $d_1(x,x')<\delta$, and $d_1(x',a_n') \to 0$ implies that $d_2(f(a_n), f(a_n')) < \varepsilon$ for all sufficiently large n.

$$\text{Now } d_2(g(x), g(x')) = \lim_{n\to\infty} d_2(f(a_n), f(a_n'))$$
$$< \varepsilon \quad \text{by the above result}$$

Hence, $d_1(x,x')<\delta \Rightarrow d_2(g(x), g(x')) < \varepsilon$

$\Rightarrow$ g is uniformly continuous.

Finally we want to show that g is unique.

Suppose f has two extensions g and g′.

Then, $f(x)=g(x)=g'(x)$ for all $x \in A$.

Then by problem 3.07,

$g(x)=g'(x)$, for all $x \in \bar{A}$.

$\Rightarrow g(x)=g'(x)$, for all $x \in X \Rightarrow g=g'$ Thus f has a unique extension.

Suppose that X is an arbitrary non-empty set, and consider the set L of all real functions defined on X. It is clear that L is a real linear space with respect to the standard operations of addition and scalar multiplication of functions. We now restrict ourselves to the subset B consisting of the bounded functions in L, B is obviously a linear subspace of L, so it is a linear space in its own right.

We next assume the underlying set X is a metric space. This enables us to consider the possible continuity of functions defined on X. We define C (X, R) to be the subset of B which consist of continuous functions. C (X, R) is thus the set of all bounded continuous real functions defined on the metric space X, and it is non-empty. The following is an enlightening result.

Theorem 3.11: Prove that C (X, R) is a closed subset of the metric space B.

Proof : Let f be a function in B which is in the closure of C (X, R). We show that f is continuous, and therefore in C (X, R), by showing that it is continuous at an arbitrary point x_0 in X. Since a set which equals its closure is closed, this will suffice to prove the lemma. Let d be the metric on X, and let $\varepsilon > 0$ be given. Since f is in the closure of C (X, R), there exists a function f_0 in C (X, R) such that $d(f, f_0) < \varepsilon/3$, from which it follows that $|f(x) - f_0(x)| < \varepsilon/3$ for every point x in X. Since f_0 is continuous, and hence continuous at x_0, we can find a $\delta > 0$ such that

$$d(x,x_0) < \delta \Rightarrow |f_0(x) - f(x_0)| = |f(x) - f_0(x) + f_0(x) - f_0(x_0) + f_0(x_0) - f(x_0)|$$

$$\leq |f(x) - f_0(x)| + |f_0(x) - f_0(x_0)| + |f_0(x_0) - f(x_0)|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

Hence, C (X, R) is a closed subset of B.

Note: C (X, R) plays an important role in many branches, particularly in functional analysis and Housdroff spaces. We will discuss later some more exciting results on this space.

PROBLEM SET 3

Ex 1 A map $f : (X, d_1) \to (Y, d_2)$ is said to be an open (a closed map) if the image of every open set (closed set) in X is open (closed) in Y. Give an example of

a continuous map which is not open and closed;

an open map which is not continuous;

a closed map which is not continuous;

Ex 2 If f is not continuous in theorem 3.04, construct a sequence $\{x_n\}$ such that $x_n \to x_0$ may not imply $f(x_n) \to f(x_0)$.

Ex 3 Refer to problem 3.08 . Show with an example that the image of a Cauchy sequence under an ordinary continuous mapping may not be a Cauchy sequence.

Ex 4 Can we extend the problem 3.09 to the n dimensional Euclidean space R^n ?

Ex 5 Produce a counter example to show that the result in theorem 3.10 may not be true if f is not uniformly continuous .

Ex 6 Let X be a metric space with metric d , and let x_0 be a fixed point in X . Show that the real function f_{x_0} defined on X by $f_{x_0} = d(x,x_0)$ is continuous . Is it uniformly continuous ?

Ex 7 Construct a sequence of continuous functions defined on [0,1] which converges pointwise but not uniformly to a continuous limit .

Ex 8 Construct a sequence of continuous functions defined on [0,1] which converges pointwise to a discontinuous limit.

Ex 9 Let X and Y be metric spaces with metrics d_1 ,d_2 and let $\{f_n\}$ be a sequence of mappings of X into Y which converges pointwise to a mapping f of X into Y , in the sense that $f_n(x) \to f(x)$ for each x in X . Define what ought to be meant by the statement that f_n converges uniformly to f , and prove that under this assumption f is continuous if each f_n is continuous.

UNIT 2

TOPOLOGICAL SPACES
FUNDAMENTAL CONCEPTS AND RESULTS

Introduction :

This chapter is devoted primarily to explain the concept of a general topological space and some elementary results in order to lay down a solid foundation so that its walls and beams should be firmly and truly placed , and each part should bear a meaningful relation to every other part. We wish initially to highlight some prominent ingredients like bases, subbases, countable spaces, separable spaces etc which play important role in ther basic theory of topological spaces. We shall also develop the point of view that there is a constant illuminating interplay between the structure of these spaces and the properties of the continuous functions which they carry.

Definition 4.01 Let X be a non-empty set . A class T of subsets of X is called a topology on X if it satisfies the following three conditions :

t_1) $\Phi\in T$, $X\in T$, that is , the empty set and the whole space belong to T;

t_2) If $A\in T$ and $B\in T$, then $A\cap B\in T$, that is the intersection of two sets in T belongs to T;

t_3) If $A_\lambda\in T$ for every $\lambda\in\Delta$, Δ being an arbitrary index set , then $\bigcup_{\lambda\in\Delta} A_\lambda \in T$, i.e. the union of every class of sets in T is a set in T.

A topology on X is thus a class of subsets of X which is closed under the formation of arbitrary unions and finite intersections . A topological space consists of two objects : a non - empty set X and a topology T on X . The sets in the class T are called the open sets of the topoogical space (X,T) and the elements of X are called its points. The complement of an open set is called a closed set . We often encounter another concept - a **metrizable space** which is defined to be a topological space X with the property that there exists at least one metric on the set X whose class of generated open sets is precisely the given topology . A metrizable space is thus a topological space which is so far as its open sets are concerned , essentially a metric space . Please note that a metric space is always made to be a topological space , but a topological space may not be a metrizable space.

4.2 Examples of topological spaces:

Given a non-empty set X , we always have two obvious topology on X, namely the indiscrete topology and the discrete topology . The students are strongly advised to carry out the detailed proof of the following examples in order to acquire the very concrete and comprehensive idea about the concept of topology.

Example1. Let X be a non-empty set and $I = \{\Phi , X \}$. Then I is a topology on X known as the **indiscrete topology** on X ,and (X,I) is called an indiscrete topological space.

Example 2. Let D consists of all subsets of X . Then (X,D) is a topological space known as a discrete topological space with the **discrete topology** D.

Example 3. Let T be the collection of all those subsets of X whose complements are finite together with the empty set . Then T is a topology on X called the **co-finite topology** or the **finite complement topology** . This is a very ideal example which may be demonstrated as a counter example to disprove many topological results.

Example 4. Let U be the collection of all those subsets G of R (the set of reals) having the property that to each $x \in G$ there exists an open interval I_x such that $x \in I_x$ [p,q)$\subset$G . Then U is a topology on R called the usual topology on R.

Example 5. Let R be the set of all real numbers and let S consists of subsets of R defined as follows :

a) $\Phi \in S$,

b) a non-empty subset G of R belongs to S if to each $x \in G$ there exists a right half open interval [p,q), where $p, q \in R$, $p < q$ such that $x \in [p,q) \subset G$. Then S is a topology on R termed as the **lower limit topology** on R.

Example 6. Let T be the class of subsets of N (the set of all natural numbers) consisting of the empty set and all subsets of N of the form

$$A_n = \{ n, n+1, n+2, \ldots\ldots \},\ n \in N .$$

Then T is a topology on N.

Some complicated examples will be percolated through time to time in later chapters , particularly in the 'separation chapter' to produce counter examples . Exciting examples may be formulated with the aid of measure theoretic concepts.

Definition 4.03 : Let (X,T) be a topological space and let Y be subset of X . Then **T-relative topology** on Y is the collection T^* given by

$$T^* = \{ G \cap Y \mid G \in T \}$$

The topological space (Y, T^*) is called a subspace of (X,T). A property of a topological space is said to be hereditary if every subspace of the space has that property.

Definition 4.04 : Let X and Y be topological spaces , and f a mapping of X into Y .f is called a continuous mapping if $f^{-1}(G)$ is open in X whenever G is open in Y , and an open mapping if f(G) is open in Y whenever G is open in X . A mapping is continuous if it pulls open sets back to open sets, and open if it carries open sets over to open sets . Any image f(X) of a topological space X under a continuous mapping f is called a continuous image of X.

A homeomorphism is a one-to- one continuous mapping of one topological space onto another which is also an open mapping. Two topological spaces X and Y are said to be homeomorphic if there exists a homeomorphism of X onto Y (and in this case, Y is called the homeomorphic image of X) If X and Y are homeomorphic , then their points can be put into one-to -one correspondence in such a way that their open sets also correspond to one another. The two spaces therefore differ only in the nature of their points, and can from the point of view of topology, be considered essentially identical. A topological property is a property which, if possessed by a topological space X, is also possessed by every homeomorphic image of X. The subject of 'topology' can now be defined as the study of all topological properties of topological spaces.

Note: The concepts, 'the closure of a set ', 'the derived set', 'the interior of a set', ' the boundary of a set', nowhere dense sets, everywhere dense sets etc. in a topological space can be extended smoothly from these concepts in a metric space, and their fundamental properties are similar to those discussed in metric spaces.

Some elementary results are now in order.

Problem 4.05 Let X be a non-empty set , Then

i) . Arbitrary intersection of topologies on X is a topology on X ;

ii) The union of two topologies may not be a topology on X ; however the result is true if one topology is weaker than the other;

iii) any intersection of closed sets in X is closed and any finite union of closed sets in X is closed.

Proofs are routine , and so, left for the readers.

Problem 4.06 Show that a subset of a topological space is dense $\Leftrightarrow$ it intersects every non-empty open sets.

Solution : Let (X,T) be a topological space , and let $A \subseteq X$.

Now, A is nowhere dense $\Leftrightarrow \overline{A} = X$ (i.e. each point of X is a closure point of A)

$\Leftrightarrow$ The only closed superset of A is X

$\Leftrightarrow$ the only open set disjoint from A is Φ.

$\Leftrightarrow$ A intersects every nonempty open set.

Problem 4.07 : Let X be a topological space A an arbitrary subset of X . Then prove that

i) $\text{int}(A^c) = \overline{A}^{\,c}$, (c denotes the complement of a set)

ii) $\overline{A} = \{ x \mid \text{each neighbourhood of } x \text{ intersects } A \}$

Solution

i) $x \in \text{int}(A^c) \Leftrightarrow x \in A^c$ and there exists an open set G_x containing x such that $G_x \subseteq A^c$

$\Leftrightarrow$ x is not a closure point of A

$\Leftrightarrow x \in \overline{A}^{c}$, therefore $\text{int}(A^c) = \overline{A}^{\,c}$

ii) we begin by proving that $\overline{A}$ is contained in the given set (the set on the right) by showing that any point not in the given set is not in $\overline{A}$. Let x be a point with a neighbourhood which does not intersect A. Then the complement of this neighbourhood is a closed superset of A which does not contain x, and since $\overline{A}$ is the intersection of all closed supersets of A , x is not in $\overline{A}$. In the same way , it can easily be shown that $\overline{A}$ contains the given set.

Problem 4.08 : Show that

i) a subset of a topological space is closed $\Leftrightarrow$ it contains its boundary ;

ii) a subset of a topological space has empty boundary $\Leftrightarrow$ it is both open and closed.

Solution

i) Let A be a subset of a topological space X.

Let A be closed.

$\Rightarrow A = \overline{A}$

Then , $b(A) = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c}$ (since $A = \overline{A}$)

$\subseteq A$

Thus , if A is closed then $b(A) \subseteq A$, i.e. A contains its boundary.

Conversely, let $b(A) \subseteq A$, we are to show that A is closed , i.e. $A = \overline{A}$

Now, $b(A) \subseteq A \Rightarrow \overline{A} \cap \overline{A^c} \subseteq A$ .... (1)

Suppose, if possible , $\overline{A}$ is strictly superset of A , that is there is at least one point $x \in \overline{A}$ such that $x \notin A$. Then $x \in A^c$. So, every open set G_x containing x intersects both A and A^c and therefore, $x \in b(A)$. By (1) , $x \in A$ which is a contradiction . Hence , we must have $\overline{A}=A$. So , A is closed.

ii) We first assume that $b(A) = \Phi$. We are to show that A is both open and closed.

$b(A) = \Phi \Rightarrow \overline{A} \cap \overline{A^c} = \Phi$ .... (2)

If $x \in \overline{A}$ is such that $x \notin A$, then $x \in A^c$. So , in that case , every open set G_x containing x intersects both A and A^c and therefore, $x \in b(A)$ which is a contradiction to (2). So, such an x can't exist . This implies that $A = \overline{A}$ i.e. A is closed . Similarly , we can show that A^c is closed . Thus both A and A^c are closed , and hence both A and A^c are open.

Conversely , let A be both open and closed . To show that $b(A) = \Phi$.

Now $b(A) = \overline{A} \cap \overline{A^c} = A \cap A^c$ (since $\overline{A} = A$, $\overline{A^c} = A^c$)

$= \Phi$.

Problem 4.09 :

i) Show that the boundary of a closed set is nowhere dense . Is it true for an arbitrary set ?

ii) Show that the Cantor set is nowhere dense.

Solution

i) Let A be closed subset of a topological space (X,T)

$$A \text{ is closed } \Rightarrow \overline{A} = A$$

Now, $b(A) = \overline{A} \cap \overline{A^c} = A \cap \overline{A^c}$

b(A) is a closed set and so $\overline{b(A)} = b(A)$.

Let x be an interior point of b(A) . Then there exists an open set G_x such that

$x \in G_x \subseteq b(A) \subseteq A$ ..... (I) (because A is closed and hence $b(A) \subseteq A$, by problem 4.08 I)

but $x \in b(A)$ and hence G_x must intersect both A and A^c. ....(II)

(I) and (II) are contradictory statements , and hence the interior of b(A) must be empty .

So , b(A) is nowhere dense.

However , the result may not be true for an arbitrary set . For example , consider the real line R with the co-finite topology .

Let A = Q , the set of all rationals.

Then $b(Q) = \overline{Q} \cap \overline{Q^c} = R \cap R = R$.

again , int(b(Q)) = int (R) = R , because R is both open and closed. So, b(Q) is not nowhere dense.

Let Γ be the contor set . Γ is closed and so $\overline{\Gamma}=\Gamma$ contains only specific rotational points in [0,1] . If $x \in \Gamma$ and G_x is an open set containing x. Then the structure of an open set in the usual topology on R implies that G_x contains infinite number of irrationals in [0,1]. Hence, G_x can't be contained in Γ. Thus no point of Γ is an interior point of Γ . This means that $\overline{\Gamma}^0=\Gamma=\phi$. Hence Γ is a nowhere dense set.

Definition 4.10 Local base at a point

Let (X,T) be a topological space. A non-empty collection $\beta(x)$ of open set containing $x \in X$ is called a **local base at the point** x if for every open set G containing x, there exists a set B in $\beta(x)$ such that $B \subseteq G$.

Example : Let X={a,b,c,d,e} and let T={ ϕ,{a},{a,,b},{a,c,d},{a,b,c,d},{a,b,e},X}. Then a local base at each of the points a,b,c,d,e is given by

$\beta(a)=\{\{a\}\}$, $\beta(b)=\{\{a,b\}\}$, $\beta(c)=\{\{a,b,c\}\}$, $\beta(d)=\{\{a,c,d\}\}$, $\beta(e)=\{\{a,b,e\}\}$.

We may, however, define other local bases at these points. For example, the collection

$\{\{a\},\{a,b\}\}$ forms a local base at a.

Definition 4.11 A topological space is said to be a **first countable space** if each point of X possesses a countable local base.

Example : A discrete space (X,D) is first countable. For , in a discrete space, every subset of X is open. In particular, each singleton $\{x\}$, $x \in X$ is open and so in a neighbourhood of x. Also every neighbourhood N (i.e. open set containing x in this case) of x must be a superset of $\{x\}$. Hence, the collection $\{\{x\}\}$ constituting of the single neighbourhood $\{x\}$ of X constitutes a local base at x. But a collection consisting of a single member is countable. Hence there exists a countable base at each point of X.

Definition 4.12 Base for a topology

Let (X,T) be a topological space. A collection β of subsets of X is said to form a base (or an open base) for T if

(i) $\beta \subseteq T$

(ii) for each point $x \in X$ and each open set G containing x there exits a member $B \in \beta$ such that $x \in B \subset G$.

Example : Consider the usual topology U for R . Let β be the collection for all open intervals on R . Then β is a base for U , because

(i) each open interval is U-open so that $\beta \subseteq U$, and

(ii) for each $x \in R$ and each open set G containing x , there exists an open interval (a,b) such that $x \in (a,b) \subseteq G$.

Definition 4.13 A topological space (X,T) is said to be a **second countable space** if there exists a countable base for T.

For example, (R,U) is a second countable space. A space (X,T) is said to be separable if there exists a countable dense subset of X. For example, the set of rationals is a countable dense subset of R , and so R is separable.

Theorem 4.14 : Let (X,T) be a topological space. Prove that s sub-collection β of T is a base for T iff every open set can be expressed as the union of sets β.

Proof :

The only if part : Let β be a base for T and let $G \in T$. Let $H = \cup\{B: B \in \beta \text{ and } B \subset G\}$, that is, let H be the union of all subsets in β which are contained in G and let $x \in G$. Since G is open and β is a base, by definition of a base there exists a set B in β such that $x \in B \subseteq G$.

Since $B \subset H$, it follows that $x \in H$.

Hence $G \subset H$ ....(1)

Also if $x \in H$, then $x \in B$ for some $B \in \beta$

Hence $H \subset G$ ....(2)

(1) and (2) together imply G=H.

This shows that every open set can be expressed as the union of sets in β.

The if part Here $\beta \subset T$ and let every open set G be the union of sets in β. We have to show that β is a base for T .We have

(i) $B \subset T$ (given)

(ii) Let $x \in X$ and let G be any open set containing x. But G is the union of sets in β. Hence there exists a set B in β such that $x \in B \subset G$. Thus β is base for T.

Theorem 4.15 Let X be a non-empty set and let β be a collection of subsets of X. Then prove that β is a base for a unique topology on X if and only if it satisfies the following conditions.

(i) $X = U\{B \mid B \in \beta\}$,

(ii) For every pair of sets, B_1, B_2 in β , and every point $x \in B_1 \cap B_2$, there exists a $B \in \beta$ such that $x \in B \subset B_1 \cap B_2$.

Proof : Straightforward , and so, left for the readers.

Definition 4.16 Let (X,T) be a topological space. A class γ of subsets of X is called a subbase for T if finite intersections for sets in γ form a base for T . It follows that γ is a subbase for T iff every open set is the union of finite intersections of sets in γ.

Example : Consider the space (R,U). Let $\gamma = \{(-\infty,b) \text{ or } (a,\infty) \mid b,a \in R\}$. Then γ is a subbase for U, because the collection of all finite intersections of γ contains all open intervals (a,b) which form a base for U .

The following theorem is simple but many pleasant applications.

Theorem 4.17 : Let X be a non-empty set, and let S be an arbitrary class of subsets of X. Prove that S can serve as an (open) subbase for a unique topology on X.

Proof : If S is empty, then the class of all finite intersections of its set is the single-element class {X}, and the class of all unions of sets in the class is the two element class $\{\phi , x\}$, which is the discrete topology on X. Next, assume that S is non-empty. Let β be the class of all finite intersections of sets in S, and let T be the class of all unions of sets on β. We must show that T is a topology. T clearly contains ϕ and x , and is closed under the formation of arbitrary unions. All that remains is to show that if $\{G_1, G_2, \ldots\ldots, G_n\}$is a non-empty class of finite sets in T then $G = \bigcap_{i=1}^{n} G_i$ is also in T. Since the empty set is in T, we may assume that G is non-empty. Let x be a point in G. Then x is in each G_i, and by the definition of T, for each i there is a set B_i in β such that $x \in B_i \subseteq G_i$. Since each B_i is a finite intersection of sets in S, the intersection of all sets in S which arise in this way is a set in β which contains x and is contained in G. This shows that G is a union of sets in β and is thus itself a set in T.

To show the uniqueness, let T and T^* be the two topologies generated by the class S. Let $G \in T$.

$$\Rightarrow G = \bigcup_{\lambda \in \Delta} B_\lambda, \quad [B_\lambda \text{ is a basic open set for each } \lambda \in \Delta, \Delta \text{ an index set}]$$

$$\Rightarrow B_\lambda = \bigcap_{i=1}^{k} C_{\lambda i}, \quad \text{where } C_{\lambda i} \in S, k \text{ being a finite integer.}$$

Again, since S also generates T^*,

$$C_{\lambda i} \in T^*, \quad \begin{cases} \forall i = 1,2,\cdots k \\ \text{and for each } \lambda \in \Delta \end{cases}$$

$$\Rightarrow \bigcap_{i=1}^{k} C_{\lambda I} \in T^* \quad \text{because } T^* \text{ is a topology}$$

$$\Rightarrow B_\lambda \in T^*, \quad \forall \lambda \in \Delta$$

$$\Rightarrow G = \bigcup_{\lambda \in \Delta} B_\lambda \in T^*,$$

Thus, $G \in T \Rightarrow G \in T^* \Rightarrow T \subseteq T^*$ Similarly, we can show that $T^* \subseteq T$, and hence $T = T^*$, i.e. S generates a unique topology as a subbase.

Problem 4.18 Let X={a,b,c}. Let S={{a},{c},{a,b}} be a class of subsets of X. What topology will be generated by S as a subbase?

Solution : Here, S= {{a},{c},{a,b}}.

Let β = finite intersection of sets in S

={ϕ,X,{a},{c},{a,b}}

which is a base for all the desired topology.

and T = all unions of sets in β

= {ϕ,X,{a},{c},{a,b},{a,c}}

which is the wanted topology.

A few theorems described below show some beautiful properties of a second countable space.

Theorem 4.19 (Lindelof's Theorem) Let X be a second countable space. If a non-empty open set G in X is represented as the union of a class $\{G_i\}$of open sets, then G can be represented as a countable union of G_i's.

Proof. X is given to be second countable and so X has a countable base β.

$$\beta=\{B_1,B_2,B_3,\ldots\ldots\ldots B_n,\ldots\ldots\ldots\}$$

Let x be a point in G . The point x is in some G_i , and we can find a basic open set B_n such that $x\in B_n\subseteq Gi$. If we do this for each point x in G, we obtain a subclass of our countable base whose union is G, and this subclass is necessarily countable. Further , for each basic open set in this subclass is necessarily countable . Further, for each basic open in this subclass we can select a G_i which contains it. The class of G_i's which arises in this way is clearly countable , and its union is G.

As an application of this result, we have the following :

Theorem 4.20 . Let X be a second countable space. Then any base for X has a countable subclass which is also a base.

The readers may consult Simon's book for its illuminating proof.

Problem 4.20 . Show that a second countable space is separable, but that the converse is not necessarily true.

What class of topological space is the converse true?

Solution. First Part : Let (X,T) be a second countable space. Then X has a countable basis, $\beta=\{B_1,B_2,\ldots\ldots B_n,\ldots\ldots\}$. Now select a point x_n in such non-empty basic o[pen set B_n, and form the countable set $A=\{x_n\}$. Let x be any point in X and G be any open set containing x. Since β is a base, ∃ a set B_n in β such that $x\in B_n\subset G$. The G contains $x_n\in A$. So x is a closure point of A. Thus $\bar{A}=X$. Thus X has a countable dense subset , and so X is separable.

Second Part: We next show with a counter example that a separable topological space is not necessarily countable.

Consider the real line R with the co-finite topology. Consider the set Q of all rationals. Then $\bar{Q}=R$, and hence Q is a countable dense subset of R, thus R is a separable topological space.

We now show that it cannot be second countable. If possible, let there exist a countable open base β for T. Let x be ant point in X. we, can claim that $\cap\{G\,|\,G$ is open , $x\in G\}=\{x\}$

For, if possible, let {x,y} be this intersection. Let H=R—{y}. Then H is an open set containing x, and so the intersection cannot be {x,y} but {x}.

Now, for each open set G containing x , we can find a $B\in\beta$ such that $x\in B\subseteq G$. As G runs through all open set containing x, G runs through all these sets in β which contain x.

Hence the intersection of all those sets in β containing x is $\{x\}$. Let D be the collection of all those sets in β which contain x. Then,

$$\cap\{H : H\in D\}=\{x\} \qquad \ldots\ldots\ldots\ldots(1)$$

Taking complements of both sides of (1), we get

$$\cap\{R - H:Hx\in D\}=RX -- \{x\}$$

$\Rightarrow$ a countable set = an uncountable set

This is an absurd result. So X is not second countable.

Third part: However, the converse is true for a metrizable space.

Let X be a separable metric space, and let A be a countable dense subset. If we consider the open spheres with rational radii centered on all the points of A, then the class B of all these open spheres is a countable class of open sets. We show that it is an open base. Let G be an arbitrary nonempty open set and x a point in G. We must find an open sphere in our class which contains x and is contained in G. $S_r(x)$ be an open sphere centered on x and contained in G, and consider the concentric open sphere $S_{\frac{r}{3}}(x)$with one third its radius. Since A is dense, there exists a point a in A which is in $S_{\frac{r}{3}}(x)$. Let r_1 be a rational number such that $r/3 < r_1 < 2r/3$. Then $x\in S_{r_1}(a)\subseteq S_r(x)\subseteq G$. Consequently X is second countable.

Some very deep and profound results on second countable space will be discussed later in the chapter of metrizable spaces.

Problem set 4

Ex 1. Let $f : X\rightarrow Y$ be a mapping of one topological space into another, and let there be given a base in X and a subbase with its generated base in Y. Then prove that

(i) f is continuous $\Leftrightarrow$ the inverse image of each basic open set is open $\Leftrightarrow$ the inverse image of each subbasic open set is open;

(ii) f is open $\Leftrightarrow$ the image of each basic open set is open.

Ex 2. (i) Can any class of subset of a nonempty set X form a base for some topology on X ?

(ii) Can any class of subsets of a nonempty set X form a subbase for some topology on X?

Ex 3. Let $f : X\rightarrow Y$ be any arbitrary mapping from X into Y. Can we always assign a topology to X so that f is continuous ? What is the weakest topology on X so that f is continuous ? If T is a fixed topology on X, what is the largest topology to be assigned to Y so that f is continuous ?

Ex 4. Let X be the set of all positive integers equipped with discrete metric. Show that C (X, R) is not separable.

Ex 5. Show that if f is any nonempty set equipped with the discrete metric, then C(X, R) is separable ⇔ x is finite.

Ex 6. Prove in detail that the open rectangles in the Euclidean plane form an open base.

Ex 7. Let X be an uncountable set equipped with the cofinite topology.

(a) Show that any infinite subset of X is dense.

(b) Show that X is not second countable.

Ex 8. A subset A of a topological space is called a **perfect set** if A=D(A). Show that a set is perfect ⇔ it is closed and has no isolated points. Show that the Cantor set is perfect.

Ex 9. Show that a set A is nowhere dense ⇔ every nonempty open set has a nonempty open subset disjoint from A.

Ex 10. Show that a closed set a nowhere dense ⇔ its component is everywhere dense. Is this true for an arbitrary set ?

Ex 11. Is every second countable space first countable ? Justify your answer.

Ex 12. Construct an example of a first countable space which is not second countable.

Ex 13. Construct a topology on an infinite set X so that X is first countable, second countable and separable.

Ex 14. Let f : X→Y be a continuous map of a topological space (X,T) into a topological space (Y,V).

Examine if a continuous image of
(a) a base for T is a base for V;
(b) a subbase for T is a subbase for V;
(c) a second countable space is second countable;
(d) a separable space is separable.

Ex 15.

(i) Is a separable space metrizable ?
(ii) Is a metrizable space always
(a) separable
(b) first countable
(c) second countable?

❖❖❖

CONTINUOUS FUNCTIONS ON TOPOLOGICAL SPACES

Introduction:

Although continuous functions on metric spaces are briefly highlighted in Chapter 3, it is the purpose of this section to define continuous functions on topological spaces and establish their elementary properties. The notion of distance could be effectively suppressed in defining continuity of functions between metric spaces, by introduction of the use of neighborhoods or open sets.

Definition 5.01. Let X and Y be topological spaces and let f: $X \to Y$ be a mapping. Then f is said to be **continuous** at $x_o \in X$ iff for each neighborhood V if $f(x_o)$ in Y there is a neighborhood U of x_o in X such that $f(U) \subseteq V$. We say f is continuous on X iff f is continuous at each $x_o \in X$.

It is left to the reader to verify that the effect of the definition is not altered if 'nbhd' is replaced by 'open set' throughout.

A map $f: X \to Y$ is said to be **an open mapping (closed mapping)** if f(G) is open (closed) in Y whenever G is open (closed) in X. A **homeomorphism** is a one-to-one continuous mapping of one topological space onto another which is also an open mapping. Two topological spaces X and Y are said to be **homeomorphic** if there exists a **homeomorphism** of X onto Y (and in this case, Y is called a homeomorphic image of X), if X and Y are homeomorphic, then their points can be put into one-to-one correspondence in such a way that their open sets also correspond to one-another. The two spaces therefore differ only in the nature of their points and can, from the point of view of topology, be considered essentially identical.

The proofs of the following few theorems can be established in the line of proofs described in Chapter 3. The first theorem provides an alternative set of characterizations of a function $f: X \to Y$ which is continuous on all of X.

Theorem 5.02. If X and Y are topological spaces and $f: X \to Y$, then the following are all equivalent:

a) f is continuous,

b) for each open set G in Y, $f^{-1}(G)$ is open in X,

c) for each closed set K in Y, $f^{-1}(K)$ is closed in X,

d) for each $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$,

e) the inverse image of each basic open set is open,

f) the inverse image of each subbasic open set is open.

Theorem 5.03. If X,Y and Z are topological spaces and $f:X\to Y$ and $g:Y\to Z$ are continuous, then $g\circ f:X\to Z$ is continuous.

Proof: If H is open in Z, then $g^{-1}(H)$ is open in Y, by continuity of g. Hence, by continuity of f, $f^{-1}[g^{-1}(H)] = (g\circ f)^{-1}(H)$ is open in X. Thus $g\circ f$ is continuous.

Definition 5.04. If $f:X\to Y$ and $A\subset X$, we will use $f|A$ (**the restriction of f to A**) to denote the map of A into Y defined by $(f|A)(a)=f(a)$ for each $a\in A$.

Theorem 5.05. If $A\subset X$ and $f:X\to Y$ is continuous, then $(f|A):A\to Y$ is continuous.

Proof: If H is open in Y, then $(f|A)^{-1}(H) = f^{-1}(H)\cap A$, and the latter is open in the relative topology on A.

Theorem 5.06. If $X = A\cup B$, where A and B are both open (or both closed) in X, and if $f:X\to Y$ is a function such that both $f|A$ and $f|B$ are continuous, then f is continuous.

Proof: Suppose A and B are open. If H is open in Y, then $f^{-1}(H)$ is open in X, since $f^{-1}(H)=(f|A)^{-1}(H)\cup(f|B)^{-1}(H)$ and each of the latter is open in an open subspace of X and so open in X. The proof is similar if A and B are closed.

Theorem 5.07. If X and Y are topological spaces and $f:X\to Y$ is one-one and onto, the following are all equivalent :

a) f is a homeomorphism,

b) if $G\subset X$, then f(G) is open in Y iff G is open in X,

c) if $F\subset X$, then f(F) is closed in Y iff F is closed in X,

d) if $A\subset X$, then $f(\overline{A})=\overline{f(A)}$.

Definition 5.08. Topological property or Topological invariant: A topological property is one which remains invariant under a homeomorphism. Examples: bases, closed sets, open sets, axioms of countability etc.

Theorem 5.09. Show that every open continuous image of a second countable space is second countable, i.e. second axiom of countability is a topological property.

Proof: Let (X,T) be a second countable topological space so that $\exists$ a countable base $\beta=\{B_n \mid n\in N\}$ for T.

Let $f:(X,T)\to(Y,U)$ be a homeomorphism. We assert that $\{f(B_n) \mid n\in N\}$ is a base for the topology U on Y.

Now β is countable $\Rightarrow\{B_n \mid n\in N\}$ is countable $\Rightarrow\{f(B_n) \mid n\in N\}$ is countable.

Since f is open ,$f(B_n)\in U, \forall n\in N$.

Let $G\in U$ be arbitrary.

f is homeomorphism$\Rightarrow$ f is continuous$\Rightarrow f^{-1}(G)$ is T-open set

$$\Rightarrow f^{-1}(G)=\cup\{B_r \mid r\in \Delta\subset N\} \qquad \text{(by a property of a base)}$$

$$\Rightarrow G= f[\cup\{B_r \mid r\in\Delta\}] =\cup\{f(B_r) \mid r\in\Delta\}$$

Therefore, the collection $\beta'=\{f(B_n) \mid n\in N\}$ is a countable base for U and hence (Y,U) is a second countable space.

Theorem 5.10. Show that

(i) the first axiom of countability,

(ii) compactness

(iii) separability and

(iv) connectedness

are topological properties.

Proof: Routine

Two another important theorems on continuity are now in order:

Theorem 5.11 If f and g are continuous real or complex functions defined on a topological space X, then f + g, αf and fg are also continuous. Furthermore, if f and g are real, then $f\vee g$ and $f\wedge g$ are continuous.

Proof: Let (X,T) be a topological space and $x_0\in X$ be arbitrary. Let f and g be continuous real a complex functions on X so that f, g are continuous at x_0. Let $\varepsilon > 0$, then by definition of continuity, $\exists\, G_1, G_2\in T$ with $x_0\in G_1$, $x_0\in G_2$ such that

$$x\in G_1 \Rightarrow |f(x)-f(x_0)|<\varepsilon$$

$$x\in G_2 \Rightarrow |g(x)-g(x_0)|<\varepsilon$$

Take $G = G_1\cap G_2\in T$. Then

$$x\in G_1\cap G_2 = G \Rightarrow x\in G_1,\ x\in G_2$$

Our first aim is to show that f + g is continuous at x_0.

$$|(f+g)(x)-(f+g)(x_0)| = |f(x)-f(x_0)+g(x)-g(x_0)| < |f(x)-f(x_0)| + |g(x)-g(x_0)|$$

$$< \varepsilon+\varepsilon = 2\varepsilon = \varepsilon_1$$

This ensures the continuity of f + g at x_0. Similar routine work will imply the continuity of fg and αf.

Next we suppose that f and g are real continuous functions. Let A=(a, + ∞), B=(−∞, b). We know that the sets of the type A and B form an open subbase for the real line. If we show that inverse image of any such set is open, then the functions will be continuous.

$(f\vee g)^{-1}(A) = \{x: \max \{f,g\}>a\}$

$=\{x : f(x) > a\}\cup\{x : g(x) > a\}$

= union of two open sets = open set.

$(f\vee g)^{-1}(B)=\{x: \max \{f(x),g(x)\}<b\}$

$=\{x : f(x) < b\}\cap\{x : g(x) < b\}$= intersection of open sets = open set.

∴ A,B are open ⇒$(f\vee g)^{-1}(A)$, $(f\vee g)^{-1}(B)$are open ⇒ $f\vee g$ is continuous.

Similarly, we can show that the function $f\wedge g$ is continuous.

Theorem 5.12. We show that f is continuous by showing that it is continuous at an arbitrary point x_0 in X. Let $\varepsilon>0$ be given. Since f is the uniform limit of the f_n's, there exists a positive integer n_0 such that $\left|f(x)-f_{n_0}(x)\right|<\varepsilon/3$ for all points x in X. Since f_{n_0} is continuous, and thus continuous at x_0, there exists a neighborhood G of x_0 such that

$x\in G\Rightarrow\left|f_{n_0}(x)-f_{n_0}(x_0)\right|<\varepsilon/3$. The continuity of f at x_0 now follows from the fact that.

$x\in G\Rightarrow\left|f(x)-f(x_0)\right|$

$=\left|\ [f(x)-f_{n_0}(x)]+[f_{n_0}(x)-f_{n_0}(x_0)]+[f_{n_0}(x_0)-f(x_0)]\ \right|$

$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$ Thus, f is continuous.

Remark: any uniform limit of continuous functions is continuous.

Problem Set 5

Exercise 1. (i) Let U denote the usual topology on R. Let α be any non-zero real number. Then each of the following maps is open as well as closed.

f:(R,U) →(R,U) such that $f(x) = \alpha + x$,

g: (R,U)→ (R,U) such that $g(x) = \alpha x$

In this case if α=0, then this map is closed but not open.

6.0 Introduction:

The great mathematician Frechet was the first to use the term 'compact' at the beginning of Twentith century (present century) and compactness is the next best thing to finiteness. Many of the most important theorems in a course in classical analysis are proved

(ii) The identity map f: (X,U)→(X,U) is open as well as closed.

(iii) A map from an indiscrete space into a topological space is open as well as closed.

(iv) A map from a topological space into a discrete space is open as well as closed.

Exercise 2. Show that characteristic function of $A \subset X$ is continuous on X iff A is both open and closed in X.

Exercise 3. Let X be a topological space defined as follows :

X consists of integers 0,1,2,: A consists of 0; B consists of 0,1. The topology T on X consists of φ, A,B and X. Let f be a continuous map of X into itself such that

f(1) = 0 and f(2) = 1. What is f(0)?

Exercise 4. Show that

(i) [a,b] is homeomorphic to [0,1]

(ii) R is homeomorphic to (0,1)

(iii) R is homeomorphic to (-1,1).

Exercise 5. Show that sequential continuity and continuity in metric spaces are identical, but it is not true in topological spaces.

Exercise 6. Give an example of a sequence of continuous functions defined on [0,1] which converges point wise but not uniformly to a continuous limit.

Exercise 7. Give an example of a sequence of continuous functions defined on [0,1] which converges point wise to a discontinuous limit.

Exercise 8. Give an example of a one-one onto map which is continuous but not homeomorphism.

Exercise 9. Let f be a continuous real or complex function defined on a topological space X, and assume that f is not identically zero, i.e. that the set Y={x: f(x)≠0}is non-empty. Prove in detail that the function $\frac{1}{f}$ defined by $(1/f)(x) = 1/f(x)$ is continuous at each point of the subspace Y.

UNIT 3

COMPACTNESS

6.0 Introduction:

The great mathematician Frechet was the first to use the term 'compact' at the beginning of Twentith century (present century) and compactness is the next best thing to finiteness. Many of the most important theorems in a course in classical analysis are proved for closed bounded intervals (e.g. a continuous function on a closed bounded interval assumes its maximum). The basis for the proof of such theorems is almost without exception the Heine Borel Theorem, that a cover of a closed bounded interval by open sets has a finite subcover. It is not surprising, then, that the (topological) property of closed bounded intervals thus expressed has been made the subject of a definition in topology, the **definition of compactness**:

This Chapter is long, but falls naturally into three parts: In this first we study compactness and their general properties, in the second locally compact spaces are alluded with examples and basic properties, and in the third, some fundamental results of compactness in Metric spaces are highlighted with a few applications.

6.01. Compact spaces: Let X be a topological space. A class $\{G_i\}_{i\in I}$,I being an index set, of open subsets of X is said to be an **open cover** of X if each point in X belongs to at least one G_i, that is, if $\bigcup_i G_i = X$. A subclass of an open cover which is itself an open cover is called **a subcover**. A **compact space** is a topological space in which every open cover has a finite subcover. A compact subspace of a topological space is a subspace which is compact as a topological space in its own right. X is said to be **countably compact** iff each countable open cover of X has a finite subcover. Evidently, X is compact iff X is countably compact and second countable. We begin by proving two simple but widely used results.

Problem 6.02. Show that any closed subspace of a compact space is compact, but a compact subspace of a compact space may **not** be closed.

Proof: Let Y be a closed subspace of a compact space X, and let $\{G_i\}$be an open cover of Y. Each G_i being open in the relative topology on Y, is the intersection with Y of an open subset H_i of X. Since Y is closed, the class composed of Y' and all the H_i's is an open cover of X, and since X is compact, this open cover has a finite subcover. If Y' occurs in this subcover, we discard it. What remains is a finite class of H_i's whose union contains X. Our conclusion that Y is compact now follows from the fact that the corresponding G_i 's forms a finite subcover of the original open cover of Y.

Second part:

Let X={1,2,3}

T={φ, X ,{1}}

Let Y={1,3}. As a finite subspace, Y is compact, but Y is **not** closed in X.

Problem 6.03: Show that any continuous image of a compact space is compact. Furthermore, show with a counter example that if f: $X \to Y$ is a continuous map from a topological space X onto a compact space Y, then X **may not** be compact.

Solution: first part:- Let f: $X \to Y$ be a continuous mapping of a compact space X into an arbitrary topological space Y. We must show that f(X) is a compact subspace of Y. Let $\{G_i\}$ be an open cover of f(X). As in the above proof, each G_i is the intersection with f(X) of an open subset H_i and Y. It is clear that $\{f^{-1}(H_i)\}$ is an open cover of X, and by the compactness of X it has a finite subcover. The union of the finite class of H_i 's of which these are the inverse images clearly contains f(X), so the class of corresponding G_i's is a finite subcover of the original open cover of f(X), and f(X) is compact.

Second part: Consider the real line R with the usual topology and its subspace Y ={1} equipped with the relative topology. Define f: $R \to Y$ by f(x) = 1,$\forall x \in$ R. Then f is continuous, Y is compact, but R is **not** compact.

The proof of the following theorem is easy and may be collected from Simmon's Book:

Theorem 6.04: Prove that a topological space is compact if

(i) Every class of closed sets with empty intersection has a finite subclass with empty intersection,

(ii) Every class of closed sets with the finite intersection property has non-empty intersection,

(iii) Every basic open cover has a finite subcover,

(iv) Every subbasic open cover has a finite subcover.

Theorem 6.05:(The Heine-Borel Theorem) Every closed and bounded subspace of the real line in compact.

Proof: A closed and bounded subspace of the real line is a closed subspace of some closed interval [a, b], and so it suffices to show that [a, b] is compact. If a = b, this is clear, so we may assume that a<b. We know that the class of all intervals of the form [a, d) and (c, b], where c and d are any real numbers such that a<c<b and a<d<b, is a subbase for [a, b]; therefore the class of all [a, c]'s and all [d, b]'s is a closed subbase. Let S={[a, c_i],[d_j, b]}be a class of these subbasic closed sets with the finite intersection property. It suffices to show that the intersection of all sets in S is nonempty. We may assume that S is non-empty. If S contains only intervals of the type [a, c_i] or only intervals of the type [d_j, b], then the intersection clearly contains a or b. We may thus assume that S contains intervals of both types. We now define d by d= sup{d_j}, and we complete the proof by showing that $d \le c_i$ for every i. Suppose that $c_{i_0} < d$ for some i_0. Then, by the definition of d, there exists a d_j such that $c_{i_0} < d_{j_0}$. Since $[a, c_{i_0}] \cap [d_{j_0}, b] = \varphi$, this contradicts the finite intersection property for S and concludes the proof.

The converse of the Heine-Borel Theorem is also true.

Theorem 6.06: Every compact subspace of the real line is closed and bounded.

Proof: Let $C=\{A_n | n\in N\}$, where $A_n=(-n, n)$. Then evidently C is an open cover of R and hence it is an open cover of any subset of R. Let now A be any compact subset of R. Since C is an open cover of A and A is compact, there exists a finite number of positive integers $n_1,n_2,\ldots\ldots,n_k$ such that the subcollection.

$C_1=\{ A_{n_1}, A_{n_2}, \ldots\ldots, A_{n_k} \}$ of C covers A.

let $n_o=\max\{n_1,n_2,\ldots,n_k\}$. Then evidently $A\subset A_{n_0}=(-n_o, n_o)$

This implies that A is bounded.

Next, we aim at proving that A is closed. We shell show that no point outside A can be a limit point of A.

Let $a\in R\text{-}A$ so that $a\notin A$. Consider the family of closed sets $F_n=\left[a-\frac{1}{n}, a+\frac{1}{n}\right]$ for each $n\in N$. Then $C_2=\{ R-F_n | n\in N\}$ is a family of open sets. It is evident that the set $\cap\{F_n | n\in N\}$ consists of the single point a, and since a is not in A, it follows that,

$$A\subset[R-\cap\{F_n| n\in N\}]$$
$$=\cup\{ R-F_n| n\in N\}$$

Thus the set A is covered by the family C′. Hence, by compactness of A, there exists a finite subcover of C′. That is there exists a positive integer n_1 such that every point of A is contained in at least one of the open sets

$$R-F_1, R-F_2,\ldots\ldots, R-F_{n_1}.$$

It then follows that n_o point of A is contained in, $F_{n_1}=\left[a-\frac{1}{n_1}, a+\frac{1}{n_1}\right]$. This implies that a is not a limit point of A. Thus, we have shown that no point outside A can be a limit point of A. It means that all limit points of A are points of A itself. Hence A is closed.

Theorem 6.07 Remarks:

(i) The Heine-Borel Theorem can be extended to a metric space as " A compact subspace of an arbitrary metric space is closed and bounded". There are many ways to prove this result. One way of arguing this is that a metric space is a Hausdorff space, and a compact subspace of a Hausdorff space is closed. Moreover, a compact metric space is totally bounded and hence bounded. These ascertain that a compact subspace of a metric space is both closed and bounded.

However, the converse of this result may not be true in a general metric space that can be formally described as follows:

(ii) A closed and bounded subspace of an arbitrary metric space is **not necessarily** compact.

For example, consider the real line R with the discrete metric d defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, y)=0$ if $x = y$. Consider the set N of natural numbers. Then $\delta(N)=\sup \{d(x, y) \mid x, y \in N\}=1$. And hence N is bounded.

Again every subset of R is both open and closed, and hence N is closed. Thus, N is both bounded and closed.

However, the class $C=\{G_n = \{n\} \mid n \in N\}$ is an open cover of N which does not have any finite subcover and hence N is **not** compact.

§ Locally Compact Spaces:

A topological space is said to be **locally compact** if each of its points has a neighborhood with compact closure. For example, the n-dimensional Euclidean space R^n is locally compact, because any open sphere centered on any point is a neighborhood of the point whose closure, being a closed and bounded subspace of R^n, is compact.

Theorem 6.08: Every compact space is locally compact.

Proof: Let (X,T) be a compact space and $x \in X$. Now X is a neighborhood of x whose closure is X. But X is given to be compact. Hence every point of X has at least one neighborhood (the whole space) whose closure is compact. It follows that (X, T) is locally compact.

Theorem 6.09: Every closed subset of a locally compact space is locally compact.

Proof: Let (X, T) be a locally compact space and let Y be a closed subset of X. We have to show that the subspace (Y, T^*) is locally compact. Let y be an arbitrary point in Y. Since X is locally compact at y, there exists a T-neighborhood N of Y such that $\overline{N}$ is compact. Since $\overline{N} \supset N$ is a T-closed and compact nbhd of Y.

Let $M = \overline{N} \cap Y$

Since $\overline{N}$ and Y are T-closed, it follows that M is also T-closed. Also M is closed in N as well as in Y. Hence M is compact in $\overline{N}$, being a closed subset of the compact set $\overline{N}$. It follows that M is compact in X and consequently compact in Y. Also M is a T*-nbhd of Y.

Since $\overline{M}$=M, it follow that M is a T*-nbhd of Y such that $\overline{M}$ is compact in Y. Hence (Y, T*) is locally compact.

A Counter Example : Show by means of an example that a locally compact space need not be compact.

Solution: Consider any discrete topological space (X, D) where X is infinite. Then X is not compact since the collection of all single sets in an infinite open cover of X which has no finite sub cover. But X is locally compact. For, let x be any point of X. Then $\{x\}$ is a nbhd of x whose closure is $\{x\}$. Also $\{x\}$ is a compact subset of X. Hence every point of x has a nbhd whose closure is compact.

Compactness For Metric Spaces:

In all candor, we must admit that the intuitive meaning of compactness for topological spaces is some what elusive. This concept, however, is so vitally important throughout topology that we consider it worth while to devote this and the next section to giving several equivalent forms of compactness for the special case of a metric space. Some if these are quite useful.

We begin with some definitions:

Definition 6.11: A metric space is said to have the **Bolzano-Weierstrass property** if every infinite subset has a limit point. A metric space is said to be **sequentially compact** if every sequence in it has a convergent subsequence. Our main purpose in this section is to prove that each of these properties is equivalent to compactness in the case of a metric space.

Theorem 6.12: Prove that a metric space is sequentially compact $\Leftrightarrow$ it has the BWP.

Proof: Let X be a metric space, and assume first that X is sequentially compact. We show that an infinite subset A of X has a limit point. Since A is infinite, a sequence $\{x_n\}$ of distinct points can be extracted from A. By our assumption of sequential compactness, this sequence has a subsequence which converges to a point x. This yields that x is a limit point of the set of points of the subsequence, and since this set is a subset of A, x is also a limit point of A.

Next, we assume that every infinite subset of X has a limit point, and we prove from this that X is sequentially compact. Let $\{x_n\}$ be an arbitrary sequence in X. If $\{x_n\}$ has a point which is infinitely repeated, then it has a constant subsequence, and this subsequence is clearly convergent. If no point of $\{x_n\}$ is infinitily repeated, then the set A of points of this sequence is infinite. By our assumption, the set A has a limit point x, and it is easy to extract from $\{x_n\}$ a subsequence which converges to x.

Theorem 6.13: Prove that every compact metric space has the Bolzano-Weierstrass property:

Proof: Let X be a compact metric space and A an infinite subset of X. We assume that A has no limit-point, and from this we deduce a contradiction. By our assumption, each point of X is not a limit point of A, so each point of X is the centre of an open sphere which contains no point of A different from its centre. The class of all these open spheres is an open cover, and by compactness there exists a finite sub cover. Since A is contained in the set of all centres of spheres in this sub cover, A is clearly finite. This contradicts the fact that A is infinite and concludes the proof.

6.14 Definition: Let (X, d) be a metric space and $\{G_i\}$ an open cover of X. A real number a>0 is called a **lebesgue number** for our given open cover $\{G_i\}$ if each subset of X whose diameter is less than a is contained in at least one G_i. If $\varepsilon>0$ is given, a subset A of X is called an **ε-net** if A is finite and $X = \bigcup_{a\in A} S_\varepsilon(a)$, that is, if A is finite and its points are scattered through X in such a way that each point of X is distant by less than ε from at least one point of A. The metric space X is said to be **totally bounded** if it has an ε–net for each ε>0. The proofs of the following theorems can be obtained from Simmon's Book.

Theorem 6.15: (Lebesgue's Covering Lemma) In a sequentially compact metric space, every open cover has a lebesgue number.

Theorem 6.16: Prove that

Every sequentially compact metric space is totally bounded.

Every sequentially compact metric space is compact.

Theorem 6.17: Any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Proof: Let f be a continuous mapping of a compact metric space (X, d_1) into a metric space (Y, d_2). Let $\varepsilon>0$ be given. For each point x in X, consider its image f(x) and the open sphere $S_{\varepsilon/2}(f(x))$ centered on this image with radius $\varepsilon/2$. Since f is continuous, the inverse image of each of these open spheres is an open subset of X, and the class of all such inverse image is an open cover of X. Since X is compact, this open cover has a lebesgue number δ. If $x_1$and x_2 are any two points in X for which $d_1(x_1, x_2)<\delta$, then the set $\{x_1, x_2\}$ is a set with diameter less than δ, both points belong to the inverse image of some one of the above open spheres both $f(x_1)$ and $f(x_2)$ belong to one these open spheres, and therefore $d_2(f(x_1), f(x_2))<\varepsilon$, which shows that f is indeed uniformly continuous.

Remark 6.18: This theorem guarantees that any real valued continuous mapping defined on a bounded closed interval [a, b] is always uniformly continuous, because of the fact that [a, b] is a compact subset of R, (prove this).

Theorem 6.19: Prove that a metric space is compact $\Leftrightarrow$ it is complete and totally bounded.

Proof: Let X be a metric space.

We first assume that X is compact. Hence it is sequentially compact. We first want to show that X is totally bounded. Let $\varepsilon>0$ be given. Choose a point a_1 in X and form the open sphere $S_\varepsilon(a_1)$. If this open sphere contains every point of X, then the single-element set $\{a_1\}$ is an ε-net. If there are points outside of $S_\varepsilon(a_1)$, let a_2 be such a point and form the set $S_\varepsilon(a_1)\cup S_\varepsilon(a_2)$. If this union contains every point of X, then the two element set $\{a_1, a_2\}$ is an ε-net. If we continue in this way, some union of the form $S_\varepsilon(a_1)\cup S_\varepsilon(a_2)\cup\ldots\cup S_\varepsilon(a_n)$ will necessarily contain every point of X; for if this process could be continued indefinitely then the sequence $\{a_1,a_2,\ldots,a_n,\ldots\}$ would be a sequence with no convergent subsequence, contrary to the assumed sequential compactness of X. This implies that some finite set of the form $\{a_1,a_2,\ldots,a_n\}$is an ε-net, so X is totally bounded.

Next we want to show that X is complete. Suppose, if possible, X is not complete. Then there exists a Cauchy sequence $\{x_n\}$ which does not converge to any point in X. Let $p\in X$. Since $\lim x_n\neq p$, there exists a number $\varepsilon_p>0$for which it is not possible to find a positive integer n_o such that $x_{n_0}\in S_\varepsilon(p)$. Now the collection $C=\{\ S_{\frac{\varepsilon_p}{2}}(p)\mid p\in X\}$is an open cover of X. Since (X, d) is compact, there exists a finite number of points $p_1, p_2, p_3, \ldots, p_k$ in X such that $\quad C_1=\{ S_{\frac{\varepsilon_{p_i}}{2}}(p_i) | i = 1,2,\ldots k \}$ is a cover of X.

Let $\varepsilon=\min\{\varepsilon p_i \mid i = 1,2,\ldots k \}$. Since $\{x_n\}$ is a Cauchy sequence and $\varepsilon>0$, there is an integer $n(\varepsilon)$ such that

$$m, n \geq n(\varepsilon) \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2} \ldots\ldots\ldots (i)$$

Since C_1 is a cover of X and $x_{n_o} \in X$, there exists a member $S_{\frac{\varepsilon_{p_j}}{2}}(p_j)$ of C_1 such that

$x_{n_o} \in S_{\frac{\varepsilon_{p_j}}{2}}(p_j)$. Hence $\qquad d(x_{n_o}, p_j) < \varepsilon_{p_j/2} \ldots\ldots\ldots (ii)$

It follows that if $m \geq n_o$, then $\quad d(x_m, p_j) \leq d(x_m, x_{n_o}) + d(x_{n_o}, p_j)$

$$< \frac{\varepsilon}{2} + \varepsilon_{\frac{p_j}{2}} \qquad \text{by using (i) and (ii)}$$

$$< \frac{\varepsilon_{p_j}}{2} + \frac{\varepsilon_{p_j}}{2} = \varepsilon p_j$$

Hence, $x_m \in S_{\varepsilon_{p_j}}(p_j)$. Thus if $m \geq n_o$, x_m is in the sphere of radius εp_j and centre p_j. But this contradicts the choice of εp_j. It follows that X is complete.

Conversely, we assume that X is complete and totally bounded, and we prove that X is compact by showing that every sequence has a convergent subsequence. Since X is complete, it suffices to show that every sequence has a Cauchy subsequence. Consider an arbitrary sequence $S_1=\{x_{11}, x_{12}, x_{13}\ldots\}$.

Since X is totally bounded, there exists a finite class of open spheres, each with radius $\frac{1}{2}$, whose union equals X; and from this we see that S_1 has a subsequence $S_2=\{x_{21}, x_{22}, x_{23}\ldots\}$ all of whose points lie in some one open sphere of radius $\frac{1}{2}$. Another application of the total boundedness of X shows similarly that S_2 has a subsequence $S_3=\{x_{31}, x_{32}, x_{33}\ldots\}$ all of whose points lie in some one open sphere of radius $\frac{1}{3}$. We continue forming successive subsequences in this manner, and we let $S=\{x_{11}, x_{22}, x_{33}, \ldots\}$ be the "diagonal" subsequence of S_1. By the nature of this construction, S is clearly a Cauchy subsequence of S_1, and our proof is complete.

We now turn to the problem of characterizing compact subspaces of C(x, R) or C(X, C). Set (X, d) be a metric space and let A be a nonempty set of continuous real or complex functions defined on X. If f is a function in A, this function is uniformly continuous that is for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, x') < \delta \Rightarrow |f(x) - f(x')| < \varepsilon$.

In general, δ depends not only an ε but also on the function f. This leads to the following definition:

Definition 6.20: Let A be a nonempty subset of C(X, R) or C(X, C). A is said to be **equicontinuous** if for each ε, a δ can be found which serves at once for all functions f in A, that is, if for each $\varepsilon>0$ there exists $\delta>0$ such that for every f in A, $d(x, x')<\delta \Rightarrow |f(x) - f(x')|<\varepsilon$.

Theorem 6.21: (Ascoli's Theorem), If X is a compact metric space, then a closed subspace of C(X, R) or C(X, C) is compact $\Leftrightarrow$ it is bounded and equicontinuous.

Proof: Let (X, d) be a metric space and let F be a closed subspace of C(X, R) or C(X, C). We first assume that F is compact, and we prove that it is bounded and equicontinuous.

F is compact and hence it is sequentially compact. Again a sequentially compact set is totally bounded and hence it is bounded. Thus, F is a bounded subset of C(X, R). We next prove that F is equicontinuous as follows:

Let $\varepsilon>0$ be given, Since F is compact, and therfore totally bounded, we can find an $\varepsilon/3$-net $\{f_1, f_2, \ldots\ldots f_n\}$ in F. Each f_k is uniformly continuous, so for each k=1,2,3….,n,there exists $\delta_k>0$ such that $d(x, x')<\delta_k \Rightarrow |f_k(x) - f_k(x')|<\varepsilon/3$. We now define δ to be the smallest of the numbers $\delta_1, \delta_2, \ldots\ldots, \delta_n$. If f is any function in F and f_k is chosen so that $\|f - f_k\| < \varepsilon/3$, then

$$d(x, x')<\delta \Rightarrow |f(x) - f(x')| \le |f(x) - f_k(x)| + |f_k(x) - f_k(x')| + |f_k(x') - f(x')| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This shows that F is equicontinuous.

We now assume that F is bounded and equicontinuous, and we demonstrate that it is compact by showing that every sequence in it has a convergent subsequence. Since F is closed, and therefore complete, it suffices to show that every sequence in it has a Cauchy subsequence. Moreover, since a compact metric space is separable, X has a countable dense subset. Let the points of this subset be arranged in a sequence $\{x_i\}=\{x_1, x_2, x_3 \ldots\ldots, x_i, \ldots\}$

Now let $\qquad S_1 = \{f_{11}, f_{12}, f_{13}, \ldots\ldots\}$

be an arbitrary sequence in F. Our hypothesis that F is bounded means that there exists a real number K such that $\|f\| \le K$ for every f in F, or equivalently, such that $|f(x)| \le K$ for every f in F and every x in X. Consider the sequence of numbers $\{f_{1j}(x_2)\}$, j=1,2,3… and observe that since this sequence is bounded, it has a convergent subsequence. Let $S_2=\{f_{21}, f_{22}, f_{23} \ldots\}$ be a subsequence of S_1 such that $\{f_{2j}(x_2)\}$ converges. We next consider the sequence of numbers $\{f_{2j}(x_3)\}$, and in the same way, we let $S_3=\{f_{31}, f_{32}, f_{33} \ldots\}$ be a subsequence of S_2 such that $\{f_{3j}(x_3)\}$ converges. If we continue this process, we get an array of sequences of the form.

$S_1=\{f_{11}, f_{12}, f_{13},\ldots\}$, $S_2=\{f_{21}, f_{22}, f_{23}\ldots\}$, $S_3=\{f_{31}, f_{32}, f_{33}\ldots\}$, ...

in which each sequence is a subsequence of the one directly above it, and for each i the sequence $S_i=\{f_{i1}, f_{i2}, f_{i3}\ldots\}$ has the property that $\{f_{ij}(x_i)\}$is a convergent sequence of numbers. If we define $f_1, f_2, f_3\ldots$ by $f_1=f_{11}, f_2=f_{22}, f_3=f_{33},\ldots$, then the sequence

$S=\{f_1, f_2, f_3\ldots\}$ is the 'diagonal' subsequence of S_1. It is clear from this construction that for each point x_i in our dense subset of X, the sequence $\{f_n(x_i)\}$ is a convergent sequence of numbers. It remains only to show that S, as a sequence of functions in C(x, R) or C(X, C), is a cauchy sequence.

Problem Set 6

Ex. 1. Show that a closed bounded interval [a, b] is compact.

Ex 2. Show that a continuous real or complex function defined on a compact space in bounded. More generally, show that a continuous mapping of a compact space into any metric space is bounded.

***Ex 3.** If X is a compact space, and if $\{f_n\}$ is a monotone sequence of continuous real functions defined on X which converges point-wise to a continuous real function f defined on X, show that f_n converges uniformly to f. [The assumption that $\{f_n\}$ is a monotone sequence means that either $f_1 \le f_2 \le f_3 \le\ldots$or $f_1 \ge f_2 \ge f_3 \ge\ldots$].

Ex 4. Show that a topological space is locally compact $\Leftrightarrow$there is an open base at each point whose sets all have compact closures.

Ex 5. Show that a continuous image of a locally compact space need not be locally compact.

Ex 6. Show that a subspace of R^n is bounded $\Leftrightarrow$it is totally bounded.

Ex 7. Prove the Bolzano-Weierstrass theorem for R^n: if X is a closed and bounded subset of R^n, then every infinite subset of X has a limit point in X.

Ex 8. Show that a compact metric space is separable.

Ex 9. By considering the sequence of functions in C[0,1] defined by $f_n(x)=nx$, for $0\le x \le \frac{1}{n}$, $f_n(x)=1$ for $\frac{1}{n}\le x \le 1$, show that C[0,1] is not locally compact.

Ex 10. Every locally compact T_2-space is a regular space.

Ex 11. Is every open subspace of a locally compact space is locally compact? Give reasons in support of your answer.

Ex 12. If a metric space (X, d) is totally bounded, then X is bounded. Produce a counter example to show that a bounded metric space may not be totally bounded.

Ex 13. Show that every totally bounded metric space is separable. Is the converse true? Justify your answer.

Ex 14. Show that a metric space is Lindelof space if and only if it is second countable.

Ex 15. Let f: X $\rightarrow$Y be a continuous map from a compact space X into a metric space Y. Prove that f is uniformly continuous. If X is not compact, Y is compact, is f still uniformly continuous?

UNIT 4

Separations

SEPARATION-AXIOMS

7.0.Introduction: The T_i-space nomenclature, for i =0,1,2,3,4,5 ... was introduced by Alexandroff and Hopf. The symbol 'T' refers to the German word 'Trennungs-axiom', which means, "separation axiom". The separation properties are of concern to us because the supply of open sets possessed by a topological space is intimately linked to its supply of continuous functions; and since continuous functions are of central importance in topology, we naturally wish to guarantee that enough of them are present to make our discussions fruitful. Hausdorff spaces and Normal spaces, and their applications in Urysoohn lemma, Tietze Extension Theorem, Metrization Theorem etc play crucial role in this chapter.

7.01. T_o-axiom of separation or Kolomogorov Space.

A topological space (X, T) is said to satisfy the T_o-axiom of separation if given a pair of distinct points x, y in X, either $\exists$ $G \in T$ s.t. $x \in G$, $y \notin G$; or $\exists$ $H \in T$ s. t. $y \in H$, $x \notin H$. In this case the space (X,T) is called a T_o-space.

7.02 Ex. show that every discrete space is a T_o-space.

Solution: Let (X, D) be a discrete topological space and let x, y be distinct points of X. Since the space is discrete, $\{x\}$ is an open nbhd. of x which does not contain y. It follow that (X, D) is a T_o-space.

7.03 Ex. Show that an indiscrete space is not a T_o-space.

Solution: Let (X, I) be an indiscrete space and let x, y be two distinct points of X. Now, the only open nbhd of x is X which also contains y. Thus there exists no open nbhd of x which does not contain y. Hence (X, I) is **not** a T_o-space.

7.04 Theorem: prove that a space being a T_o-space is both hereditary and topological property.

The proof is routine and so left for the readers.

7.05 T_1-axiom or Frechet axiom of separation:

A topological space (X, T) is said to satisfy the T_1-axiom of separation if given a pair of distinct point x, $y \in X$, $\exists$ G, $H \in T$ such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$. In this case the space (X, T) is called a **T_1-space or Frechet space.**

7. 06 Ex: Show that the real line 3 with the usual topology U is a T_1-space.

Solution: Let x, y be any two distinct real numbers and y>x. Let y - x = k. Then G=$(x-\frac{k}{4}, x+\frac{k}{4})$ and H=$(y-\frac{k}{4}, y+\frac{k}{4})$ are U-open sets such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$. Hence (R ,U) is a T_1-space.

7. 07 Theorem: Show that a topological space (X, T) is a T_1-space if and only if every finite subset of X is closed.

Proof: We first assume that (X, T) is a T_1-space. Let $A=\{x_1,x_2....x_n\}$ be a finite subset of X. We wish to show that A is closed.

If A = X, then obviously A is closed. Next suppose that $A \neq X$. Let $x \in X - A$. Since X is T_1-space, for each i=1,2,...n. there exists an open set G_i such that $x \in G_i$ and $x_i \notin G_i$. Put $G=\bigcap_{i=1}^{n} G_i$. Then G is an open set such that $x \in G$, $x_i \notin G$, $\forall i = 1, 2, \ldots, n$. This implies that x is not limit point of A. Hence A contains all its limit points and so A is closed.

Conversely, suppose that every finite subset of X is closed. We are to show that X is T_1-space. Let x, y be any two distinct points of X. Then X-{x} is an open set which contains y but does not contain x. Similarly, X-{y} is an open set which contains x but does not contain y. Hence (X, T) is a T_1-space.

7.08 Theorem: Show that every finite T_1-space is discrete.

Proof: Let (X, T) be a T_1-space where X is finite. Since the space is T_1, every singleton subset of X is closed and consequently every finite subset of X is closed. Since X is finite, it follows that every subset of X is closed and hence open. Therefore, the space must be discrete.

7.09 Theorem: Prove that for any set X there exists a unique smallest topology **T** such that (X,T) is a T_1-space.

Proof: Left for the readers.

7.10 Definition: A topological space (X, T) is said to be a **Hausdorff space** or a **T_2-space** if for every pair of distinct points x, y of X, there exist disjoint neighborhoods of x and y, that is, there exists neighborhoods N of x and M of y such that $N \cap M=\phi$. If (X, T) is a Hausdorff space, then T is said to be a **Hausdorff topology on X.**

7.11 Example: Let X={a,b,c} and let

$$T_1=\{\phi,\{a\},\{b\},\{c\},\{a, b\},\{b, c\},\{c,a\},X\} \text{ and } T_2=\{\phi,\{a\},\{b, c\},X\}.$$

Then (X, T_1) is a Hausdorff space since each singlton set is T-open so that distinct points have disjoint nbhds. But (X, T_2) is not a Hausdorff space since there exists no disjoint nbhds of b and c.

7.12 Example: Consider the cofinite topology on an infinite set X and show that it is **not** T_2.

Proof: Straightforward.

7.13 Example: If f and g are continuous functions on a topological space X with values in a Hausdorff space Y, then prove that the set of all points x in X such that f(x)=g(x) is closed. Deduce that if f and g agree on a dense subset of X, then f = g.

Proof: Let $A=\{x\in X | f(x)=g(x)\}$. To prove that A is closed, that is, to prove that $X-A=\{x\in X | f(x)\neq g(x)\}$is open. Let x_o be any point of X-A. Then $f(x_o)\neq g(x_o)$ with $f(x_o)$, $g(x_o)\in Y$. Since Y is Hausdorff, there exist open sets G and H in Y such that $f(x_0)\in G$, $g(x_0)\in H$ and $G\cap H=\phi$ ......(1)

Since f,g are continuous, $f^{-1}(G)$and $g^{-1}(H)$ are open sets in X such that $x_o\in f^{-1}(G)$ and $x_o\in g^{-1}(H)$, and so their intersection $f^{-1}(G)\cap g^{-1}(H)$ is also an open set in X containing x_o.

Let $E= f^{-1}(G)\cap g^{-1}(H)$. We now prove that $E\subset X-A$. Suppose, if possible, $E\not\subset X-A$. Then there is at least one point in E, say z, such that $z\in E$ and $z\notin X-A$.

Now, $z\in E\Rightarrow z\in f^{-1}(G)$ and $z\in g^{-1}(H)\Rightarrow f(z)\in G$ and $g(z)\in H$........(2)

and $z\notin X\text{-}A\Rightarrow z\in A\Rightarrow f(z)=g(z)$

now (2) and (3) show that $f(z)\in G$ and $f(z)\in H$, and hence $f(z)\in G\cap H$ which contradicts (1) since $G\cap H=\phi$. It follows that $x_o\in F\subset X\text{-}A$. We have thus shown that X-A contains a nbhd of each of its points and consequently X-A is open, that is, A is closed.

Deduction: Let f and g agree on a dense subset B of X, that is, let $f(x)=g(x)$, $\forall\ x\in B$.

To prove that f =g. Let x be any arbitrary point of X. Then $\exists$ a sequence $\{x_n\}$ in B such that $x_n\rightarrow x$.

Now, $f(x_n)= g(x_n)$, $\forall n\in N$

$$\Rightarrow \lim_{n\to\alpha} f(x_n)= \lim_{n\to\alpha} g(x_n)\Rightarrow f[\lim_{n\to\alpha} x_n]=g[\lim_{n\to\alpha} x_n]\Rightarrow f(x)= g(x),\ \forall x\in X\Rightarrow f = g.$$

7.14Theorem: Prove

Every compact subset of a Hausdorff space is closed,

A one-one continuous map of a compact space onto a Hausdorff space is a homeomorphism.

For proofs see Simmon's Book.

7.15 Theorem: Prove that in a Hausdorff space, any point and disjoint compact subspace can be separated by open sets, in the sense that they have disjoint neighborhoods.

Proof: Let X be a Hausdorff space, x a point in X, and C a compact subspace of X which does not contain x. We construct a disjoint pair of open sets G and H such that $x\in G$ and $C\subseteq H$. Let y be a point in C . Since X is a Hausdorff space, x and y have disjoint neighborhoods G_x and H_y. If we allow y to vary over C, we obtain a class if H_y's whose union contains C; abd sunce C is compact, some finite subclass, which we denote by $\{H_1,H_2,\ldots,H_n\}$, is such that $C\subseteq\bigcup_{i=1}^{n} H_i$. If $G_1,G_2,\ldots,G_n$. are the nbhds of x which correspond to the H_i's, we put $G=\bigcap_{i=1}^{n} G_i$ and $H=\bigcup_{i=1}^{n} H_i$, and observe that these two sets have the required properties.

7.16 Definition: Let X be an arbitrary topological space and $\{x_n\}$ a sequence of points in X. This sequence is said to be **convergent** if there exists a point x in X such that for each neighborhood G of x a positive integer n_o can be found with the property that x_n is in G for all $n \geq n_o$. The point x is called **a limit** of the sequence, and we say that x_n converges to x (and symbolize this by $x_n \to x$),

7.17 Theorem: Every convergent sequence in a Hausdorff space has a unique limit.

Proof: Let (X, T) be a Hausdorff space and let $\{x_n\}$ be a convergent sequence in X. We want to show the limit of this sequence is unique. If possible, let the sequence $\{x_n\}$ converge to two distinct points x and y. Since the space is Hausdorff, there exist two open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \phi$.

Since $\{x_n\}$ converges to x, there exists $n_o \in N$ such that $n \geq n_o \Rightarrow x_n \in G$.

Again since $\{x_n\}$ converges to y, there exists $m_o \in N$ such that $n \geq m_o \Rightarrow x_n \in H$

Let $n_1 = \max\{n_o, m_O\}$. Then $x_{n_1} \in G \cap H$. But this contradicts the fact $G \cap H = \phi$.

Hence, the limit of the sequence must be unique.

7.18 Example: Give an example to show that the converse of the above theorem is not true.

Solution: Consider the co-countable topology T on an uncountable set X. We know that this space is not T_2. But in this space every convergent sequence has a unique limit.

As another example, consider the indiscrete topological space X consisting of at least two points. This space isn't T_2 but in this space any sequence converges to every point of the space.

7.19 Theorem: Let (X,T) be a first countable space. Then (X, T) is a Hausdorff space if and only if every convergent sequence in X has a unique limit.

Proof: We first assume that (X,T) is a Hausdorff space. To prove that every convergent sequence in X has a unique limit. For proof, see theorem 7.17,

Next, let us assume that (X,T) be a first axiom space in which every convergent sequence has a unique limit. To prove that the space is Hausdorff. Suppose, if possible, (X,T) is not Hausdorff. Then there must exist two distinct points x and y such that every open set containing x has a non-empty intersection with every open set containing y. Since the space is first countable, there exist nested (monotone decreasing) local bases at x and y (prove this)

Let $\beta(x)=\{B_n(x) \mid n \in N\}$and

$\beta(y)=\{ B_n(y) \mid n \in N\}$ be the nested local bases at x and y respectively. Then we must have $B_n(x) \cap B_n(y) \neq \phi$, $\forall n \in N$, and so there exist $x_n \in X$ such that

$x_n \in B_n(x) \cap B_n(y), \forall n \in N.$

Therefore, $x_n \in B_n(x)$ and $x_n \in B_n(y), \forall n \in N.$

Let G and H be arbitrary open sets such that $x \in G$ and $y \in H$. Then by the definition of nested base, there exists an integer n_0, such that ,

$B_n(x) \subset G$ and $B_n(y) \subset H, \forall n \geq n_0$

This means that $x_n \in G$ and $x_n \in H, \forall n \geq n_0$. It follows that $x_n \to x$ and $x_n \to y$. But this contradicts the hypothesis that every convergent sequence in X has a unique limit. Therefore, (X, T) is a Hausdorff space.

7.20 Definition: A topological space (X, T) is said to be a **regular space** if given an element $x \in X$ and closed set $F \subset X$ s. t. $x \notin F$, $\exists$ disjoint open sets $G_1, G_2, \subset X$ s. t. $x \in G_1$, $F \subset G_2$.

A regular T_1- space is called a **T_3- space.** A topological space (X, T) is called a **completely regular space** if given a closed set $F \subset X$ and a point $x \in X$ s. t. $x \notin F$, $\exists$ a continuous map $f: X \to [0,1]$ with the property,

$f(x)=0, f(F)=\{1\}.$

A completely regular T_1-space is called a **Tychonoff space**. Tychonoff space is also called a $T_{3\frac{1}{2}}$ space.

7.21 Examples:

(i) Let $X=\{a, b, c\}$ and $T=\{\phi,\{a\},\{b, c\},X\}$ show that (X, T) is regular, but not T_1 and T_3.

(ii) Show that (R, U) is T_3

7.22 Theorem: A topological space (X, T) is regular iff for every point $x \in X$ and every nbhd N of x, there exists a neighborhood M of x such that $\overline{M} \subset N$.

Proof: It is enough to prove the theorem for open neighborhoods.

The only if part. Let (X, T) be a regular space and let N be an open neighborhood of x. Then X-N is a T-closed set which does not contain x. Since the space is regular, there exists two open sets L and M such that $X-N \subset L$, $x \in M$ and $L \cap M = \phi$. Since $L \cap M = \phi$ we have $M \subset X-L$

$$\therefore \subset \overline{X-L} = X-L \text{..........(1)}$$

Also, $X-N \subset L \Rightarrow X-(X-N) \supset X-L \Rightarrow N \supset X-L$

Then (1) gives $\overline{M} \subset N$.

If part: Let the condition hold. Let F be a T-closed set and x a point of X such that $x \notin F$.

Then, $x \in X$-F. Since X-F is an open set containing x, by hypothesis, there must exist an open set M such that.

$$x \in M \text{ and } \overline{M} \subset X\text{-}F.$$

But $\overline{M} \subset X\text{-}F \Rightarrow X\text{-}\overline{M} \supset X\text{-}(X\text{-}F)$, that is, $F \subset X\text{-}\overline{M}$.

Further, since $\overline{M}$ is closed, $X\text{-}\overline{M}$ is an open set such that $M \cap (X\text{-}\overline{M}) = \phi$.

Thus, M and $X\text{-}\overline{M}$ are two disjoint open sets containing x and F respectively. It follows that the space is regular.

7.23 Theorem: Let F be a closed subset of a completely regular space (X,T) and $x_0 \in F^c$, then prove that there exist a continuous map $f:X \rightarrow [0,1]$ s. t. $f(x_0)=1, f(F)=\{0\}$.

Solution: Suppose (X, T) is a completely regular space and F a closed subset of X s. t. $x_0 \in X$-F so that $x_0 \notin F$. By definition of a completely regular space, $\exists$ a continuous map $g: X \rightarrow [0,1]$ s. t. $g(x_0)=0$, $g(F)=\{1\}$. Now, we define a map

$f:X \rightarrow [0,1]$ s. t. $f(x)=1-g(x)$.

Then $f(x_0)=1-g(x_0)=1-0=1$

$\because f(x)=1-g(x), \forall x \in X$.

$\therefore f(x)=1-g(x), \forall x \in F$.

$=1-1=0$.

$\therefore f(x)=0, \forall x \in F$ This $\Rightarrow f(F)=\{0\}$.

g is continuous $\Rightarrow$ 1-g is continuous $\Rightarrow$ f is continuous

Finally, we have a continuous map $f:X \rightarrow [0,1]$ s. t. $f(x_0)=1$ and $f(F)=\{0\}$.

7.24 Normal space and Completely Normal Space:

A topological space (X, T) is said to be a **normal space** if given a pair of disjoint closed sets F_1 and $F_2 \subset X$ there exists disjoint open sets G_1 and $G_2 \subset X$ such that $F_1 \subset G_1$, $F_2 \subset G_2$. A T_1-normal space is called a **T_4-space.** Two sets A, B, are called **separated sets** if, $A \neq \phi$, $B \neq \phi$, $\overline{A} \cap B = \phi$, $A \cap \overline{B} = \phi$. A space (X, T) is said to be **completely normal** iff given a pair of separated sets, there are disjoint open sets U and V s. t. $A \subset U$, $B \subset V$. A T_1-completely normal space is called a **T_5-space.**

7.25. Show that

i) (X, T) where X={a, b, c}and T={ϕ, x, {a}, {b, c}} is a normal space.

ii) Every metric space is a T_3-space.

Proof: Routine

7.26 Theorem: Complete normality is a topological property:

Proof: Let (X, T) be a completely normal space and let (Y, V) be its homeomorphic image under a homeomorphism f. To show that (Y, V) is also completely normal.

Let A, B be any two separated subsets of Y so that $A\cap \overline{B}=\phi$ and $\overline{A}\cap B=\phi$. Since f is a continuous map, we have

$$\overline{(f^{-1}(A))}\subset f^{-1}(A) \text{ and } \overline{(f^{-1}(B))}\subset f^{-1}(\overline{B}) \;\ldots.(1)$$

Hence, $f^{-1}(A)\cap \overline{(f^{-1}(B))}\subset f^{-1}(A)\cap f^{-1}(\overline{B})= f^{-1}(A\cap \overline{B})=f^{-1}(\phi)=\phi$

and $\overline{f^{-1}(A)}\cap f^{-1}(B)\subset f^{-1}(A)\cap f^{-1}(B)=f^{-1}(\overline{A}\cap B)=f^{-1}(\phi)=\phi$,

Thus, $f^{-1}(A)$ and $f^{-1}(B)$ are two separated subsets of X. Since (X, T) is completely normal, there exists T-open sets G, H such that

$$f^{-1}(A)\subset G,\ f^{-1}(B)\subset H \text{ and } G\cap H=\phi.$$

These relations imply that $A= f[f^{-1}(A)]\subset f(G)$, $B= f[f^{-1}(B)]\subset f(H)$

and $f(G)\cap f(H)=f(G\cap H)=f(\phi)=\phi$

Note that since f is onto, we have

$f(f^{-1}(A))=A$, $f(f^{-1}(B))=B$ and since f is one-one, we have,

$$f(G)\cap f(H)=f(G\cap H)$$

Also since f is an open map, f(G) and f(H) are V-open sets. Thus, we have shown that for any two separated subsets A, B of Y there exist V-open subsets $G_1=f(G)$ and $H_1=f(H)$ such that $A\subset G_1$, $B\subset H_1$ and $G_1\cap H_1=\phi$. It follows that (Y, V) is also a completely normal space. Hence, complete normality is a topological property.

7.27 Theorem: Prove that a topological space (X, T) is normal if and only if for any closed set F and open set G containing F, there exists an open set V such that $F\subset V$ and $\overline{V}\subset G$.

proof: The only if part: Let (X, T) be a normal space and let the closed set F be contained in the open set G, that is $F\subset G$. Then X-G is a closed set such that

$$F\cap (X-G)\subset G\cap (X-G)=\phi$$

Thus X-G and F are disjoint closed subsets of X. Since the space is normal, there exist open sets U and V such that $X-G\subset U$, $F\subset V$ and $U\cap V=\phi$, But $U \cap V = \phi \Rightarrow V \subset X\text{-}U \Rightarrow \overline{V} \subset \overline{X-U} \Rightarrow \overline{V} \subset X\text{-}U$ ....(1)

Also, $X\text{-}G \subset U \Rightarrow X\text{-}(X\text{-}G) \supset X\text{-}U \Rightarrow G \supset X\text{-}U$ ....(2)

From (1) and (2), we get $\overline{V} \subset G$. Thus there exists an open set V such that $F \subset V$ and $\overline{V} \subset G$.

The if part: Let the condition hold. Let L and M be closed subsets of X such that $L \cap M = \phi$.

Now, $L \cap M = \phi \Rightarrow L \subset X\text{-}M$.

Thus, the closed set L is contained in the open set X-M. By hypothesis there exists an open set V such that $L \subset V$ and $\overline{V} \subset X\text{-}M$

Now, $\overline{V} \subset X\text{-}M \Rightarrow X - \overline{V} \supset X\text{-}(X\text{-}M) \Rightarrow X - \overline{V} \supset M$. Also $V \cap (X - \overline{V}) = \phi$

Thus, V and $X - \overline{V}$ are two disjoint open sets such that $L \subset V$ and $M \subset X - \overline{V}$. It follows that the space is normal.

7.28,Theorem: Show that every compact Hausdorff space is normal (T_4).

Proof: Let (X, T) be a compact Hausdorff space, and let A, B be a pair of disjoint closed subsets of X. Since every normal space is regular, for each $x \in A$, there exists T-open sets G_x and H_x such that $x \in G_x$, $B \subset H_x$ and $G_x \cap H_x = \phi$,

Then the collection $C = \{G_x \cap B \mid x \in A\}$ is a T*-open cover of A, where T* denotes the relative topology for A.

Now, (A, T*) is a compact subspace of (X, T) and hence there exists a finite subcover of C, say

$C^* = \{ G_{x_i} \cap A \mid i=1,2,\ldots,n\}$ so that $\cup \{ G_{x_i} \cap A \mid i=1,2,\ldots n \} = A$

Let $G = \cup \{ G_{x_i} \mid i=1,2,\ldots n \}$ and $H = \cap \{ H_{x_i} \mid i=1,2,\ldots n \}$

Then, G, H are T-open sets such that $A \subset G$, $B \subset H$ and $G \cap H = \phi$,

Hence (X, T) is normal. Since every Hausdorff space is a T_1-space, it follows that (X,T) is also a T_4-space.

A topological space is not only rich in open sets, but also rich in continuous functions. The following are some fundamental theorems in this direction.

7.29 Definition: Dyadic Fraction. A real number of the form $t = {}^{m}/_{2^n}$, n=1,2,..., and m=1,2,...,2^n-1, is called a **Dyadic Fraction.** For example, the dyadic fractions in the interval (0,1) are

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \ldots\ldots$$

7.30 Theorem: State and prove Urysohn's lemma.

Urysohn's lemma: A topological space (X, T) is normal iff given a pair of disjoint closed sets A, B $\subset$ X, there is a continuous function f: X$\to$[0,1] such that f(A) ={0} and f(B)={1}.

Proof: We assume first that for given a pair of disjoint closed sets A, B$\subset$ X; there exists a continuous map f:X$\to$[0,1] s. t. f(A)={0}, and f(B)={1}.To prove that (X, T) us a normal space.

Let G= $f^{-1}\left([0,\frac{1}{2}]\right)$and H= $f^{-1}\left([\frac{1}{2},1]\right)$. We shall show that G and H are disjoint open sets in X such that A$\subset$G and B$\subset$ H.

Since $[0,\frac{1}{2})$ and $\left(\frac{1}{2},1\right]$ are U*-open subsets of [0,1] and f is T-U* continuous map, it follows that G and H are T-open subsets of X. Further,

$$f(A)=\{0\}\Rightarrow f^{-1}(\{0\})\supset A \quad \text{and} \quad \{0\}\subset[0,\tfrac{1}{2})\Rightarrow f^{-1}(\{0\})\subset\left(\left(0,\frac{1}{2}\right]\right)\Rightarrow f^{-1}(\{0\})\subset G$$

Hence, we obtain A$\subset$G. Similarly, we can show that B$\subset$H.

$$\text{Also } G\cap H=f^{-1}([0,\tfrac{1}{2}))\cap f^{-1}((1,\tfrac{1}{2}])=\phi.$$

Therefore, (X, T) is a normal space. Conversely, we suppose that (X, T) is a normal space. Now, B′ is a neighborhood of the closed set A, and so by the normality of X, and Theorem 7.27, A has a neighborhood $U_{1/2}$ such that

$$A\subseteq U_{1/2}\subseteq\overline{U_{1/2}}\subseteq B',$$

$U_{1/2}$ and B′ are neighborhoods of the closed sets A and $\overline{U_{1/2}}$, so in the same way there exist open sets $U_{1/4}$ and $U_{3/4}$ such that $A\subseteq U_{1/4}\subseteq\overline{U_{1/4}}\subseteq U_{1/2}\subseteq\overline{U_{1/2}}\subseteq U_{3/4}\subseteq\overline{U_{3/4}}\subseteq B'$.

If we continue this process for each dyadic rational number $t=m/2^n$, we have an open set of the form U_t such that $t_1< t_2\Rightarrow A\subseteq U_{t_1}\subseteq\overline{U}_{t_1}\subseteq U_{t_2}\subseteq\overline{U}_{t_2}\subseteq B'$. We now define our function f by f(x)=0 if x is in every U_t and f(x)=sup{t:x$\notin U_t$}, otherwise. It is clear that the values of f lie in [0,1], and that f(A) =0 and f(B)=1. All that remains to be proved is that f is continuous. All intervals of the form [0,a) and (a,1] where 0<a<1, constitute an open subbase for [0,1]. It, therefore, suffices to show that $f^{-1}([0,a))$ and $f^{-1}((a,1])$ are open. It is easy to see that f(x)<a$\Leftrightarrow$x is in some U_t for t < a; and from this it follows that $f^{-1}([0,a))=\{x:f(x)<a\}=\bigcup_{t<a}U_t$, which is an open set. Similarly f(x)>a$\Leftrightarrow$x is outside of $\overline{U}_t$ for some t >a; and therefore $f^{-1}((a,1])=\{x\mid f(x)>a\}=\bigcup_{t>a}\overline{U}_t{}'$, which is an open set.

7.31 Theorem: State and prove the Tietze Extension Theorem.

Solution: Statement: A topological space (X, T) is normal if and only if for every real valued continuous mapping f of a closed subset F of X into the closed interval [a, b], there exists a real valued continuous mapping g of X in to [a, b] such that g is a continuous extension of f over X.

Proof: The 'if part' We suppose that for every real-valued continuous mapping f of a closed subset F of X into [a, b], there exists a continuous extension of f over X. To show that (X, T) is normal.

Let A, B be two closed subsets of X such that $A \cap B = \phi$ and let [a, b] be any closed interval. We define a mapping.

f: $A \cup B \rightarrow [a, b]$ such that f(x)=a if $x \in A$ and f(x)=b if $x \in B$. This mapping is certainly continuous over the subspace $A \cup B$. For if H be any closed subset of [a, b], then

$$f^{-1}(H) = \begin{cases} A & \text{if } a \in H \text{ and } b \notin H, \\ B & \text{if } b \in H \text{ and } a \notin H, \\ A \cup B & \text{if } a \in H \text{ and } b \in H, \\ \phi & \text{if } a \notin H \text{ and } b \notin H. \end{cases}$$

It follows that $f^{-1}(H)$ is closed in $A \cup B$. Hence f is a continuous map over the subspace $A \cup B$. Therefore, by hypothesis, f can be extended to a continuous map g over X. This means that there exists a continuous map, g: $X \rightarrow [a, b]$ such that g(x)=a if $x \in A$ and g(x)=b if $x \in B$. The mapping g now satisfies the condition stated in Urysohn's lemma, and hence (X, T) is normal.

The 'only if part' :

Suppose X is a normal space, F a closed subspace, and f a continuous real function defined on F whose values lie in the closed interval [a, b]. To prove that f has a continuous extension g defined on all of X whose values also lie in [a, b],

If a = b, then the constant function $g(x)=a, \forall x \in X$ will serve the purpose. Next, assume that a<b. For numerical convenience, we take [a, b] to be [-1,1]. We begin by defining f_o to be f. The domain of f_o is our closed subspace F, and we define two subsets A_o and B_o of F by $A_o=\{x:f_o(x)\leq -\frac{1}{3}\}$ and $B_o=\{x:f_o(x)\geq \frac{1}{3}\}$. A_o and B_o are disjoint, nonempty and closed in F; and since F is closed, they are closed in X. A_o and B_o are thus a disjoint pair of closed subspaces of X, and by the Urysohn lemma there exists a continuous function $g_o:X\rightarrow\left[-\frac{1}{3},\frac{1}{3}\right]$ such that $g_o(A_o)=-\frac{1}{3}$ and $g_o(B_o)=\frac{1}{3}$. We next define f_1 on F by $f_1= f_0-g_o$, and we observe that $|f_1(x)|\leq\frac{2}{3}$. If $A_1=\{x \mid f_1(x)\leq(-\frac{1}{3})\frac{2}{3}\}$ and $B_1=\{x \mid f_1(x)\geq(\frac{1}{3})(\frac{2}{3})\}$, then in the same way as above there exists a continuous function $g_1:X\rightarrow[(-\frac{1}{3})(\frac{2}{3}),\frac{1}{3}(\frac{2}{3})]$ such that $g_1(A_1)=(-\frac{1}{3})(\frac{2}{3})$, and $g_1(B_1)=(\frac{1}{3})(\frac{2}{3})$. We next define f_2 on F by $f_2 = f_1-g_1 = f_0-(g_0+g_1)$

and we observe that $|f_2(x)| \leq \left(\frac{2}{3}\right)^2$. By continuing in this manner, we get a sequence $\{f_0, f_1, f_2 \ldots\}$ defined on F such that $|f_n(x)| \leq \left(\frac{2}{3}\right)^n$ and a sequence $\{g_0, g_1, g_2 \ldots\}$ defined on X such that $|g_n(x)| \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$ with the property that on F we have, $f_n = f_0 - (g_0 + g_1 + g_2 + \ldots + g_{n-1})$.

We now define s_n by $s_n = g_0 + g_1 + g_2 + \ldots + g_{n-1}$ and we regard the s_n's as the partial sums of an infinite series of functions in C(X, R). We know that C(X, R) is complete, so by $|g_n(x)| \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$ and the fact that $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n = 1$, we see that s_n converges uniformly on X to a bounded continuous real function g such that $| g(x) | \leq 1$. We conclude our proof by noting that since $| f_n(x)| \leq (2/3)^n$, s_n converges on F to $f_0 = f$, and that therefore g equals f on F and is a continuous extension of f to the full space X which has the desired property.

7.32 Remark: The above theorem is false if we omit the assumption that the subspace F is closed. For example, let X be the closed unit interval [0,1], F the subspace (0,1] and f the function defined on F by $f(x)=\sin\left(\frac{1}{x}\right)$. X is clearly normal, F is not closed as a subspace of X, and f cannot be extended continuously to X in any manner whatsoever.

Problem Set 7

Ex 1: prove that $T_5 \Rightarrow T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. Furthermore, construct counter examples to show that the converse route of the above implication is not true in general.

Ex 2: Prove that the T_i-spaces i=1,2,3,4,5 have both hereditary and topological properties.

Ex 3: Construct a bijective continuous map that is not a homeomorphism.

Ex 4: If f is a continuous mapping if a topological space X into a Hausdorff space Y, prove that the graph of f is a closed subset of the product $X \times Y$.

Ex 5: Show that if $f: X \rightarrow Y$ is a continuous mapping of one topological space into another, then $x_n \rightarrow x$ in $X \Rightarrow f(x_n) \rightarrow f(x)$ in Y. prove that the converse of this is true if each point in X has a countable open base.

Ex 6: If T is the cofinite (cocountable) topology on an uncountable set X, prove that (X, T) is a T_1-space but **not** a T_2-space.

Ex 7: Show that a topological space (X, T) is a T_1-space iff T contains the cofinite topology on X.

Ex 8: Prove that a closed subspace of a normal space is a normal space. Show further with a counter example that an open subspace of a normal space may not be normal.

Ex 9: Prove that a regular second countable space is normal.

Ex 10: Prove Urysohn's lemma by using the Tietze's Extension theorem.

Ex 11: State and prove a generalization of Tietze's theorem which relates to functions whose values lie in 3^n.

Ex 12: Let X be a normal space, and let A and B be distinct closed subspaces of X. If [a, b] is any closed interval on the real line, then there exists a continuous real function f defined on X, all of whose values lie in [a, b], such that f(A)={a}. and f(B)={b}.

UNIT 5

CONNECTEDNESS

8.0 Introduction:

The notion of 'connectedness' is fundamental in higher analysis, geometry and topology and is originally derived from the property of the intermediate value theorem in calculus. The modern notion of connectedness was proposed by Jordan and Schoenfliesz in 1893 and put on firm footing by Riesz. The definition of connectedness for a topological space is a quite natural one. One says that a space can be "disconnected" if it can be broken up into two "globs"- disjoint open sets. Otherwise, one says that it is connected. From this simple idea much follows:

8.01 Separated Set: Two subsets A and B of a topological space (X, T) are said to be **separated** if $A \neq \phi$, $B \neq \phi$, $A \cap \overline{B} = \phi$ and $\overline{A} \cap B = \phi$. In other words, we can also say that the nonempty sets A and B are separated if $A \cap D(B) = \phi$, $D(A) \cap B = \phi$.

8.02 Disconnectedness: Let (X, T) be a topological space and $A \subset X$. The set A is said to be **disconnect subset** of X if $\exists$ G, $H \in T$ such that $A \cap G \neq \phi$, $A \cap H \neq \phi$, $(A \cap G) \cap (A \cap H) = \phi$ and $A = (A \cap G) \cup (A \cap H)$. In this case, $G \cup H$ is called a **disconnection** of A. The set A is **connected** if it is not disconnected.

A topological space (X, T) is said to be **disconnected** if $\exists G, H \in T$ such that $G \neq \phi$, $H \neq \phi$, $G \cap H = \phi$ and $X = G \cup H$. X is said to be **connected** if it is not disconnected.

8.03 Examples:

Every indiscrete space is connected,

Every discrete space is disconnected if the space contains more than one point.

If X={a, b, c, d,} and T={ϕ, X, {a}, {a, b},{a,b,c}}. Then the topological space (X, T)is

connected.

8.04 Arcwise connected: Let (X, T)be a topological space. Let I be the set of all real numbers belonging to the closed interval [0,1] with usual topology.Let a, $b \in X$ be arbitrary.

A continuous map $f: I \rightarrow X$ with the properties f(0) = a, f(1) = b is called a **path** from the point a to the point b. In this case, a is called the **initial point** and b is called the **terminal or final** point of the path. Let (X, T) be a topological space and $A \subset X$. A is called **arcwise connected** if for any points a, $b \in X$, there is a path f: $I \rightarrow X$, from a to b such that $f(I) \subset A$.

8.05 Theorem: Prove that X is disconnected if and only if $\exists$ a nonempty proper subset of X which is both open and closed.

Proof: Let (X, T) be a disconnected topological space so that $\exists$ a disconnection $G \cup H$ of X. Then, G, $H \in T$ such that $G \neq \phi$, $H \neq \phi$, $G \cap H = \phi$, $X = G \cup H$. To prove that $\exists$ a nonempty proper subset of X which is both open and closed.

$G \cap H = \phi$, $H \in T \Rightarrow G = X-H$, X-H is closed $\Rightarrow$ G is closed.

$X = G \cup H$, $G \neq \phi \neq H \Rightarrow G \subset X$, $G \neq X$ $\Rightarrow$ G is a nonempty proper subset of X.

Finally, we have shown that $\exists$ a nonempty proper subset G of X which is both open and closed.

Conversely, let A be a nonempty proper subset of a topological space (X, T) such that A is both open and closed in X. To prove that X is disconnected .

$A \neq \phi$, A is closed $\Rightarrow X-A \neq \phi$, X-A is open

$\Rightarrow B \neq \phi$, B is open, where $B = X-A$.

$X-A = B \Rightarrow A \cup B = X$, $A \cap B = \phi$.

Thus, the sets A and B are nonempty, disjoint subsets of X which are also open in X such that $X = A \cup B$. Consequently, $A \cup B$ is a disconnection of X. So, X is disconnected.

Remark 8.06: From above we can conclude that X is connected if and only if X has no proper subset which is both open and closed in X. This means that ϕ and X are the only subsets of X which are both open and closed in X.

8.07 Theorem: Prove that if f: $X \rightarrow Y$ is a continuous map from a connected space X onto a topological space Y, then Y is also connected.

If f is continuous, Y is connected, does this imply that X must be connected?

Proof: First part:

Assume that Y is disconnected. Then $\exists$ two open sets G and H such that $Y = G \cup H$ and $G \cap H = \phi$. Then f(G) and f(H) are two open subsets of X such that $X = f(G) \cup f(H)$ and $F(G) \cap f(H) = \phi$. This asserts that X is disconnected, a contradiction. Hence Y is also a connected space.

Second part:

The result is not necessarily true. For example, let

$X = \{1,2,3\}$, $T = \{\phi, X, \{1\}, \{2,3\}\}$,

$Y = \{4\}$, $D = \{\phi, Y\}$.

Let f: $(X, T) \rightarrow (Y, D)$ be a map such that $f(x) = 4$ $\forall x \in X$. Then as a constant map f is continuous, and further obviously Y is connected. However, (X, T) is not connected.

Theorem 8.08: Let (X, T) be a topological space and let E be a connected subset of X such that $E \subset A \cup B$ where A and B are separated sets. Then $E \subset A$ or $E \subset B$.

Proof: The proof is every easy and so left for the readers.

Theorem 8.09: Let (X, T) be a topological space and let E be a connected subset of X. If F is a subset of X such that $E \subset F \subset \overline{E}$, then F is connected. In particular, $\overline{E}$ is connected.

Proof: Suppose if possible, F is disconnected. Then there exists nonempty separated sets A and B such that $F = A \cup B$. Since $E \subset F$, we have $E \subset A \cup B$.

Since the connected set E is contained in the union of two separated sets A and B, it follows from the preceding theorem that $E \subset A$ or $E \subset B$. Let $E \subset A$.

Now, $E \subset A \Rightarrow \overline{E} \subset A \Rightarrow \overline{E} \cap B \subset A \cap B = \phi$

Since ϕ is a subset of every set, we have $\overline{E} \cap B = \phi$

Again $F = A \cup B$ and $F \subset \overline{E} \Rightarrow B \subset F \subset \overline{E}$. Hence $\overline{E} \cap B = B$

The above results imply that $B = \phi$, which is a contradiction, since B is nonempty. Therefore, F is connected.

Since $E \subseteq \overline{E} \subseteq \overline{E}$, it follows that $\overline{E}$ is connected.

Theorem 8.10: If every two points of a subset E of a space X are contained in some connected subset of E, then E is a connected set.

Proof: Straightforward.

Theorem 8.11: Let $\{C_\lambda \mid \lambda \in \Delta, \Delta$ an index set$\}$ be a nonempty collection of connected subsets of X such that $\cap\{C_\lambda \mid \lambda \in \Delta\} \neq \phi$. Then $\cup\{C_\lambda \mid \lambda \in \Delta\}$ is a connected set.

Proof: Let $E = \cup\{C_\lambda \mid \lambda \in \Delta\}$, and suppose E is not connected. Then the definition of disconnectedness is equivalent to the existence of two nonempty sets A and B such that

$$E = A \cup B, A \cap \ \ = \phi, A \cap B = \phi.$$

By hypothesis, there exists some point

$$p \in \cap\{C_\lambda \mid \lambda \in \Delta\}$$

Then, p must belong to E. Since $E = A \cup B$, either $p \in A$ or $p \in B$. Without loss of generality, we may suppose that $p \in A$. Since $p \in C_\lambda$, $\forall \lambda \in \Delta$, we have $C_\lambda \cap A \neq \phi$, $\forall \lambda \in \Delta$. Now each C_λ is a connected subset of X, so that we have.

$$C_\lambda \subset A \text{ or } C_\lambda \subset B.$$

Since A and B are disjoint sets and $C_\lambda \cap A \neq \phi$, $\forall \lambda \in \Delta$, we must have $C_\lambda \subset A$ $\forall \lambda \in \Delta$ and consequently $E \subset A$, that is, $A \cup B \subset A$ which yields that $B = \phi$. But this is a contradiction since B is nonempty. Hence E must be connected.

Connectedness on the real line.

Theorem 8.12: Prove that a subset E (containing at least two points) of the real line 3 is connected iff it is an interval. In particular, R is connected.

Proof: Let E, be connected, and suppose, if possible, E is not an interval. Then there exist real numbers a, p, b with $a < p < b$ such that $a, b \in E$ but $p \notin E$.

Let $G = (-\infty, p)$, $H = (p, \infty)$. Then $a \in G$ and $b \in H$ so that G, H are nonempty disjoint open sets. Let $A = E \cap G$, and $B = E \cap H$. Then A, B are nonempty sets since $a \in A$ and $b \in B$.

$$\text{Also, } A \cap B = (E \cap G) \cap (E \cap H) = E \cap (G \cap H) = E \cap \phi = \phi$$

$$\text{And } A \cup B = (E \cap G) \cup (E \cap H) = E \cap (G \cup H) = E \cap [(-\infty, p) \cup (p, \infty)] = E \cap (R - \{p\})$$

$$= E \ (\because p \notin E, \text{ we have } E \subset R - \{p\})$$

Thus, $G \cup H$ is a disconnection of E which contradicts the hypothesis that E is connected. Hence E must be an interval.

Conversely, let E be an interval. Then, we have to show that E is connected. Assume the E is disconnected and let $G \cup H$ be a disconnection of E so that G, H are closed sets.

Let $A = E \cap G$ and $B = E \cap H$. Then A, B are nonempty disjoint sets where union is E that is, $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$, $A \cup B = E$. Since A, B are nonempty, we may choose a point $a \in A$ and a point $b \in B$. Now, $a \neq b$ since $A \cap B \neq \phi$. Thus either $a < b$ or $a > b$. Without loss of generality, we may assume $a < b$. Since E is an interval, we have $[a, b] \subset E$ and since $E = A \cup B$, each point of $[a, b]$ is either in A or in B.

Let $u = \sup([a, b] \cap A)$,

Evidently, $a \leq u \leq b$ so that $u \in E$. Since a is closed in E, the definition of u shows that $u \in A$. This give us the strict inequality $u < b$.

Moreover, the definition of u shows that $u + \varepsilon$ belongs to B for $\varepsilon > 0$ such that $u + \varepsilon \leq b$. Since B is closed in E, $u \in B$. Thus, we have shown that u belongs to both A and B which is a contradiction since A, B are disjoint.

Hence E is connected.

8.13 Problem: The range of a continuous real-valued map defined on a connected space is an interval.

Solution: Let f be a continuous real valued map defined on a connected set A.

To prove that f(A) is an interval. Our assumption implies that

(i) f: $A \to R$, (ii) f is continuous, (iii) A is connected

(i) and (ii) $\Rightarrow$ f(A) is connected. $f(A) \subset R$ is connected $\Rightarrow$ f(A) is an interval.

8.14 Theorem: Continuous image of an arcwise connected set is arcwise connected.

Solution: Let f: $(X, T) \to (Y, V)$ be a comtinuous map and let $A \subset X$ be arcwise connected.

To prove that f(A) is arcwise connected.

Let $p, q \in f(A)$ be arbitrary, then $\exists\ p_1, q_1 \in A$ s. t. $f(p_1)=p$, $f(q_1)=q$.

Let I=[0,1]. A is arcwise connected implies that $\exists$ a path $g: I \to X$ from p_1 to q_1 such that $g(I) \subset A$. It means that $g(0)=p_1$, $g(1)=q$,

g is a continuous map.

g: $I \to X$ and f: $X \to Y$ are continuous maps and hence so is the map f o g: $I \to Y$.

$$(f \circ g)(o)=f(g(0))=f(p_1)= p$$

$$(f \circ g)(1)=f(g(1))=f(q_1)=q$$

$$(f \circ g)(1)=f(g(I)) \subset f(A)$$

Finally, f o g: $I \to Y$ is a continuous map such that for any arbitrary point $p, q \in f(A)$, we have

$(f \circ g)(0) = p$, $(f \circ g)(1) = q$, and $(fog)(I) \subset f(A)$. This proves that f(A) is arcwise connected.

8.15 Theorem: Prove that arcwise connectedness is a topological property.

Solution: Suppose $f:(X, T) \to (Y, U)$ is a homeomorphism and X is arcwise connected, the result will follow.

f is a homeomorphism $\Rightarrow$ f is one-one onto and f is continuous. Then, by theorem 8.07, the required result follows:

The components of a Space

If a space is not itself connected, then the next best thing is to be able to decompose it into a disjoint clan of maximal connected subspaces. Our present objective is to show that this can always be done.

8.16 Definition: A maximal connected subspace C of a topological space X is called a **component** of the space.

Thus, C is a component of X iff C is connected and C is not properly contained in any larger connected subspace. A component of a subset Y of X is a component of Y with its relative topology, since singlton sets of a topological space are connected, it is evident that components are nonempty.

8.17 Example:

(i) Let (X, T) be connected. Then X has only one component, namely X itself.

(ii) Let (X, T) be a discrete space. Then singleton sets are the only connected subsets of X and consequently they are the maximal connected sets. Hence, each singleton set is a component for a discrete space.

8.18 Problem: Show that every component of a topological space (X, T) is closed. Is it also necessarily open? Justify your answer.

Solution: First part:

Let C be a component of X. Since C is connected, its closure $\overline{C}$ is also connected. Now C, being a component, is a maximal connected set. Hence, $\overline{C} \subset C$. Also, we always have $C \subset \overline{C}$. Therefore, $C = \overline{C}$, If follows that C is closed.

Second part:

A component may not be open. This can be illustrated with the following example.

Let X be the subspace of the real line which consists of all rational numbers. We observe two facts about X. First if x and z are any two distinct rationals and of $x<z$, then there exists an irrational y such that $x<y<z$, and therefore

$X=[X \cap (-\infty, y)] \cup [X \cap (y, +\infty)]$ is a disconnection of X which separates x and z. We see from this that any subspace of X with more than one point is disconnected, so the components of X are its points. Second, the points of X are not open, for any open subset of R which contains a given rational number also contains others different form it. Here, then, is a space whose components are its points and whose points are not open. This example also shows that a space need not be discrete in order that each of its points be a component.

Theorem 8.19: Prove the following for any arbitrary topological space (X, T).

Each points in X is contained in exactly one component of X

Each connected subspace of X is contained in a component of X; and

A connected subspace of X which is both open and closed is a component of X

Proof: (1) Let x be any point in X. Consider the class $\{C_i\}$of all connected subspaces of X which contain x. This class is nonempty, since, x itself is connected. Now $C = \bigcup_i C_i$ is a connected subspace of X which contains x.C is clearly maximal, and therefore a component of X, because any connected subspace of X which contain C is one of the C_i's and is thus contained is C. Finally, C is the only component of X which contains x. For if C^* is another, it is clearly among the C_i's and is therefore contained in C and since C^* is maximal as a connected subspace of X, we must have $C^*=C$.

(2) Let C be any connected subset of X. If $\dot{C} = \phi$, then C is contained in every component. If $C \neq \phi$, then C contais a point $x_0 \in X$. Then by the construction in (1), $\exists$ a component C_{x_0} such that $x_0 \in C_{x_0}$ and C_{x_0} is the union of all connected subsets of X containing x_0. Hence, $c \subset C_{x_0}$.

(3) Let A be a connected subset of X which is both open and closed. By (2), A is contained in some component C. If A is a proper subset of C, then it is easy to see that

$C = (C \cap A) \cup (C \cap A')$ is a disconnection of C. This contradicts the fact that C, being a component, is connected, and we conclude that A = C.

Totally Disconnected Space.

8.20 Definition: A totally disconnected space is a topological space X in which every pair of distinct points can be separated by a disconnection of X. This means that for every pair of points x and y in X such that $x \neq y$, there exists a disconnection $X = A \cup B$ with $x \in A$ and $y \in B$

8.21 Example: (i) show that every discrete space is totally disconnected.

Solution: Let, (X, D) be any discrete space and let x, y be any two distinet points of X. Then $A = \{x\}$ and $B = X - \{x\}$, are both nonempty open disjoints sets whose union is X such that $x \in A$ and $y \in B$. Hence $A \cup B$ is a disconnection of X with $x \in A$ and $y \in B$. It follows that (X, D) is totally disconnected.

(ii) Show that (a) the set of all rational numbers, (b) the set of all irrational numbers, (c) the Cantor set, are all totally disconnected spaces.

Proofs are simple, and so left for the readers,

8.22 Theorem: The components of a totally disconnected space are its points.

Proof: IF X is a totally disconnected space, it suffices to show that every subspace Y of X which contains more than one point is disconnected. Let x and y be distinct points in Y, and let $X = A \cup B$ be a disconnection of X with $x \in A$ and $y \in B$. If is obvious that

$Y = (y \cap A) \cup (y \cap B)$ is a disconnection of y. Hence, the required result.

8.23 Theorem: Let X be a Hausdorff space. If X has an open base whose sets are also closed, then X is totally disconnected.

Proof: Let x and y be distinct points in X. since X is Hausdorff, x has a neighborhood G which does not contain y. By our assumption, there exists a basic open set B which is also closed such that $x \in B \subseteq G$.

$X = B \cup B^c$ is clearly a disconnection of X which separates x and y

8.24 Theorem: Let x be a compact Hausdorff space. Then X is totally disconnected iff it has an open base whose sets are also closed.

Proof: The 'if part' follows from the theorem 8.23.

To prove the 'only if part' we assume that X is totally disconnected. We want to show that the class of all subsets of X which are both open and closed forms an open base.

Let x be a point and G an open set which contains it. We must produce a set B which is both open and closed such that $x \in B \subseteq G$. We may assume that G is not the full space, for of G=X then we can satisfy our requirement by taking B=X. G' is thus a closed subspace of X, and since X is compact, G' is also compact. By the assumption that X is totally disconnected, for each point y in G' there exist a set H_y which is both open and closed and contains y but not x. G' is compact, so ther exists some finite class of H_y's which we denote by $\{H_1, H_2, H_3 \ldots\ldots H_n\}$, with the property that its union contains G' but not x. We define H by $H = \bigcup_{i=1}^{n} H_i$, and we observe that since this is a finite union and all the H_y's are closed as well as open, H is both open and closed, it contains G', and it does not contain x. If we now define B to be H', then B clearly has the properties required of it.

Locally Connected Space:

8.25 Definition: A **locally connected** space is a topological space with the property tht if x is any point in it and G any neighborhood of x,then G contains a connected neighborhood of x. This is evidently equivalent to the condition that each point of the space have an open base whose sets are all connected subspaces.

8.26 Example: Show that every discrete space (X, D) is locally connected.

Solution: Let x be any arbitrary point of X. Then $\{x\}$is a connected neighborhood of X. Also every neighborhood of x must contain $\{x\}$. Hence (X, D) is locally connected.

8.27 Theorem: Every component of a locally connected space is both closed and open.

Proof: We have already verified that a component of any space is closed (theorem 8.18)

Next, to show that a component C of a locally connected space X is open, let $a \in C$. Since X is a locally connected, a must belong to at least one connected open set G_a. Since C is a component, we must have

$$a \in G_a \subset C$$

It follows that $C = \cup\{G_a \mid a \in C\}$. Hence C is open being a union of open sets.

8.28 Show that a space X is locally connected if and only if for every open set U of X, each component of U is open in X.

Proof: Suppose that X is locally connected; let U be on open set in X; let C be a component of U. If x is a point of C, we can choose a connected nbhd V of X such that $V \subset U$. Since V is connected, it must lie entirely in the component C of U. Therefore, C is open in X.

Conversely, suppose that component of open sets in X are open. Given a point x of X and a neighbourhood U of x. Let C be the component of U containing x. Now C is connected; since it is open in X by hypothesis, X is locally connected at x.

8.29 Problem:

Give an example of a locally connected space which is not connected.

Give an example of a connected space which is **not** locally connected.

Proof: Consider the real line R with the discrete topology. Let $x \in R$. Then the singtlon set $\{x\}$ is both open and connected. If G is any open set containing x, then $\exists$ a connected open set $C=\{x\}$such that $x \in C \subseteq G$. Since x is arbitrary, it follows that R is locally connected at each point $x \in R$, and therefore (R,D) is locally connected.

Since the singleton set $\{x\}$is a nonempty proper subset of R which is both open and closed, it follows that (R,D) is not connected. Alternatively if $A=(1,2)\cup(3,5)$ is the union of two disjoint open intervals, then A is locally connected, but not connected.

Let X be the subspace of the Euclidean place defined by $X=A\cup B$, where

$A=\{(x, y) \mid x=0 \text{ and } -1\leq y \leq 1\}$ and $B=\{(x, y) \mid 0<x\leq 1 \text{ and } y = \sin(\frac{1}{x})\}$. Now, B is the image of the interval (0,1] under the continuous mapping f defined by $f(x)=(x, \sin(\frac{1}{x}))$, so B is connected (a continuous image of a connected space is connected). If (0,a) is any point in A, then $\exists$ a sequence.

$$\left(\frac{1}{2n\pi+\sin^{-1}a}, \sin\frac{1}{x_n}\right), \text{ where } x_n=\frac{1}{2n\pi+\sin^{-1}a} \text{ in B such that}$$

$\left(x_n, \sin\frac{1}{x_n}\right)\to(0,a)$. Hence every points of A is a limit point of B, and so $\overline{B}=X$.

Since B is connected, $\overline{B}$ is also connected. Thus, X is a connected space.

Next consider the point $(0,\frac{1}{2})$ in A and consider an open sphere $S_{\frac{1}{4}}(0,\frac{1}{2})$ with center $(0,\frac{1}{2})$ and radius $\frac{1}{4}$. Then this open sphere contains some line segments of B, but the union of all those line segments is **not** a connected open set containing $(0,\frac{1}{2})$. Thus, the open sphere does not contain any connected open subset of X. Hence X is not locally connected at $(0,\frac{1}{2})$ and therefore X is not locally connected.

Problem set 8

Example 1. Can we have a topological space which is both

Connected and disconnected,

Connected and totally disconnected,

Connected and locally connected,

Locally Connected and totally disconnected.

Example 2. Consider the real line R with the usual topology. Is each of the following subsets of 3 connected? Justify your answer.

(i) (0,5) (ii) $(-7,2)\cup(0,3)$ (iii) $(-11,-2)\cup(2,11)$ (iv) A = {1, ½, 1/3, ...} (v) ∠, (vi) Q (vii) The Cantor set.

Example 3. Let X be a connected space. Can we construct a continuous map form X onto (1,3)?

Example 4. Prove that a topological space X is disconnected ⇔ there exists a continuous mapping of X onto the discrete topological space {0,1}.

Example 5. Show that the spaces R^n and C^n are connected.

Example 6. Show that a topological space is connected ⇔every nonempty proper subset has a nonempty boundary.

Example 7. Show that the graph of a continuous real function defined on an interval is a connected subspace of the Euclidean plane.

Example 8. Prove that the component of a topological space X form a partition of X, that is, any two component are either disjoint or identical and the union of all the components is X.

Example 9. Prove that the product space X×Y . Is connected if X and Y are connected, Is totally disconnected if X and Y are totally disconnected, Is totally connected if X and Y are locally connected.

Example 10. Show that an open subspace of the complex plane is connected ⇔ every two points in it can be joined by a polygonal line.

Example 11. Prove that the deleted comb space

$C=([0,1]\times 0)\cup(K\times[0,1])$, where $K=\{\frac{1}{n} \mid n\in Z_+\}$, is connected but not locally connected.

Example 12. If f: X→Y is continuous and X is locally connected, is f(X) necessarily locally connected? What if f is both continuous and open?

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