M201

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Paper I
Complex Analysis

Contents:

Course Co-Ordination

Prof. Amit Choudhury	Director, GU, IDOL
Dr. Chandra Rekha Mahanta	Associate Professor, Dept. of Mathematics Gauhati University

Contributors:

Prof. Rita Choudhury	Dept. of Mathematics Gauhati University

Editorial Team

Prof. Kuntala Patra	Dept. of Mathematics Gauhati University
Prof. Rita Choudhury	Dept. of Mathematics Gauhati University
Dipankar Saikia	Editor (SLM) G.U, IDOL

Cover Page Designing:

Kaushik Sarma	:	Graphic Designer CET, IITG

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UNIT 1

ANALYTIC FUNCTIONS

Introduction:

In this unit we shall consider the notion of differentiability applicable to the complex valued functions of a complex variable. Here following Cauchy (1789-1857), the notion of Analytic functions will be given in terms of differentiability.

Definition 1.1 If a function f(z) be defined and single-valued function in a neighbourhood of $z = z_0$ with the possible exception of $z = z_0$ itself, then we say that the number l is the limit of f(z) as z approaches to z_0 and write $\lim_{z \to z_0} f(z) = l$ if for every positive number ϵ (however small) we can find some positive number δ (which depends on ϵ) such that

$$|f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Definition 1.2 If f(z) be defined and single valued function in a neighbourhood of $z = z_0$ as well as at $z = z_0$, then this function f(z) is said to be continuous at $z = z_0$ and we write

$$\lim_{z \to z_0} f(z) = f(z_0)$$

A function f(z) is said to be continuous in a region if it is continuous at all points of the region.

A function f(z) is said to be uniformly continuous in a region if for $\varepsilon > 0$ we have $\delta > 0$ such that

$|f(z_1) - f(z_2)| < \varepsilon$ whenever $0 < |z_1 - z_2| < \delta$ where z_1 and z_2 are any two points of the region.

Definition 1.3 If a function f(z) is defined and single valued in some region R of the z-plane, then the derivative of f(z) at $z = z_0$ is denoted by $f'(z_0)$ and is defined as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (1.1)$$

provided that the limit exists. If we consider $\Delta z = z - z_0$ then we can write (1.1) as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad (1.2)$$

We often drop the subscript on z_0 and introduce the number $\Delta\omega = f(z + \Delta z) - f(z)$. Thus if we write $\frac{dw}{dz}$ for $f'(z)$, equation (1.2) becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Although differentiability implies continuity , but the converse is not true.

Definition 1.4 If the derivative $f'(z)$ exists at all points z of a region R, then f(z) is said to be analytic in R and is referred to as an analytic function in R or a function analytic in R. The terms regular and holomorphic are sometimes used as synonyms for analytic.

A function f(z) is said to be analytic at a point z_0 if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists.

Theorem 1.1. The necessary and sufficient condition that a function $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region R is that the Cauchy -Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ are satisfied in R where it is supposed that these partial derivatives are continuous in R.

Proof: Necessity. In order for f(z) to be analytic , the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x+\Delta x, y+\Delta y) + i\, v(x+\Delta x, y+\Delta y)\} - \{u(x,y) + i\, v(x,y)\}}{\Delta x + i\Delta y} \quad (1.3)$$

must exist independent of the manner in which Δz (or , Δx and Δy) approaches zero. We consider two possible approaches.

Case 1 $\Delta y = 0, \Delta x \to 0$. .In this case (1.3) becomes

$$\lim_{\Delta x \to 0} \{\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i(\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x})\}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$ provided the partial derivatives exist.

Case 2 $\Delta x = 0, \Delta y \to 0$. .In this case (1.3) becomes

$$\lim_{\Delta y \to 0} \{\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}\}$$

$$= \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now , f(z) cannot possibly be analytic unless these two limits are identical . Thus a necessary condition that f(z) be analytic is

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Comparing real and imaginary parts we have

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x}=-\frac{\partial v}{\partial y} \qquad (1.4)$$

b) Sufficiency: Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed continuous, we have

$$\begin{aligned}\Delta u &= u(x+\Delta x, y+\Delta y)-u(x,y)\\ &=\{u(x+\Delta x,y+\Delta y)-u(x,y+\Delta y)\}+\{u(x,y+\Delta y)-u(x,y)\}\\ &=\left(\frac{\partial u}{\partial x}+\epsilon_1\right)\Delta x+\left(\frac{\partial u}{\partial y}+\eta_1\right)\Delta y=\frac{\partial u}{\partial x}\Delta x+\frac{\partial u}{\partial y}\Delta y+\epsilon_1\Delta x+\eta_1\Delta y\end{aligned}$$

where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x\to 0$ and $\Delta y \to 0$.

Similarly , since $\frac{\partial v}{\partial x'}$ and $\frac{\partial v}{\partial y}$ are supposed continuous, we have

$$\Delta v=\left(\frac{\partial v}{\partial x}+\varepsilon_2\right)\Delta x+\left(\frac{\partial u}{\partial y}+\eta_2\right)\Delta y=\frac{\partial v}{\partial x}\Delta x+\frac{\partial v}{\partial y}\Delta y+\varepsilon_2\Delta x+\eta_2\Delta y$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x\to 0$ and $\Delta y\to 0$. Then

$$\begin{aligned}\Delta w &= \Delta u+i\Delta v\\ &=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\Delta y+\varepsilon\Delta x+\eta\Delta y \qquad \cdots\cdots\cdots(1.5).\end{aligned}$$

where $\varepsilon=\varepsilon_1+i\varepsilon_2 \to 0$ *and* $\eta=\eta_1+i\eta_2\to 0$ *as* $\Delta x\to 0$ *and* $\Delta y \to 0$

By the Cauchy-Riemann equations (1.5) can be written as

$$\begin{aligned}\Delta w &=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial v}{\partial x}\right)\Delta y+\varepsilon\Delta x+\eta\Delta y\\ &=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\Delta x+i\Delta y)+\varepsilon\Delta x+\eta\Delta y\end{aligned}$$

Then on dividing by $\Delta z = \Delta x + i\,\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique , i.e f(z) is analytic in R.

Definition 1.5 If $f(z) = u(x,y) + iv(x,y)$ is an analytic function , then the real functions u and v of the two real variables x and y are called conjugate functions.

Definition 1.6 If the second partial derivatives of u and v with respect to x and y exist and are continuous in a region R, then we find from (1.4) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 , \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

It follows that under these conditions the real and imaginary parts of an analytic function satisfy Laplace's equation denoted by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ \text{ or } \ \nabla^2 \phi = 0, \text{ where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The operator ∇^2 is often called the Laplacian. The functions u(x,y) and v(x,y) which satisfy Laplace's equation in a region R are called harmonic functions and are said to be harmonic in R.

Definition 1.7 Elementary functions of a complex variable.

Consider the series $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, where the coefficients a_n and z, z_0 may be complex . Since the latter series may be obtained from the former by a simple change of origin , the former may be regarded as a typical power series.

So far as absolute convergence is concerned , everything that has been proved for absolutely convergent series of real terms extends at once to complex series ,for the series of moduli

$|a_0| + |a_1||z| + |a_2||z|^2 + \text{------}$ is a series of positive terms . The most useful convergence test for power series is Cauchy's root test , which states that a series of positive terms $\sum u_n$ is convergent or divergent according as $\lim (u_n)^{1/n}$ is less than or greater than unity. If we write $\lim |u_n|^{1/n} = 1/R$, then we easily see that the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent if $|z| < R$, divergent if $|z| > R$, and if $|z| = R$ we can give no general verdict and the behaviour of the series may be of the most diverse nature. The number R is called the radius of convergence and the circle , center the origin , and radius R, is called the circle of convergence of the power series. Clearly there are three cases to consider (i) R=0 ,

(ii) R is finite , (iii)R infinite . The first case is trivial , since the series is then convergent only when z= 0 . In the third case the series converges for all values of z. In the second case the radius of the circle of convergence is finite and the power series is absolutely convergent at all points within the circle, and divergent at all points outside it.

We now consider briefly the definitions of the so called elementary functions of a complex variable.

I. Rational functions

A polynomials in z, $a_0 + a_1z + \text{-------} + a_m z^m$, may be regarded as a power series which converges for all values of z. Since such functions are analytic in the whole plane , rational functions of the type

$$f(z) = \frac{a_0 + a_1 z + \text{------} + a_m z^m}{b_0 + b_1 z + \text{------} + b_m z^m}$$

are analytic at all points of the plane at which the denominator does not vanish. If we choose a point z_0, at which the denominator does not vanish , and replace z by $z_0 + (z - z_0)$, the function f(z) becomes

$$\frac{A_0 + A_1(z - z_0) + \cdots + A_m(z - z_0)^m}{B_0 + B_1(z - z_0) + \cdots + B_k(z - z_0)^k},$$

in which $B_0 \neq 0$. It readily follows that f(z) may be expanded in a power series of the form $\sum_0^\infty C_n(z - z_0)^n$

II. The exponential function

The exponential function of z can be defined by exp z as the sum-function of the series of complex terms

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots$$

Since the series converges for all values of z, it defines a function analytic in the whole z-plane. Such functions are called integral function.

III. The trigonometric and hyperbolic functions.

We define sin z and cos z , when z is complex, as the sum-functions of power series as

$$\sin Z = \sum_{n=0}^{\infty} (-1)^n \frac{Z^{2n+1}}{(2n+1)!} \quad , \quad \cos Z = \sum_{n=0}^{\infty} (-1)^n \frac{Z^{2n}}{(2n)!} ;$$

and, since each of these power series has an infinite radius of convergence, sin z and cos z are integral functions.

The other trigonometrical functions are then defined by

$$\tan z = \frac{\sin z}{\cos z}\ ,\ \cot z = \frac{\cos z}{\sin z}\ ,\ \sec z = \frac{1}{\cos z}\ ,\ \operatorname{cosec} z = \frac{1}{\sin z}$$

If we denote exp.iz by e^{iz}, then we obtain the results

$$\cos z + i\sin z = e^{iz}\ ,\cos z - i\sin z = e^{-iz}\ ;$$

leading to Euler's formulae

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right),\ \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

The hyperbolic functions of a complex variable are also defined as

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right)\ ,\ \cosh z = \frac{1}{2i}\left(e^{z} + e^{-z}\right)$$

There two functions are clearly analytic in any bounded domain.

The important relations

$$\sin(iz) = i\sinh(z)\ ,\ \cos(iz) = \cosh(z)$$

$$\sinh(iz) = i\sin(z)\ ,\ \cosh(iz) = \cos(z)$$

are of great usefulness for deducing properties of the hyperbolic functions from the corresponding properties of the trigonometrical functions.

IV. The logarithmic function.

If z is complex, however, but no zero, the corresponding equation exp w = z has an infinite number of solutions, each of which is called a <u>logarithm of z</u>. If w = u + iv we have

$$e^{u}\ (\cos v + i\sin v\ ) = z.$$

Hence we that v is one of the values of arg z and $e^{u} = |z|$. Hence $u = \log|z|$. Every solution of exp(w)=z is thus of the form

$$w = \log|z| + i\arg z.$$

Since arg z has an infinite number of values, there is an infinite of logarithms of complex number z, each pair differing by $2\pi i$. We write

$$\text{Log} z = \log|z| + i\arg z ,$$

so that Log z is an infinitely many valued function of z.

The principal value of Log z , which is obtained by giving arg z its principal value, will be denoted by log z , since it is identical with the ordinary logarithm when z is real and positive.

V. The general power ζ^z.

If z and ζ denote any complex numbers we define the <u>principal value</u> of the power ζ^z, with $\zeta \neq 0$ as the only condition, to be the number uniquely determined by the equation

$$\zeta^z = e^{z \log \zeta},$$

where log ζ is the principal value of Log ζ. By choosing other values of Log ζ we obtain other values of the power which may be called its <u>subsidiary values</u>. All these are contained in the formula

$$\zeta^z = \exp\{z(\log\zeta + 2k\pi i)\}.$$

Hence ζ^z has an infinite number of values, in general, but one, and only one, principal value.

The functions considered in I to V above, together with functions derived from them by a finite number of operations involving addition, subtraction, multiplication, division and roots are called <u>elementary functions</u>.

Definition 1.6. Many valued functions.

To illustrate the idea of many valuedness, let us consider the simple case of the relation $w^2=z$. On putting $z=re^{i\theta}$, $w=Re^{i\phi}$ we get

$$R^2 e^{2i\phi} = re^{i\theta}$$

For given r and θ ($<2\pi$),two obvious solutions are

$$w_1 = \left|\sqrt{r}\right| e^{\frac{1}{2}i\theta} \text{ and } w_2 = \left|\sqrt{r}\right| e^{i\left(\frac{1}{2}\theta+\pi\right)} = -\left|\sqrt{r}\right| e^{\frac{1}{2}i\theta},$$

and these are the only continuous solutions for fixed θ, since $\left|\sqrt{r}\right|$ and $-\left|\sqrt{r}\right|$ are the only continuous solutions of the real equation $x^2=r$, $r>0$.

We see that the equation $w^2=z$ has no continuous one valued solution defined for the whole complex plane, but $w^2=z$ defines a two-valued function of z. The two functions w_1 and w_2 are called the two branches of the two-valued function $w^2=z$. Each of these branches is a one-valued function in the z-plane if we make a narrow slit, extending from the origin to infinity along the positive real axis, and distinguish between the values of the functions at points on the upper and lower edges of the cut. This idea may be easily extended. For example, the function $z^{1/3}$ is a triple-valued function and log z is a many valued function.

Note : Construction of analytic function by Milne - Thomson's method :

We have $z = x+iy$

So that $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$.

and $w = f(z) = u(x, y) + iv(x, y)$

Therefore, $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

we put $x = z$ and $y = 0$ so that $z = \bar{z}$

and thus $f(z) = u(z,0) + i\,v(z,0)$

now, $f(z) = u + iv \therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ by Cauchy - Riemann equations.

Henceif we write $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial y} = \phi_2(x,y)$

we have $f'(z) = \phi_1(x,y) - i\phi_2(x,y)$

$= \phi_1(z,0) - i\phi_2(z,0)$

Integrating,

$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)] + c$ wherec is arbitraryconstant.

Thus f(z) is constructed when u is given.

Similarly, if v is given, it can be shown that

$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)] + c$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ and $\psi_2(x,y) = \frac{\partial v}{\partial x}$.

Solved problems

Exp 1. Show that the function

$$f(z)=\sqrt{|xy|}$$

is not analytic at the origin, although the Cauchy-Riemann equations are satisfied at the point.

Solution :

Let $w = f(z) = u(x,y) + iv(x,y) = \sqrt{|xy|}$

Thus $u = \sqrt{|xy|}$, $v = 0$

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 0$$

Consequently, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann equations are satisfied at the origin.

Again,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sqrt{|xy|}}{x + iy}$$

$$= \lim_{x \to 0} \frac{|mx^2|^{1/2}}{x + imx} \quad \text{letting } z \to 0 \text{ along } y = mx$$

$$= \lim_{x \to 0} \frac{\sqrt{m}}{1 + im}$$

This limit depends on m and so it is not unique i.e. $f'(0)$ is not unique. Hence $f(z)$ is not analytic at the origin although Cauchy-Riemann equations are satisfied there.

Exp.2 Find the analytic function of which the real part is

$$u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

Solution :

Here $u_x = \dfrac{2(1 + \cos 2x \cosh 2y)}{(\cosh 2y + \cos 2x)^2} = \phi_1(x, y)$ say

and $u_y = \dfrac{\sin 2x(-2 \sinh 2y)}{(\cosh 2y + \cos 2x)^2} = \phi_2(x, y)$ say

$\therefore$ The analytic function f(z)=u+iv is given by

$$\begin{aligned} f(z) &= \int [\phi_1(z,0) - i\phi_2(z,0)]dz + c \\ &= \int \frac{2(1+\cos 2z)}{(1+\cos 2z)^2} dz + c \\ &= \int \frac{dz}{(1 + \cos 2z)} + c \\ &= 2\int \frac{1}{2} \sec^2 z dz + c \\ &= \tan z + c \end{aligned}$$

Exp.3 Show that an analytic function with a constant real part (or imaginary part) is itself a constant function.

Solution. Let w=f(z)=u(x,y)+iv(x,y) be an analytic function.

Let u(x,y)=constant.

$\therefore u_x = 0, u_y = 0$

By Cauchy Reimann equation we have,

$u_x = v_y = 0, u_y = -v_x = 0$

But $f'(z) = u_x + iv_x \quad \forall z \in$ domain of f(z)

$= 0 \quad \forall z \in$ domain of f(z)

$\therefore$ f(z) itself is a constant function.

Exp. 4 a) Prove that $u=e^{-x}(x \sin y - y \cos y)$ is harmonic.

b) Find v such that $f(z)=u+iv$ is analytic.

Solution. Given

$$u = e^{-x}(x \sin y - y \cos y)$$

$$\therefore \frac{\partial u}{\partial x} = e^{-x} \sin y - xe^{-x} \sin y + ye^{-x} \cos y$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -2e^{-x} \sin y + xe^{-x} \sin y - ye^{-x} \cos y \qquad (1)$$

$$\text{and } \frac{\partial u}{\partial y} = xe^{-x} \cos y + ye^{-x} \sin y - e^{-x} \cos y$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -xe^{-x} \sin y + 2e^{-x} \sin y + ye^{-x} \cos y \qquad (2)$$

Adding (1) and (2) we have $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow$ u is harmonic.

(b) From Cauchy-Riemann equations

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=e^{-x}\sin y-xe^{-x}\sin y+ye^{-x}\cos y \quad \text{.........}(3)$$

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=e^{-x}\cos y-xe^{-x}\cos y-ye^{-x}\sin y \quad \text{.........}(4)$$

Integrating (3) with respect to y, keeping x constant, we have

$$v=-e^{-x}\cos y+xe^{-x}\cos y+e^{-x}(y\sin y+\cos y)+F(x)$$
$$=ye^{-x}\sin y+xe^{-x}\cos y+F(x) \quad \text{.........}(5)$$

where F(x) is an arbitrary real function of x. Substituting (5) in (4),

$$-ye^{-x}\sin y-xe^{-x}\cos y+e^{-x}\cos y+F'(x)=e^{-x}\cos y-xe^{-x}\cos y-ye^{-x}\sin y$$
$$\text{or } F'(x)=0$$
$$\therefore F(x)=c, \text{ a constant.}$$

Then from (5) $\quad v=ye^{-x}\sin y+xe^{-x}\cos y+c.$

Exp 5: Prove that if $u=x^2-y^2, v=-\frac{y}{(x^2+y^2)}$ both u and v satisfy Laplace's equation, but u+iv is not an analytic function of z .

Solution :

We have $u=x^2-y^2, v=-\frac{y}{(x^2+y^2)}$

$$\therefore \frac{\partial u}{\partial x}=2x, \frac{\partial u}{\partial y}=-2x$$

and $\frac{\partial^2 u}{\partial x^2}=2, \frac{\partial^2 u}{\partial y^2}=-2$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \Rightarrow \text{u is harmonic}$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{\left(x^2+y^2\right)^2}$$

Hence $\frac{\partial^2 v}{\partial x^2}=\frac{2y\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}$

And $\frac{\partial v}{\partial y}=\frac{x^2-y^2}{\left(x^2+y^2\right)^2}$. Hence $\frac{\partial^2 v}{\partial y^2}=\frac{2y\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3}$

$$\therefore \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \Rightarrow \text{ v is hermonic}$$

Hence both u and v satisfy Laplace's equation .

But $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$

that is, Cauchy-Reimann equation are not satisfied.

Hence u+iv is not an analytic function .

Supplementary Problems:

1. Verify that the real and imaginary parts of the following functions satisfy the Cauchy-Riemann equations and thus deduce the analyticity of each function :

   (a) $f(z)=z^2+5iz+3-i$

   (b) $f(z)=ze^{-z}$

   (c) $f(z)=\sin 2z$

2. If $f'(z)=0$ in a region R, prove that f(z) must be a constant in R.

3. If $f(z) = \dfrac{x^3y(y-ix)}{x^6+y^2}$ $(z \neq 0)$, $f(0) = 0$,

   prove that $\dfrac{\{f(z) - f(0)\}}{z} \to 0$ as $z \to 0$ along any radius vector, but not as $z \to 0$ in any manner.

4. Prove that the function $u+iv=f(z)$, where

$$f(x)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}(z\neq 0), f(0)=0,$$

   is continuous and that the Cauchy-Reimann equations are satisfied at the origin; yet $f'(0)$ does not exist .

5. If f(z) is an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(x)|^2=4|f'(x)|^2.$$

6. If $w=f(z)$ is an analytic function of z such that $f'(0)\neq 0$, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(x)|=0$$

   If $|f'(z)|$ is the product of a function x and a function of y , show that

$$f'(z)=\exp(\alpha z^2+\beta z+\gamma)$$

   where a is a real and β and γ are complex constants .

7. Prove that $u=y^3-3x^2y$ is a harmonic function . Determine its harmonic conjugate and find the corresponding analytic function f(z) in terms of z.

8. If ϕ and Ψ are functions of x and y satisfying Laplace's equation, show that (s+i t) is analytic,

where $s=\frac{\partial\phi}{\partial y}-\frac{\partial\Psi}{\partial x}$ and $t=\frac{\partial\phi}{\partial x}+\frac{\partial\Psi}{\partial y}$.

9. Show that the function

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

satisfies Laplace's equation and determine the corresponding analytic function

f((z)=u+iv .

10 Determine which of the following functions (u) are harmonic . For each harmonic function find the conjugate harmonic function v and express u+iv as an analytic function of z.

(a) $3x^2y+2x^2-y^3-2y^2$

(b) $2xy+3xy^2-2y^3$

(c) $xe^x\cos y - ye^x\sin y$

(d) $e^{-2xy}\sin(x^2-y^2)$

(e) $\frac{\sin 2x}{\cosh 2y+\cos 2x}$

ANALYTIC FUNCTIONS AS MAPPINGS

Introduction :

The set of equations

$$\left.\begin{aligned} u &= u(x,y) \\ v &= v(x,y) \end{aligned}\right\} \text{---} > (1.2.1)$$

defines ; in general , a transformation or mapping which establishes a correspondence between points in the uv and xy planes . The equation (1.2.1) are called **transformation equations** . If to each point of the **uv-plane** there corresponds one and only point of the **xy-plane**, and conversely, we speak of a one to one transformation or mapping . In such cases a set of points in the xy-plane [such as a curve or region] is mapped into a set of points in the uv plane[curve or region] and conversely . The corresponding set of points in the two planes are called images of each other.

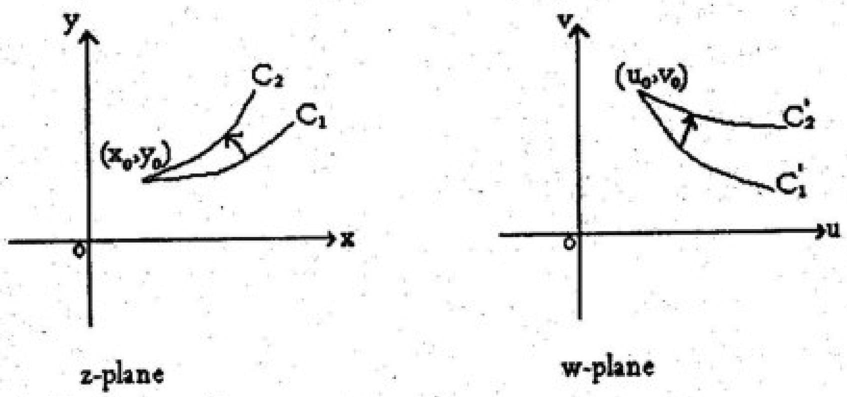
Definition 1.2.1 Isogonal and conformal transformation

Suppose the transformation

$$u = u(x,y),\ v = v(x,y)$$

maps the two curves(intersecting at the point z_0) of the z plane on the two curves C_1', C_2'

(intersecting at the point w_0) of w-plane.

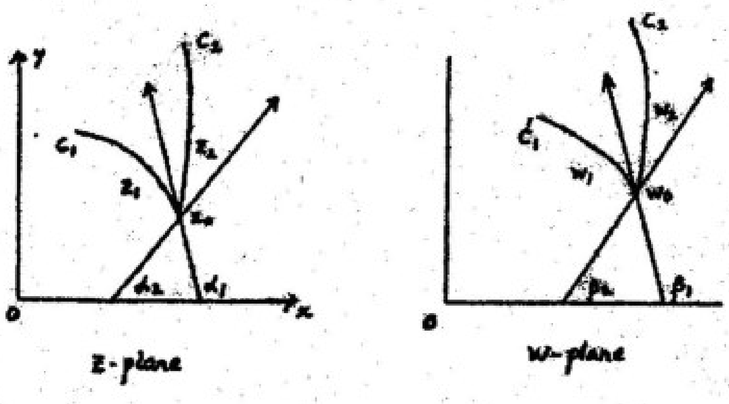


Now if the transformation is such that the angle between C_1 and C_2 at z_0 is equal to the angle between C_1' and C_2' at w_0 both in magnitude and sense, the transformation or mapping is said to be conformal at $z_0 = (x_0, y_0)$. Again, a mapping which preserves the magnitudes of angles but not necessarily the sense is called **isogonal**.

Theorem 1.2.1 The necessary and sufficient condition for the transformation w=f(z) to be conformal is that f(z) is analytic.

Necessary condition : Let w = f(z) be Analytic analytic function in a domain D on the z-plane and z_0 be an interior point of D. Also, let C_1' and C_2' be the curves (intersecting at w_0) in the w-plane corresponding to the curves C_1 and C_2 (intersecting at z_0) of the z-plane. Let w_1 and w_2 be points on C_1' and C_2' respectively corresponding to the points z_1 and z_2 on C_1 and C_2 respectively near to z_0 and distance between z_2 and z_0 =r(say) so that we can write

$$z_1 - z_0 = re^{i\theta_1}, \qquad z_2 - z_0 = re^{i\theta_2}$$



Let the tangents at z_0 to the curves C_1 and C_2 make angles α_1 and α_2 with the real axis so that

and $\theta_2 \to \alpha_2$ as $r \to 0$

Also let the tangents at w_0 to the curves C_1' and C_2' make angles β_1 and β_2 with the real axis and let

$$w_1 - w_0 = \rho_1 e^{i\phi_1}, \quad w_2 - w_0 = \rho_2 e^{i\phi_2}$$

where $\varphi_1 \to \beta_1$ as $\rho_1 \to 0$ and $\varphi_2 \to \beta_2$ as $\rho_2 \to 0$

Now, the derivative of the function f(z) and (z_0) is given by

$$f'(z_0) = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{r_1 e^{i\theta_1}} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\phi_1 - \theta_1)}$$

Since $f'(z_0) \neq 0$ therefore we can write $f'(z_0) = Re^{i\lambda}$, then

$$Re^{i\lambda} = \lim_{z_1 \to z_0} \frac{\rho_1}{r_1} e^{i(\phi_1 - \theta_1)}$$

Equating modulus and arguments we have

$$R = \lim\left(\frac{\rho_1}{r_1}\right) \text{ and } \lambda = \lim(\phi_1 - \theta_1) = \lim\phi_1 - \lim\theta_1$$

$$= \beta_1 - \alpha_1 \text{ or } \beta_1 = \lambda + \alpha_1$$

Similarly, it can be shown that $\beta_2 = \lambda + \alpha_2$

$$\therefore \beta_1 - \beta_2 = \alpha_1 - \alpha_2$$

i.e. the angle between C_1' and C_2' at w_0 is equal in magnitude and as well as in sign to the angle between the curves C_1 and C_2 at z_0.

Therefore the transformation is conformal.

Sufficient condition

Let $w = f(z) = u(x,y)+iv(x,y)$ and also let $u = u(x,y)$, $v=v(x,y)$ are equations defining conformal transformation from z-plane to w-plane.

Let ds and $d\sigma$ be the elements in z-plane and w-plane respectively so that

$$ds^2 = dx^2 + dy^2 \quad \text{........(1.2.2)}$$

$$d\sigma^2 = du^2 + dv^2 \quad \text{........(1.2.3)}$$

Since u and v are functions of x and y therefore we have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

squaring and adding,

$$d\sigma^2 = du^2 + dv^2 = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]dx^2 + \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dy^2$$

$$+2\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}\right]dxdy \quad \text{.....(1.2.4)}$$

Since transformation is conformal, therefore the ratio $d\sigma{:}ds$ is independent of direction, so that comparing the coefficients of (1.2.2) and (1.2.4) we have

$$\frac{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}{1} = \frac{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}{1} = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}}{0}$$

i.e. $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \quad \text{....................(1.2.5)}$$

and $$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0 \quad \text{..............................(1.2.6)}$$

From (1.2.6) we can write

$$\frac{\left(\frac{\partial u}{\partial x}\right)}{\left(\frac{\partial v}{\partial y}\right)} = -\frac{\left(\frac{\partial v}{\partial x}\right)}{\left(\frac{\partial u}{\partial y}\right)} = \lambda \text{ (say)}$$

$$\therefore \frac{\partial u}{\partial x} = \lambda \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\lambda \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots(1.2.7)$$

Substituting (1.2.7) in (1.2.5) we have

$$(\lambda^2 - 1)\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = 0$$

which shows that

$$\lambda^2 - 1 = 0 \text{ i.e. } \lambda = \pm 1.$$

When $\lambda = 1$, we have from (1.2.7)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots(1.2.8)$$

when $\lambda = -1$, we have from (1.2.7)

$$\therefore \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots(1.2.9)$$

The equations (1.2.7) are **Cauchy-Riemann equations** and hence $w = f(z)$ is an **analytic function.**

The equations(1.2.9) can be reduced to (1.2.7) by writing $-v$ in place of v i.e. by taking an image figure found by reflection in the real axis of w-plane . So equations (1.2.9) correspond to an **isogonal** but not **conformal transformation .**

Definition 1.2.2 Fixed or invariant points of a transformation

Suppose that we superimpose the w-plane on the z-plane so that the coordinate axes coincide and there is essentially only one plane. Then we can think of the transformation $w = f(z)$ as taking certain points of the plane into other points . Points for which $z = f(z)$ will however remain fixed, and for this reason we call them the **fixed or invariant points of the transformation.**

Example: The fixed or invariant points of the transformation $w = z^2$ are solutions of $z^2 = z$ i.e. $z = 0,1$.

Definition 1.2.3 Some general transformations

In the following α, β are given complex constants while a, θ_0 are real constants.

A. Translation $w = z + \beta$

By the transformations, figures in the z-plane are displaced or translated in the direction of vector β .

B. Rotation $w = e^{i\theta_0} z$

By this transformation, figures in the z-plane are rotated through an angle θ_0 . If $\theta_0 > 0$, the rotation is counterclockwise, while if $\theta_0 < 0$, the rotation is clockwise.

C. Stretching. $W = az$

By this transformation, figures in the z-plane are stretched (contracted) in the direction z if $a > 1$ (or $0 < a < 1$). We consider contraction as a special case of stretching.

D. Inversion $w = \frac{1}{z}$

Definition 1.2.4 The successive transformation

If $w = f_1(\zeta)$ maps region $R\zeta$ of the ζ-plane into the region R_w of the w-plane while $\zeta = f_2(z)$ maps region Rz of the z-plane , then $w = f_1[f_2(z)]$ maps R_z into R_w . The functions f_1 and f_2 define successive transformations from one plane to another which are equivalent to a single transformation. These ideas are easily generalized.

Definition 1.2.5 The linear transformation

The transformation $$W = \alpha z + \beta \qquad ...(1.2.9)$$

where α and β are given complex constants, is called a **linear transformation**. Since we can write (1.2.9) in terms of successive transformations $w = \zeta + \beta$, $\zeta = e^{i\theta_0}\tau, \tau = az$ where $\alpha = e^{i\theta_0}$, we see that a general linear transformation is a combination of the transformation of translation, rotation and stretching.

Definition 1.2.6 The Bilinear transformation

The transformation

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \alpha\delta - \beta\gamma \neq 0 ....(1.2:.11)$$

which is linear in z as well as w is called **bilinear transformation** where $\alpha, \beta, \gamma, \delta$ are called complex constants. Bilinear transformations are also called **Mobius transformation** since the first study of such transformations goes back to A.F. Mobius(1790-1868).

Bilinear transformations are also sometimes called Fractional or **homographic transformations.**

The expression $\alpha\delta - \beta\gamma$ is called the determinant of the transformation . The bilinear transformation can be considered as combinations of the transformations of translation, rotation, stretching and inversion.

The transformation inverse to (1.2.11) is also a Mobius transformation

$$z = \frac{-\delta w + \beta}{\gamma w - \alpha}, (-\delta)(-\alpha) - \beta\gamma \neq 0 \ldots\ldots\ldots\ldots\ldots\ldots(1.2.12)$$

Definition 1.2.7 Geometric Inversion

There is an intimate relation between Bilinear transformation and geometrical inversion.

Let S be a circle of centre k and radius r. Then two points P and P_1, collinear with k, such that $kP \cdot kP_1 = r^2$, are called **inverse points** with respect to the circle S, and it is known from the geometry that any circle passing through P and P_1 is orthogonal to S. In the case of a straight line S,P and P_1 are inverse points with respect to S, if P_1 is the image of P in S. If P,P_1 and k are the points z,$z_1$and k we have

$$|(z_1 - k)(z - k)| = r^2, \arg(z_1-k) = \arg(z-k) \ldots\ldots\ldots(1.2.13)$$

the second equation expressing the collinearity of the points k,P,P_1. The two equations (1.2.13) are satisfied, if and only if,

$$|(z_1 - k)(\bar{z} - \bar{k})| = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2.14)$$

If S is the circle

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2.15)$$

which may be written as

$$\left(z + \frac{\bar{B}}{A}\right)\left(\bar{z} + \frac{B}{A}\right) = \frac{B\bar{B} - AC}{A^2}, \ldots\ldots\ldots\ldots(1.2.16)$$

we see that (1.2.15) is a circle with centre $\frac{\bar{B}}{A}$ and radius $\sqrt{\frac{B\bar{B} - AC}{A^2}}$

Hence equation (1.2.14) becomes

$$\left(z_1 + \frac{\bar{B}}{A}\right)\left(\bar{z} + \frac{B}{A}\right) = \frac{B\bar{B} - AC}{A^2}$$

which on simplification is

$$Az_1\bar{z} + Bz_1 + \bar{B}\bar{z} + C = 0 \qquad \ldots\ldots\ldots(1.2.17)$$

We thus get the relation between z and its inverse z_1 from the equation of S by substituting z_1 for z and leaving $\bar{z}$ unchanged. On solving (1.2.17). the transformation is

$$z_1 = \frac{-\bar{B}\bar{z} - C}{A\bar{z} + B} \quad \text{...............(1.2.18)}$$

The inversion (3.18) can be written as a succession of two transformations

$$w = \bar{z}, z_1 = \frac{-\bar{B}w - C}{Aw + B} \quad \text{.......(1.2.19)}$$

The first is a reflection in the real axis and the second is a Bilinear transformation. The first preserves the angles but reverses their signs; the second is conformal. Hence inversion is an isogonal but not conformal transformation.

Definition 1.2.8 The critical points

Let us consider the bilinear transformation

$$w = T(z) = \frac{\alpha z + \beta}{\gamma z - \delta} \text{..........(1.2.20)}$$

then solving for z, we get the inverse map as

$$z = T^{-1}(w) = \frac{\beta - w\delta}{w\gamma - \alpha} \text{........(1.2.21)}$$

The transformation T associates a unique point of the w-plane to any point of z-plane except the point $z = -\frac{\delta}{\gamma}$ where $\gamma \neq 0$. The transformation T^{-1} associates a unique point of the z-plane to any point of the w-plane except the point $w = \frac{\alpha}{\gamma}$ where $\gamma \neq 0$. These exceptional points $z = -\frac{\delta}{\gamma}$ and $w = \frac{\alpha}{\gamma}$ are mapped into the points $w = \propto$ and $z = \propto$ respectively.

Now from (3.20),

$$\frac{dw}{dz} = \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2}$$

which shows that

$$\frac{dw}{dz} = \infty \text{ if } z = -\frac{\delta}{\gamma}$$

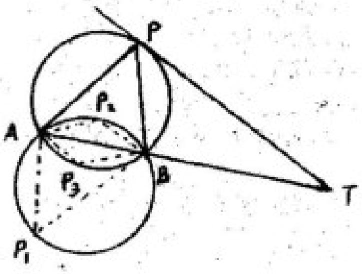
$$= 0 \quad \text{if } z \to \infty$$

The points $z = -\frac{\delta}{\gamma}$, $z = \propto$ are called critical points where the conformal property does not hold good.

There two critical points cease to be exceptional if we extend the definition of conformal representation in the following manner. A function w=f(z) is said to transform the neighbourhood of a point z_0 conformally into a neighbourhood of $w = \infty$, if the function $t = 1/f(z)$ transforms the neighbourhood of z_0 conformally into a neighbourhood of t=0. Also w=f(z) is said to transform the neighbourhood of $z = \infty$ conformally into a neighbourhood of $\zeta = 0$ conformally into a neighbourhood of w_0. In this definition w_0 may have the value ∞.

With these extension of the definition we may now say that every bilinear transformation gives a one-one conformal representation of the whole closed z-plane on the whole closed w-plane. In other words, the mapping is biuniform for the complete planes of w and z.

Definition 1.2.9 Coaxal circles



Let a,b,z be the coordinates of the three point A,B and P of the z-plane. Then

$$\arg\frac{z-b}{z-a} = \angle APB$$

if the principal value of the argument be chosen. Let A and B be fixed and P is a variable point.

If the two circles in figure are equal and z_1, z_2, z_3 are the coordinates of the points P_1, P_2, P_3 and $\angle APB = \theta$, we see that

$$\arg\frac{z_2-b}{z_2-a} = \pi - \theta, \arg\frac{z_1-b}{z_1-a} = -\theta, \arg\frac{z_3-b}{z_3-a} = -\pi + \theta$$

The locus defined by the equation

$$\arg\frac{z-b}{z-a} = \theta \ldots\ldots\ldots\ldots\ldots(1.2.20)$$

when θ is a constant, is the arc APB. By writing $-\theta$, $\pi-\theta$, $-\pi+\theta$ for θ we obtain the arcs AP_1B, AP_2B, AP_3B respectively. The system of equations obtained by varying θ from $-\pi$ to π represents the system of circles which can be drawn through the points A,B. It should be observed that each circle must be divided into two parts, to each of which corresponds different values of θ.

Let T be a point at which the tangent to the circle APB at P meets AB. Then the triangles TPA, TBP are similar and

$$\frac{AP}{PB}=\frac{PT}{BT}=\frac{TA}{TP}=k \qquad \text{........(1.2.21)}$$

Hence $\frac{TA}{TB}=k^2$ and so T is a fixed point for all positions of P which satisfy

$$\left|\frac{z-a}{z-b}\right|=k$$

where k is a constant. Also $TP^2 = TA\cdot TB$ and so is constant. Hence the locus of P is a circle whose centre is T.

The system of equations obtained by varying k represents a system of circles. The system given by (1.2.20) is a system of coaxal circles of the common point kind, and that given by (1.2.21) a system of the limiting point kinds with A and B as the limiting points of the system. If $k\to\infty$ or if $k\to 0$ then the circle becomes a point circle at A or B. All the circles of one system intersect all the circles of the other system orthogonally.

The above important result is of frequent application in problems involving bilinear transformations.

It may be used to prove that the bilinear transformation transforms circles into circles.

Definition 1.2.10 : Fixed points of a bilinear transformation

We consider any bilinear transformation

$$w=\frac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta-\beta\gamma\neq 0$$

and suppose that w and z are represented by points on the same plane.

The invariant points of this transformation are given by

$$z=\frac{\alpha z+\beta}{\gamma z+\delta}$$

$$\text{or, } z=\frac{(\alpha-\delta)\pm\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}$$

We have the following four possibilities for fixed point or invariant points of a bilinear transformation.

(i) $\gamma\neq 0$, $(\alpha-\delta)^2+4\beta\gamma\neq 0$, two finite fixed points.

(ii) $\gamma\neq 0$, $(\alpha-\delta)^2+4\beta\gamma=0$, one finite fixed point.

(iii) $\gamma=0$, $\alpha-\delta\neq 0$, one finite and the other infinite fixed points.

(iv) $\gamma=0$, $\alpha-\delta=0$, one fixed point i.e. ∞

Definition 1.2.11 : Cross Ratio

If z_1, z_2, z_3, z_4 are four distinct points then the ratio

$$\frac{(z_4 - z_3)(z_2 - z_1)}{(z_2 - z_3)(z_4 - z_1)}$$

is called the cross ratio of the points z_1, z_2, z_3, z_4. This ratio is invariant under the bilinear transformation and this property can be used in obtaining specific bilinear transformation mapping three points into three other points.

Theorem 1.2.2 : Invariance of the cross ratio

If z_1, z_2, z_3, z_4 be any four points of the z-plane and let w_1, w_2, w_3, w_4 be the points which correspond to them by the bilinear transformation.

$$w = \frac{\alpha z + \beta}{\gamma z + \delta}, \alpha\delta - \beta\gamma \neq 0$$

If we suppose that all the numbers z_r, w_r are finite, we have

$$w_r - w_s = \frac{\alpha z_r + \beta}{\gamma z_r + \delta} - \frac{\alpha z_s + \beta}{\gamma z_s + \delta}$$

$$= \frac{\alpha\delta - \beta\gamma}{(\gamma z_r + \delta)(\gamma z_s + \delta)}(z_r - z_s)$$

and hence it follows that

$$\frac{(w_1 - w_4)(w_3 - w_2)}{(w_1 - w_2)(w_3 - w_4)} = \frac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$$

The right hand side of the above expression is the cross ratio of the four points, z_1, z_2, z_3, z_4 **and so we have the result that the cross ratio is invariant under bilinear transformation i.e.**

$$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

Definition 1.2.12 : Some special bilinear transformations

Here we consider the general bilinear transformation which map the

(1) real axis on itself.

(2) unit circle on itself

(3) real axis on the unit circle

(1) A bilinear transformation which maps the real axis in the z-plane on the real axis in the w-plane is such that some three points x_1, x_2, x_3 on the real axis in the z-plane are mapped

on 0, 1, ∞ respectively lying on the real axis on the w-plane. The bilinear transformation mapping x_1, x_2, x_3 on 0, 1, ∞ is given by

$$(0,1,\infty,w)=(x_1,x_2;x_3,z)$$

$$i.e.\ w=\frac{(z-x_1)(x_2-x_3)}{(z-x_3)(x_2-x_1)}$$

$$=\frac{\alpha z+\beta}{\gamma z+\delta} \qquad \text{...(1.2.22)}$$

where $\alpha = x_2 - x_3$, $\quad \beta = -x_1(x_2 - x_3)$

$\gamma = x_2 - x_1$, $\quad \gamma = -x_3(x_2 - x_1)$

Here $\alpha\delta - \beta\gamma = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \neq 0$ for x_1, x_2, x_3 are distinct.

The bilinear transformation (1.2.22) for all possible choices of three distinct real numbers constitute the totality of bilinear transformation.

Every member of this totality is of the form (1.2.22), where $\alpha,\beta,\gamma,\delta$ are any real numbers such that $\alpha\delta-\beta\gamma \neq 0$. Clearly also every transformation (1.2.22) wherefor α, β, γ, δ are real and $\alpha\delta-\beta\gamma \neq 0$ is a bilinear transformation mapping the real axis onto the real axis.

Again from (1.2.22) we have

$$\overline{w}=\frac{\alpha\overline{z}+\beta}{\gamma\overline{z}+\delta}$$

so that

$$w-\overline{w}=\frac{\alpha\delta-\beta\gamma}{|\gamma z+\delta|^2}(z-\overline{z})$$

i.e.

$$I(w)=\frac{\alpha\delta-\beta\gamma}{|\gamma z+\delta|^2}\ I(z)$$

Thus the transformation (1.2.22) which maps the real axis on itself will map the region $I(z)>0$ i.e. the upper half plane, onto $I(w)>0$ or $I(w)<0$ according as the determinant $\alpha\delta-\beta\gamma$ of the transformation is positive or negative.

(2) Let us suppose that

$$w=\frac{\alpha z+\beta}{\gamma z+\delta}$$

is a transformation mapping $|z|=1$ onto $|w|=1$ .Now, $w = 0$ and $w = \infty$ are inverse points for $|w|=1$ and they are the transform of

$$z=-\frac{\beta}{\alpha}, z=-\frac{\delta}{\gamma} \text{ respectively}.$$

Therefore, $-\frac{\beta}{\alpha}$ and $-\frac{\delta}{\gamma}$ are inverse points for $|z|=1$ so that if we write $a=-\frac{\beta}{\alpha}$ then $\frac{1}{\bar{a}}=-\frac{\delta}{\gamma}$ and so

$$w=\frac{\alpha}{\gamma}\frac{(z-a)}{\left(z-\frac{1}{\bar{a}}\right)}=\frac{\alpha\bar{a}}{\gamma}\frac{(z-a)}{(\bar{a}z-1)}$$

The point z=1 corresponds to a point $|w|=1$ and so

$$\left|\frac{\alpha\bar{a}}{\gamma}\frac{(1-a)}{(\bar{a}-1)}\right|=\left|\frac{\alpha\bar{a}}{\gamma}\right|=1$$

Thus we can consider that every bilinear transformation which maps $|z|=1$ onto $|w|=1$ must necessarily be of the form

$$w=k\frac{z-a}{\bar{a}z-1}$$

where a is any complex number and k is unimodular.

Again, inverting the above transformation we can show that every point of the circle is the image of some point of $|z|=1$.

Now, we have $\bar{w}=k\frac{\bar{z}-\bar{a}}{a\bar{z}-1}$

$$\therefore w\bar{w}-1=\frac{z-a}{\bar{a}z-1}\frac{\bar{z}-\bar{a}}{a\bar{z}-1}-1$$

i.e. $|w|^2-1=\frac{(1-|z|^2)(1-|a|^2)}{|\bar{z}a-1|}$

This shows that the bilinear transformation which maps $|z|=1$ onto $|w|=1$ will map

$|z|<1$ onto $|w|<1$ or $|w|>1$.

according as $|a|<1$ or $|a|>1$.

(3) Let us suppose that the bilinear transformation

$$w = \frac{\alpha z+\beta}{\gamma z+\delta}$$

transforms the real axis $I(z) = 0$ on to the unit circle $|w| = 1$.

Now the points $w = 0$, $w = \infty$ are inverse points for are the images of the $z = -\frac{\beta}{\alpha}$ and $z = -\frac{\delta}{\gamma}$ points respectively.

Thus the points $-\frac{\beta}{\alpha}$ and $-\frac{\delta}{\gamma}$ are the inverse points for the real axis. Thus if we take

$$a = -\frac{\beta}{\alpha} \text{ then } \bar{a} = -\frac{\delta}{\gamma}$$

Now we write

$$w = k\frac{z-a}{z-\bar{a}} \text{ where } k = \frac{\alpha}{\gamma}$$

we consider $|w| = 1$ that when z is real therefore

$$|w| = |k|\left|\frac{z-a}{z-\bar{a}}\right|$$

i.e. $1=|k|$

Hence we see that a bilinear transformation which maps $I(z)=0$ onto $|w| = 1$ is necessarily of the form

$$w = k\frac{z-a}{z-\bar{a}} \text{ where k is unimodular.}$$

Conversely also we can show that the above transformation where a is any number and k is unimodular does map $I(z) = 0$ onto $|w| = 1$.

Again ,

$$w\bar{w} - 1 = \frac{(z-\bar{z})(a-\bar{a})}{|z-\bar{a}|^2}$$

i.e. $$|w|^2 - 1 = \frac{4I(z)I(a)}{|z-\bar{a}|^2}$$

This shows that the bilinear transformation which maps

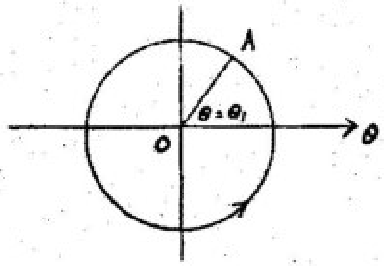
$I(z) = 0$ onto $|w| = 1$

will map

$I(z)>0$ onto $|w| > 1$ or $|w| < 1$

According as $I(a)<0$ or $I(a)>0$

Definition 1.2.13 Branch points and Branch lines



z-plane

We consider the function $w = z^{1/2}$. We allow z to make a complete circuit (counterclockwise) around the origin starting from the point A. Let us take $z=re^{i\theta}$ $\therefore \omega = \sqrt{r}e^{i\frac{\theta}{2}}$ and hence if at A, $\theta = \theta_1$ then $w = \sqrt{r}e^{\frac{i\theta_1}{2}}$.

Now after a complete circuit when z comes back to A then $\theta = \theta_1 + 2\pi$ and $w = \sqrt{r}e^{\frac{i(\theta_1+2\pi)}{2}}$

$= \sqrt{r}e^{i\theta_1/2}$

i.e. the same value of w with which we started.

We can describe the above by starting that if $0 \le \theta \le 2\pi$ we are on one branch of the multiple valued function $z^{1/2}$, while if $2\pi \le \theta \le 4\pi$ we can on the other branch of the function.

It is clear that each branch of the function is single valued. In order to keep the function single-valued we set an artificial barrier such as OB where B is at infinity (although any other line from O can be used) which we agree not to cross. This barrier OB is called a **branch line** or branch cut, and the point O is called a **branch point**. Since a circuit around any point other then $z = 0$ does not lead to different values therefore $z = 0$ is the only finite branch point.

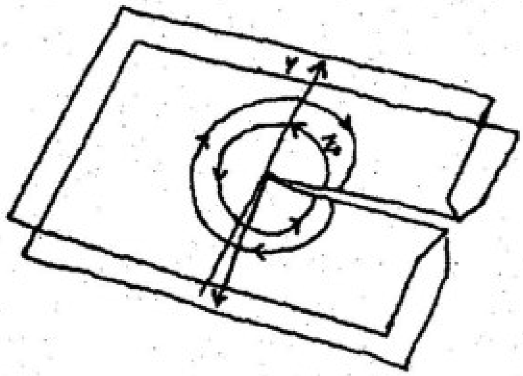
Definition 1.2.14 Riemann Surface

If $w = f(z)$ is a single valued function, a single value of z and a single value of w can be affixed to each point of the z-plane. If the function is many-valued, again a single value of z but several values of w correspond to each point of the z-plane.

Sometimes it is convenient to have a surface such that a single value of z and a single value of w correspond to each of its points, even when w is many-valued.

It is evident that the value of z will be repeated on it as many times as different values of w correspond to same value of z. Such a surface is called a **Riemann surface.**

The simplest example of a Riemann surface is obtained for the function $w=z^{1/2}$ (which is double valued, but whose inverse $z = w^2$ is single-valued). It can be obtained as follows:



Let us cut the z-plane along the positive axis. This cut join the branch points O and ∞. Now let a second slit plane lie upon the first. We will tie the two plane together along their cuts, gluing the upper and lower edges of their cuts together as shown in figure.

Looking at the upper plane from above, we can draw the x and y axis as in the figure, and consider it as a slit z-plane to which all the values of z and all the values of $w_1 = \sqrt{r}e^{\frac{i\theta}{2}}$ $(0 < \theta < 2\pi)$ are affixed.

Now, looking at the lower plane from below, and without changing the x-axis, we draw a y-axis as the broken line as shown in figure. This plane can be considered as another slit z-plane to which another set of values of z and all the values of $w_2 = -w_1$ are affixed.

It is evident that when a point moving on the upper plane reaches the edges of the cut, it is forced to go onto the lower plane and that starting from z_0 and attempting a complete turn around the origin, z_0 is reached again only after two complete turns have been made, one on the upper plane and one on the lower plane.(If the first is counter clockwise as seen in figure, the second is counter clockwise too, provided that the lower plane is observed from the right direction , that is from below.). Each one of the planes, which contains a whole branch of w and nothing else, is a Riemann sheet. The collection of two sheets is called a Riemann surface corresponding to the function $z^{1/2}$

The concept of Riemann surface has the advantage is that the various values of multiple-valued function are obtained in a continuous fashion.

The ideas are easily extended. For example, for the function $z^{1/3}$ the Riemann surface has three sheets, for logz the Riemann surface has infinitely many sheets.

Solved problems

1. Find the bilinear transformation which maps the points $z = \infty$, i, 0 into points $w = 0$, i, ∞ respectively.

Solution: The bilinear transformation mapping $z = z_1, z_2, z_3$ into $w = w_1, w_2, w_3$ respectively, is

$$\frac{(z-z_3)(z-z_2)}{(z-z_1)(z_2-z_3)} = \frac{(w-w_3)(w_2-w_1)}{(w-w_1)(w_2-w_3)}$$

Here, $\dfrac{(w-w_3)(i-0)}{(w-0)(i-w_3)} = \dfrac{(z-0)(i-z_1)}{(z-z_1)(i-0)}$ when $w_3 \to \infty$ and $z_1 \to \infty$

i.e. $\dfrac{i}{w} = \dfrac{z}{i}$ i.e. $w = -\dfrac{1}{z}$.

2. Find the image of the rectangle

$x = 0, y = 0, x = 2, y = 1$ in the plane under the map $w = z + (1 - 2i)$

Solution: Given $w = z + (1 - 2i)$ where $z = x + iy$

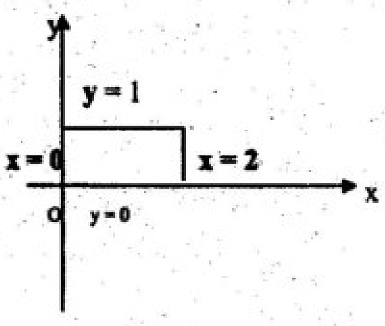
$= u + iv$ when $u = x + 1, v = y - 2$

The line $x = 0$ is mapped into $u = 1$

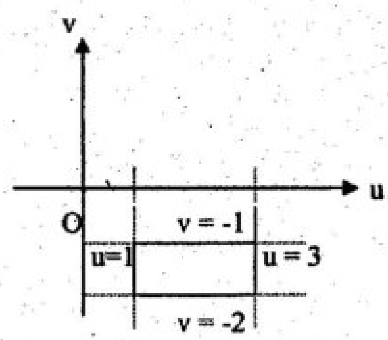
The line $y = 0$ is mapped into $v = -2$

The line $x = 2$ is mapped into $u = 3$

The line $y = 1$ is mapped into $v = -1$



z-plane



w-plane

Similarly we can show that each point of the region R in the z-plane is mapped into on and only one point of the region R$'$ in the w-plane and conversely. This mapping accomplishes a translation of any region.

3. Show that both the transformation

$$w=\frac{z-i}{z+i} \text{ and } w=\frac{i-z}{i+z}$$

transform $|w|\le 1$ into the upper half plane $I(z)\ge 0$.

Solution: If $w=\dfrac{z-i}{z+i}$

Then $w\overline{w}-1=\dfrac{z-i}{z+i}\cdot\dfrac{\overline{z}+i}{\overline{z}-i}-1=\dfrac{i-z}{i+z}\cdot\dfrac{-i-\overline{z}}{-i+\overline{z}}-1$ which is same as if $w=\dfrac{z-i}{z+i}$

$$=\frac{(z-i)(\overline{z}+i)-(z+i)(\overline{z}-i)}{(z+i)(\overline{z}-i)}$$

$$=\frac{2i(z-\overline{z})}{|z+i|^2}, \text{ the denominator being positive}$$

or, $|w|^2-1=\dfrac{-4I(z)}{|z+i|^2}$ for both the transformations.

Hence for both the transformations $|w|^2-1\le 0$ according as $I(z)\ge 0$.

This means that the circle $|w|=1$ corresponds to the real axis and the region interior to this circle corresponding to the upper half z-plane.

4. Discuss the transformation

$$w=\frac{i(1-z)}{1+z}$$

and show that it transforms the circle $|z|=1$ into the real axis of the w-plane and the interior of the circle $|z|<1$ into the upper half of the w-plane.

Solution: The given transformation is

$$w=\frac{i(1-z)}{1+z}$$

i.e. $u+iv=i\dfrac{1-x-iy}{1+x+iy}$

$$= \frac{\{y + i(1-x)\}\{(1+x) - iy\}}{(1+x)^2 + y^2}$$

Equating real and imaginary parts, we have

$$u = \frac{2y}{(1+x)^2 + y^2} \quad ......(1)$$

$$v = -\frac{x^2 + y^2 - 1}{(1+x)^2 + y^2} \quad ....(2)$$

From (2), we see that when $v = 0$, we have $x^2 + y^2 - 1 = 0$ i.e. $x^2 + y^2 = 1$ i.e. $|z| = 1$

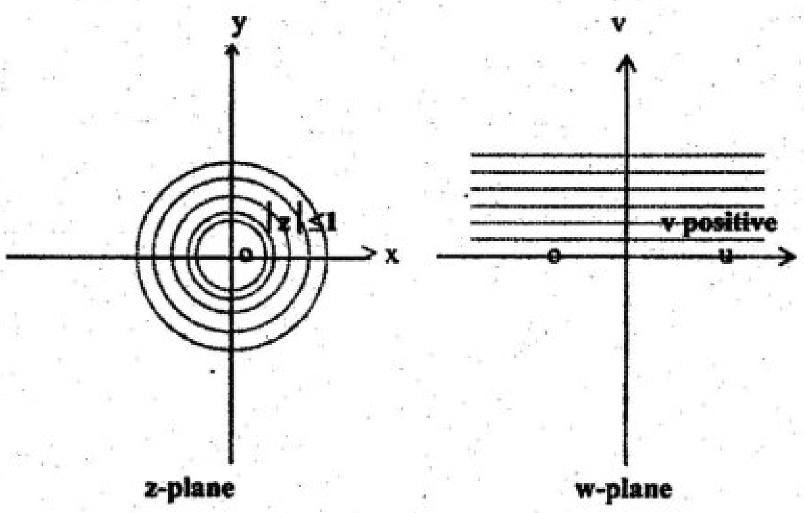
This shows that the unit circle $|z| = 1$ in the z-plane corresponds to the real axis in the w-plane. From (1) it is seen that y is positive when u is positive, so upper semi-circle in the z-plane corresponds to the positive half of the real axis on the w-plane, also it is seen from (1) that y is negative, when u is negative , so lower half of the semi-circle in the z-plane corresponds to the negative half of the real axis in the w-plane.

Again, if $x^2 + y^2 = 1$ is negative, i.e. if $x^2 + y^2 < 1$

i.e. if $|z| < 1$

Thus interior of the circle $|z| = 1$ in the z-plane corresponds to the half of the w-plane above the real axis.

The correspondence between the region is shown below:



5. Discuss the application of the transformation $w = z^2$ to the area in the first quadrant of the z-plane bounded by the axes and the circles $|z| = a, |z| = b$ $(a > b > 0)$. Is the transformation conformal?

Solution: The given transformation is $w = z^2$

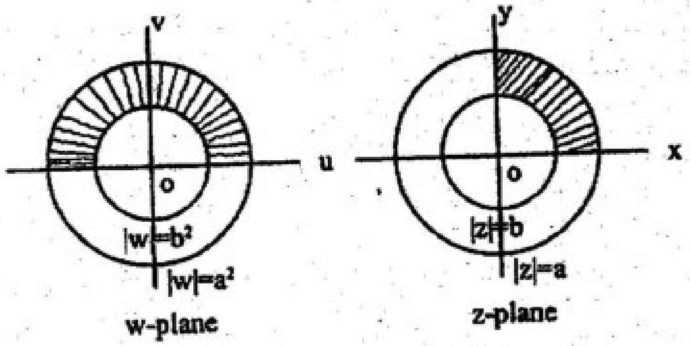
We put $w = e^{i\phi}$ and $z = re^{i\theta}$, we have $Re^{i\phi} = r^2 e^{i\theta}$

$\therefore R = r^2$ and $\phi = 2\theta$, i.e. $|w| = |z|^2$ and $\phi = 2\theta$.

Thus the circles $|z| = a, |z| = b$ in the z-plane correspond to the circles $|w| = |a|^2$ and $|w| = |b|^2$ respectively in the w-plane.

If $b < |z| < a$, then $b^2 < |z| < a^2$.

This shows that the region in z-plane included between the circles $|z| = a, |z| = b$ corresponds to the region in w-plane included between $|w| = |a|^2$ and $|w| = |b|^2$.



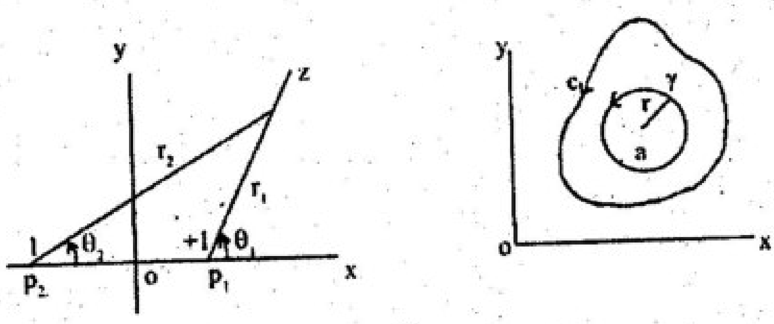
Since $\phi = 2\theta$, therefore the region $0 \le \phi \le \frac{1}{2}\pi$ in the z-plane maps on the region $0 \le \phi \le \pi$ in the w-plane.

Thus area in the first quadrant of the z-plane between the two circles and the axes corresponds to the area in the w-plane between the two corresponding circles and the real axis.

The transformation is conformal, because $\frac{dw}{dz} = 2z$ is not zero at any point within the area.

6. Describe a Riemann surface for the multiple-valued function $f(z) = (z^2 - 1)^{1/2}$

Solution : Let us describe a Riemann surface for the double-valued function



$$f(z) = (z^2 - 1)^{1/2} = \sqrt{r_1 r_2}\, e^{i\frac{(\theta_1+\theta_2)}{2}}$$

where $z-1 = r_1 e^{i\theta_1}$ and $z+1 = r_2 e^{i\theta_2}$ as shown in fig. The point $z = \pm 1$ are branch points of the function.

A Riemann surface for this double-valued function must consists of two sheets R_0 and R_1. Let both sheets be cut along the segment P_1P_2. The lower edge of the slit in R_0 is then joined in the upper edge of the slit in R_1, and the lower edge in R_1 is joined to the upper edge in R_0.

On the sheet R_0 let the angles θ_1 and θ_2 range from 0 to 2π. If a point on the sheet R_0 describes a single closed curve which encloses the segment P_1P_2 once in the counter clockwise direction, then both θ_1 and θ_2 change by the amount 2π upon the return of the point to its original position. The change in $(\theta_1+\theta_2)/2$ is also 2π and the value of f is unchanged. If a point starting on the sheet R_0 describes a path which passes twice around just the branch points $z=1$, it crosses from the sheet R_0 on the sheet R_1 and then back onto the sheet R_0 before it returns to its original position. In this case the value of θ_1 changes by the amount 4π while the value of θ_2 does not change at all. Similarly, for a circuit twice around the $z = -1$, the value of θ_2 changes by 4π while the value of θ_1 remains unchanged. Thus on the sheet R_0 the range of the angles θ_1 and θ_2 may be extended by changing both θ_1 and θ_2 by the same integral multiple of 2π or by changing just one of the angles by a multiple of 4π. In either case the total change in both angles is an even integral multiple of 2π.

To obtain the range of values for θ_1 and θ_2 on the sheet R, we note that if a point starts on the sheet R_0 describes a path around just one of the branch points once, it crosses onto the sheet R_1 and does not return to the sheet R_0. In this case the values of one of the angles is changed by 2π while the value of the other remain unchanged. Hence, on the sheet R_1 one angle can range from 2π to 4π while other angle ranges from 0 to 2π. Their sum then ranges from 2π to 4π, and the value of $(\theta_1+\theta_2)/2$, the argument of f(z), ranges from π to 2π. Again the range of the angles is extended by changing the value of just one of the angles by an integral multiple of 4π or by changing the value of both of the angles by the same integral multiple of 2π.

The given double-valued function now be considered as a single-valued function of the points on the Riemann surface just considered. The transformation w = f(z) maps each of the sheets used in the construction of that surface onto the entire w-plane

Supplementary Problems

1. Prove that the mapping given by w = f(z) from the z-plane to w-plane is conformal at z_0 if f(z) is analytic at z_0 and $f'(z_0) \neq 0$. What goes wrong when $f'(z_0) = 0$?

2. Find the fixed or invariants points of the transformation

$$w = \frac{2z-5}{z+4}$$

3. Find the bilinear transformation which maps $z = 1, -i, 2$ onto $w = 0, 2, -i$ respectively.

4. Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$.

For this transformation find the images (a) $|z| < 1$, (b) $|z| \le \rho < 1$.

5. Find the image of the rectangle $x = 0, y = 0, x = 1, y = 2$ in the z-plane under the map

$$w = (1 + i)z + (2 - i).$$

6. Show that a bilinear transformation leaves a cross-ratio invariant.

7. If $(w+1)^2 = \frac{4}{z}$, show that the unit circle in the w-plane corresponds to a parabola in the z-plane and the inside of the circle corresponds to the outside of the parabola.

8. Find the bilinear transformation which maps $R(z) \ge 0$ onto the unit circle $|w| \le 1$.

9. Find all the bilinear transformation which map $|z| \le 1$ onto $|w| \ge 1$.

10. Show that the relation $w = \frac{5-4z}{4z-2}$ transforms the circle $|z| = 1$, into a circle of radius unity in the w-plane and find the centre of the circle.

11. Determine the image in the w-plane of a circle in the z-plane under the transformation $w = \frac{1}{z}$.

12. Prove that the equation

$$\left|\frac{z-a}{z-b}\right| = k,$$

k is a non-negative parameter $\neq 1$, represents a family of circles for every member of which a and b are inverse points.

13. Show that the equations

$$\left|\frac{z-a}{z-b}\right| = \lambda, \quad \arg\left|\frac{z-a}{z-b}\right| = \alpha$$ where λ and α are variable parameters, represent two orthogonal families of coaxal circles.

14. Describe briefly the Riemann surface for the function $f(z) = \log z$.

15. Discuss the Riemann surface for the double-valued function $w = z^{1/2}$.

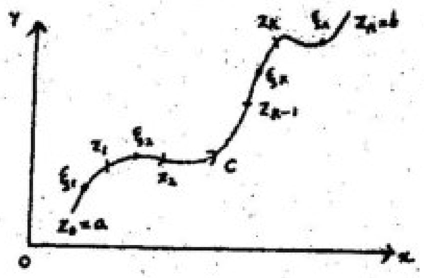
UNIT 2

COMPLEX INTEGRATION

Introduction

This unit will deal with the notion of integrability and integral of a complex function along an oriented curve in the complex plane followed by the fundamental theorem of function theory which was first discovered by Cauchy (1789-1857) in the year 1814. The theorems presented here constitute one of the pillar's of Mathematics and have far ranging applications.

Definition 2. 1 Let f(z) be continuous at all points of a curve C having a finite length i.e. C is a rectifiable curve.



We divide C into n parts by means of points z_0, z_1 ...., z_n, chosen arbitrarily. Also let $z_0 = a$ and $z_n = b$. On each are joining z_{k-1} to z_k (where k goes from 1 to n) we choose a point z_k and form the sum

$$S_n = f(\xi_1)(z_1 - z_0) + f(\xi_2)(z_2 - z_1) + \ldots\ldots f(\xi_n)(z_n - z_{n-1})$$

$$= \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$

Suppose the maximum value of $(z_k - z_{k-1}) \to 0$ as $n \to \infty$. Then the sum S_n tends to a fixed limit which does not depend upon the mode of subdivision and we denote this limit by

$$\int_a^b f(z)dz \text{ or} \int_c f(z)dz$$

which is called the **complex line integral or line integral** of f(z) along C. an evaluation of integral by such method is also called **ab-initio method.** Thus

$$\int_c f(z)dz = \lim_{n \to \infty} \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) \ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

Note 4.1. Properties of integrals

If f(z) and g(z) are integrable along C, then

1. $\int_c \{f(z)+g(z)\}dz = \int_c f(z)dz + \int_c g(z)dz.$

2. $\int_c Af(z)dz = A\int_c f(z)dz$ where A is any constant.

3. $\int_a^b f(z)dz = -\int_b^a f(z)dz$

4. $\int_a^b f(z)dz = \int_a^m f(z)dz + \int_m^b f(z)dz$ where the points a, b, m, are on C.

5. $\left|\int_c f(z)dz\right| \le ML$ where $|f(z)| \le M$ i.e. M is an upper bound of $|f(z)|$ on C, and L is the length of C.

Theorem 2.1. Cauchy's Theorem

If f(z) is an analytic function of z and if f'(z) is continuous at each point within and on a closed contour C then

$$\int_c f(z)dz = 0$$

Proof : Let R be the region which consists of all points within and on the contour C. Now if P(x , y), Q(x , y),

$\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are all continuous functions of x and y in R then by Green's theorem we have

$$\int_c (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \quad \text{.................(2.2)}$$

Since f(z) = u + iv is continuous on the simple curve C and f '(z) exists and is continuous in , therefore u, v, $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are all continuous in R. Thus the conditions of Green's theorem are satisfied and hence $\int_c f(z)dz = \int_c (u+iv)(dx+idy)$

$= \int_c \{(udx - vdy) + i(vdx + udy)\} = -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$. Thus $\int_c f(z)dz = 0$

Note 2.2 It was first shown by Goursat in 1900 that if is unnecessary to assume the continuity of $f'(z)$ and that Cauchy's theorem holds if we only assume that $f'(z)$ exists at all points within and on C.

Theorem 2.2 Cauchy- Goursat Theorem

If a function f(z) is analytic and single valued inside and on a simple closed contour C, then $\int_c f(z)dz = 0$.

Proof : We shall first prove two lemmas :

Lemma 1. If C is a closed contour, $\int_c dz = 0, \int_c zdz = 0$. These results both follow from the definition of the integral, for

$$\int_c dz = \lim_{n\to\infty}\sum_{k=1}^{n}(z_k - z_{k-1})\cdot 1 = 0 \text{ as max } |z_k - z_{k-1}| \to 0 \text{........(2.4)}$$

$$\int_c zdz = \lim_{n\to\alpha}\sum_{k=1}^{n} z_k(z_k - z_{k-1}) = \lim_{n\to\alpha}\sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1})$$

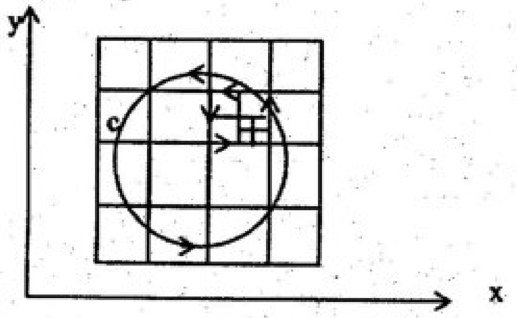
$$= \frac{1}{2}\lim_{n\to\alpha}\sum_{k=1}^{n}\{(z_k + z_{k-1})(z_k - z_{k-1})\} = \frac{1}{2}\lim_{n\to\alpha}\sum_{k=1}^{n}\{z_k^2 - z_{k-1}^2\} = 0 \text{ ....(2.5)}$$

Lemma 2. Goursat lemma: Given $\varepsilon > 0$, it is possible to divide the region inside C into a finite number of meshes, either complete squares C_n, such that, with each mesh there exists a point Z_0 for which

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon \;\; \forall \text{ z in the mesh } \text{ ....(2.6)}$$

Proof : Suppose the lemma is false, then however the interior of C is subdivided, there will he at least one mesh for which (2.6) is untrue.

We shall show that this necessarily implies the existence of a point within or on C at which f(z) is not differentiable.



We subdivide the mesh [for which (2.6)is false] by means of lines joining the middle points of the opposite sides. If there is still at least one part which does not satisfy the condition (2.6), again subdivide this part in the same manner. This process comes to end after a finite number of steps, or the process may go on indefinitely. In the second case, we obtain a sequence of squares (each contained in the preceding one) whose limit point is z_0 which lies inside C and at which the condition (2. 6) is not satisfied. Since the condition (2.6) is not satisfied at z_0, therefore

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right| \text{not less than } \varepsilon \text{ when} \left|z-z_0\right|<\delta,$$

δ being a small number depending on ε.

This shows that f (z) is not differentiable at z_0 so that f (z) is not analytic at z_0. This contradicts the hypothesis that f(z) is analytic. Hence the lemma is true i.e.

$$\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)=\eta(z) \text{ where} |\eta|<\varepsilon$$

and $\eta \to 0$ as $z \to z_0$.

Thus $f(z)=(z-z_0)\eta(z)+f(z_0)+(z-z_0) f'(z_0)$ ....(2.7)

Proof of the Theorem

We devide the interior of C into squares $C_1, C_2, \ldots\ldots\ldots C_n$ and partial squares D_1, $D_2, \ldots\ldots D_m$ in each of which (4.6)is satisfied.

We consider the integral $\sum_{r=1}^{n}\int_{C_r} f(z)dz+\sum_{r=1}^{m}\int_{D_r} f(z)dz$

where the path of every integral being in anti-clockwise direction.

In the complete sum, integration along each straight side of each square (whether complete or partial)happens to be taken twice in opposite directions and so all the integrals along straight sides of squares cancel. The integrals which remain, are taken along curved boundaries of partial squares because these are described only once. The integrals which are left behind sum to $\int_c f(z)dz$.

Thus $$\int_c f(z)dz=\sum_{r=1}^{n}\int_{C_r} f(z)dz+\sum_{r=1}^{m}\int_{D_r} f(z)dz \qquad \text{.............(2.8)}$$

In view of (2.7),

$$\int_c f(z)dz=\int_{C_r}\left[f(z_0)+(z-z_0)\eta+(z-z_0)f'(z_0)\right]dz$$

$$= \int_{c_r} [f(z_0) - z_0 f'(z)] \int_{c_r} dz + f'(z_0) \int_{c_r} z dz + (z - z_0)\eta(z) dz = \int_{c_0} (z - z_0)\eta dz$$

Thus (2.8)becomes

$$\int_c f(z) dz = \sum_{r=1}^{n} \int_{c_r} (z - z_0)\eta dz + \sum_{r=1}^{m} \int_{D_r} (z - z_0)\eta dz.$$

$$\therefore \left| \int_c (z) dz \right| \leq \sum_{r=1}^{n} \left| \int_{c_r} (z - z_0)\eta dz \right| + \sum_{r=1}^{n} \left| \int_{D_r} (z - z_0)\eta dz \right|$$

$$< \sum_{r=1}^{n} \varepsilon \int_{c_r} |z - z_0||dz| + \sum_{r=1}^{m} \varepsilon \int_{D_r} |z - z_0||dz| \qquad \text{..........(2.9)} \qquad [as|\eta| < \varepsilon]$$

Let ℓ_n ,A_n be respectively the length of the side and area of square C_n. similarly, ℓ'_n ,A_n',denote respectively length and area of square D_r .then (2.9)becomes.

$$\left| \int_c f(z) dz \right| < \sum_{r=1}^{n} \varepsilon \ell_r \sqrt{2} \int_{c_r} |dz| + \sum_{r=1}^{m} \varepsilon \ell_r \sqrt{2} \int_{D_r} |dz|$$

[since $|z - z_0| \leq \ell_r \sqrt{2}$ = diagonal of square c_r $< \sum_{r=1}^{n} \varepsilon \ell_r \sqrt{2}.4 \ell_r + \sum_{r=1}^{m} \varepsilon \ell'_r \sqrt{2}(4\ell'_r + S_r)$

where S_r is that length of the are C which forms curved boundary of D_r .

$$= 4\varepsilon\sqrt{2} \left[\sum_{r=1}^{n} A_r + \sum_{r=1}^{m} A'_r \right] + \varepsilon\sqrt{2} \sum_{r=1}^{m} \ell'_r S_r$$

Thus $\left| \int_c f(z) dz \right| < 4\varepsilon\sqrt{2}A + \varepsilon\sqrt{2}\, \ell \sum_{r=1}^{m} S_r,\; L'_r < \ell$

and A is the combined area of squares of length ℓ with which the region was originally covered.

Again, let $L = \sum_{r=1}^{m} S_r$ be the total length of boundary of C therefore,

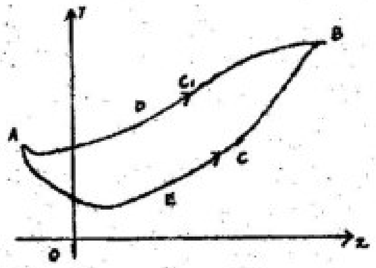
$\left| \int_c f(z) dz \right| < 4\varepsilon\sqrt{2}A + \varepsilon\sqrt{2}\, \ell L = \varepsilon\left[4\sqrt{2}.A + \ell.\sqrt{2}.L\right]$. Since ε is arbitrary and so making

$\varepsilon \to 0$we get $\int_c f(z) dz = 0$

Definition 2.2 A region R is called simply-connected if any simple closed curve which lies in R can be shrunk to a point without leaving R. A region which is not simply-connected is called multi-connected.

Theorem 2. 3 : Extension of Cauchy's Theorem

Theorem 2. 3.A : If f(z) is analytic in a simply connected region R then the integral along any rectifiable curve in R joining any two given points of R is the same i.e. it does not depend



upon the curve joining two points .

Proof : Let the two points A and B of the simply connected region R arc joined by the curves C_1 and C_2 as shown in figure.

By Cauchy's Theorem,

$$\int_{ADBEA} f(z)dz = 0 \qquad \text{or,} \int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0$$

Hence

$$\int_{ADB} f(z)dz = -\int_{BEA} f(z)dz = \int_{AEB} f(z)dz$$

Thus,

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz = \int_a^b f(z)dz$$

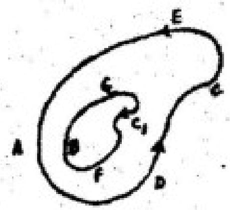
This yields the required result.

Theorem 2. 3.B Let f(z) be analytic in a region bounded by two simple closed curves C and C_1(where C_1 lies within C) and on these curves then

$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

where C and C_1 are bothe traversed in the positive sense relative to their interiors.

Proof : Let us construct cross cut AB. Then since f(z) is analytic in the region R, we have by



Cauchy's Theorem,

$$\int_{ADEABGFBA} f(z)dz = 0 \quad \text{or,} \int_{ADEA} f(z)dz + \int_{AB} f(z)dz + \int_{BGFB} f(z)dz + \int_{BA} f(z)dz = 0$$

$$\text{or,} \int_{ADEA} f(z)dz = -\int_{BGFB} f(z)dz = \int_{BFGB} f(z)dz \qquad \text{or,} \int_{C} f(z)dz = \int_{C_1} f(z)dz$$

Theorem 2.3.C Let f(z) be analytic in a region bounded by the non-overlapping simple closed curves C, $C_1, C_2, \ldots, C_n$(where C, $C_1, C_2, \ldots, C_n$ are inside C) and on these curves then

$$\int_{C} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

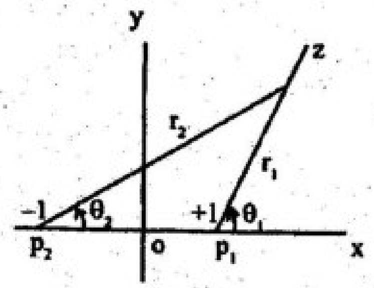
where integral along each curve is taken in the anti clockwise direction.

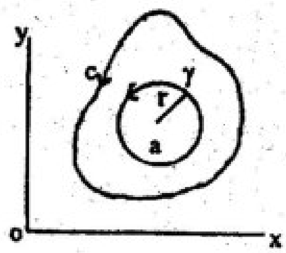
Theorem 2.4 Cauchy's integral formula

If f(z) is analytic within and on a closed contour C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_{0} \frac{f(z)dz}{z-a}$$

Proof : If f(z) is analytic within and on a closed contour C and if a is any interior point of C.





To prove the Theorem we describe a circle γ about the centre $z = a$ of small radius r lying entirely within C. In the region between C an γ the function $\phi(z) = \frac{f(z)}{z-a}$ is analytic . Hence by Cauchy's Theorem for multi-connected region we have

$$\int_C \frac{f(z)dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a}$$

$$\int_C \frac{f(z)dz}{z-a} - \int_\gamma \frac{f(a)dz}{z-a} = \int_\gamma \frac{f(z)dz}{z-a} - \int_\gamma \frac{f(a)dz}{z-a} = \int_\gamma \frac{f(z)-f(a)}{z-a}dz$$

Now, since f(z) is analytic within C and so it is continuous at $z = a$ so that given $\varepsilon > 0$, $\exists$,$\delta > 0$ such that $|f(z) - f(a)| < \varepsilon$ for $|z - a| < \delta$. ....(2.11)

If we take $r < \delta$ then (2.11) is satisfied $\forall$ z on the circle γ. For any point z on γ

$$z - a = r\,e^{i\theta}$$

$$\therefore \int_\gamma \frac{f(a)}{z-a}dz = \int_0^{2\pi} \frac{f(a)}{re^{i\theta}} ire^{i\theta}d\theta = 2\pi i f(a)$$

Therefore from (2.10) we have

$$\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a) = \int_\gamma \frac{f(z)-f(a)}{z-a}dz$$

$$\therefore \left|\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a)\right| \le \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}|dz| < \frac{\varepsilon}{r}\int_\gamma |dz| = \frac{\varepsilon}{r}2\pi r = 2\pi\varepsilon$$

i.e. $\therefore \left|\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a)\right| \le 2\pi\varepsilon$

Since ε is arbitrary so making $\varepsilon \to 0$ we get

$$\int_C \frac{f(z)}{z-a}dz - 2\pi i f(a) = 0. \text{ So } \quad f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Theorem 2.5 If f(z) is analytic in a region R, then its derivative is given by

$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$ where C is any simple closed contour in R surrounding the point $z = a$.

Proof : Let a + h be a point in the neighbourhood of the point a, then by Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

and $$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+h)}dz$$

$$f(a+h) - f(a) = \frac{1}{2\pi i}\int_C \left[\frac{1}{z-a-h} - \frac{1}{z-a}\right] f(z)dz$$

$$= \frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a-h)(z-a)}dz$$

i.e. $$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a-h)(z-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2} + \frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2(z-a-h)} \quad ...(2.12)$$

Now the result follows on taking the limit as $h \to 0$ if we can show that the last term of (2.12) approaches to zero. Let us consider a circle Γ of radius ε and a centre a which lies entirely in R, then

$$\frac{h}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2(z-a-h)} = \frac{h}{2\pi i}\int_\Gamma \frac{f(z)dz}{(z-a)^2(z-a-h)}$$

Now, we choose h so small in absolute value that a + h lies in Γ then

$$|z-a-h| \geq |z-a| - |h|$$
$$= \varepsilon - |h|$$

Again, since f(z) is analytic in R, we can find a possible number M, such that $f(z) \leq M$. Then since the length of $2\pi\varepsilon$, we have

$$\left|\frac{h}{2\pi i}\int_\Gamma \frac{f(z)dz}{(z-a)^2(z-a-h)}\right| \leq \frac{|h|}{2\pi}\frac{M2\pi\varepsilon}{\varepsilon^2(\varepsilon-|h|)} = \frac{|h|}{(\varepsilon-|h|)\varepsilon}$$

which approaches to zero as $h \to 0$.

Thus when $h \to 0$, then from (2.12) we have

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

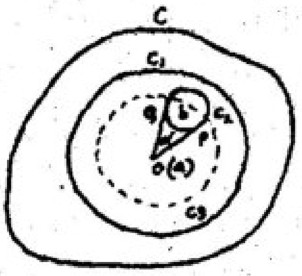
Theorem 2.6 If f(z) is analytic within and on a closed contour C and 'a' is any point with C then derivatives of all orders of f(z) are analytic and given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz, n = 0,1,2,...$$

Theorem 2.7 Maximum Modulus Theorem

If f(z) is analytic inside and on a simple closed curve C and is not identically equal to a constant then the maximum value of $|f(z)|$ occurs on C.

Proof : Since f(z) is analytic inside and on C therefore f(z) is continuous inside and on C .



Consequently $|f(z)|$ attains the maximum value M at some point inside or on C. We want to show that $|f(z)|$ attains the value M at a point lying on the boundary of C and not inside C.

Suppose if possible, this value is not attained on the boundary of C but is attained at a points z = a inside C so that

$$\max|f(z)| = |f(a)| = M$$

Let C_1 be a circle with 'a' as centre lying within C. Now f(z) is not constant and its continuity implies that there exists a point 'b' inside C such that

$$|f(b)| < M \quad \text{or,} \ |f(b)| = M - \varepsilon \text{ where } \varepsilon > 0$$

Again, by the continuity of $|f(z)|$ at b, we see that for $\varepsilon > 0$ we can find the $\delta > 0$ such that

$$\big||f(z)| - |f(b)|\big| < \frac{\varepsilon}{2} \quad \text{whenever} |z - b| < \delta$$

i.e. $|f(z)| < |f(b)| + \frac{\varepsilon}{2} \;\; = M - \varepsilon + \frac{\varepsilon}{2} \;\; = M - \frac{\varepsilon}{2}$

i.e. $|f(z)| < M - \frac{\varepsilon}{2} \ \forall z$ such that $|z - b| < \delta$ for all points interior to a circle C_2 with centre at b and radius δ.

Again , we draw a circle C_3 with centre at a (and which passes through the point b) and radius $|b - a| = r$. Therefore by Cauchy integral formula

$$f(a)=\frac{1}{2\pi i}\int_{C_3}\frac{f(z)}{(z-a)}dz \qquad \text{on } C_3 \text{ where } |z-a|=r \text{ i.e. } z-a=re^{i\theta}$$

$$\therefore f(a)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}f(a+re^{i\theta})d\theta$$

If we measure θ in counter clockwise direction from OP and if $\angle POQ=\alpha$ then

$$f(a)=\frac{1}{2\pi}\left[\int_0^{\alpha}f(a+re^{i\theta})d\theta+\int_{\alpha}^{2\pi}f(a+re^{i\theta})d\theta\right]$$

$$\therefore |f(a)|\le\frac{1}{2\pi}\int_0^{\alpha}\left|f(a+re^{i\theta})\right|d\theta+\frac{1}{2\pi}\int_{\alpha}^{2\pi}\left|f(a+re^{i\theta})\right|d\theta$$

$$<\frac{1}{2\pi}\int_0^{\alpha}(M-\frac{\varepsilon}{2})d\theta+\frac{1}{2\pi}\int_0^{\alpha}Md\theta=\frac{\alpha}{2\pi}(M-\frac{\varepsilon}{2})+\frac{M}{2\pi}(2\pi-\alpha)=M-\frac{\alpha\varepsilon}{4\pi}$$

i.e. $|f(a)|<M-\frac{\alpha\varepsilon}{4\pi}$ i.e. $M=f(a)<M-\frac{\alpha\varepsilon}{4\pi}$ which is a contradiction. Thus we can conclude that $|f(z)|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Theorem 2.8 Minimum Modulus Theorem

If f(z) is analytic inside and m a simple closed curve C and $f(z)\neq 0$ inside C, then $|f(z)|$ assumes its minimum on C.

Proof : Since f(z) is analytic inside and on a closed curve C and $f(z)\neq 0$ inside C therefore $\frac{1}{f(z)}$ is analytic within C. By maximum modulus Theorem $\frac{1}{|f(z)|}$ cannot assume its maximum value inside C so that $|f(z)|$ cannot assume its minimum value inside C. Then since $|f(z)|$ has a minimum therefore this minimum must attained on C.

Theorem 2.9 Morera's Theorem (Converse of Cauchy's Theorem)

If f(z) is continuous in a simply connected region R and if $\int_c f(z)dz=0$ around every simple closed curve in R, then f(z) is analytic in R.

Proof: Let 'a' be a fixed point and z be a variable point in the region R, then the value of the integral $\int_a^z f(u)du$ is independent of the path joining a and z, so long this path is in R. Let $\phi(z)=\int_a^z f(u)du$. We shall show that $\phi(z)$ is analytic function in R.

As z is a point of the region R, there exists a circle Γ with the centre at z which entirely lies in R. Also, let $z + h$ be any point in the neighbourhood of z.

Therefore we have,

$$\phi(z+h)-\phi(z)=\int_a^{z+h} f(u)du-\int_a^{z} f(u)du$$

$$=\int_a^{z+h} f(u)du+\int_z^{a} f(u)du=\int_a^{z+h} f(u)du \qquad \text{....(2.13)}$$

This integral being independent of the curve joining z to z + h. We take the same along the straight segment from z to z + h.

$$\therefore \frac{\phi(z+h)-\phi(z)}{h}-f(z)=\frac{1}{2}\left[\int_z^{z+h} f(u)du-\int_z^{z+h} f(z)dz\right]$$

$$=\frac{1}{h}\int_z^{z+h}[f(u)-f(z)]du \qquad \text{...(2.14)}$$

Let $\varepsilon>0$ be any given number. As f(u) is continuous at z therefore $\exists, \delta>0$ such that

$$|f(u)-f(z)|<\varepsilon \text{ for } |u-z|<\delta.$$

Now, we suppose that $|h|<\delta$, we see that for every point u on the line segment joining z to $z+h$, we have

$$|f(u)-f(z)|<\varepsilon \qquad \text{...(2.15)}$$

$\therefore$ when $|h|<\delta$, then from (2.14) and (2.15) we have

$$\left|\frac{\phi(z+h)-\phi(z)}{h}-f(z)\right|=\left|\frac{1}{h}\int_z^{z+h}[f(u)-f(z)]du\right|<\frac{\varepsilon}{|h|}\cdot|h|$$

which tend to zero as $\varepsilon \to 0$.

Thus we have,

$$\lim_{h\to 0}\frac{\phi(z+h)-\phi(z)}{h}=f(z) \text{ so that } \phi'(z) \text{ exists and } \phi'(z)=f(z)$$

This equality holds for every $z \in R$.

Now, since the derivative of $\phi(z)$ exists therefore $\phi(z)$ is analytic in R. But we know that the derivative of analytic function is analytic therefore $\phi'(z)$ i.e. f(z) is analytic in R.

Definition 2.3 A function $\phi(z)$ is said to be indefinite integral of a function f(z) in a region R, if for every $z \in R$, $\phi'(z) = f(z)$.

Theorem 2.10 Cauchy's Inequality

If f(z) is analytic within and on a circle C of radius r and centre at z = a then

$$|f^{(n)}(a)| \le \frac{Mn!}{r^n}, n = 0, 1, 2, ...$$

where M is a constant such that $|f(z)| \le M$ on C.

Proof : By Cauchy's integral formula,

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

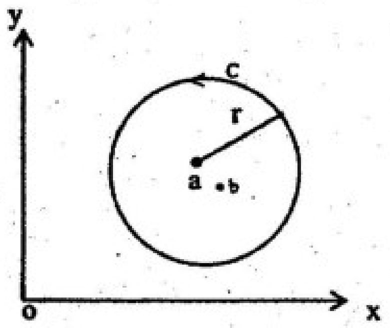
Now, $$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}\right| \le \frac{n!}{2\pi}\int_0^{2\pi} \frac{Mrd\theta}{r^{n+1}} \qquad [\because |z-a| = r]$$

$$= \frac{n!M \cdot 2\pi}{2\pi r^n} = \frac{n!M}{r^n}, \; n = 0, 1, 2, ...$$

Theorem 2.11 Liouville's Theorem

If for all z in the entire complex plane, (i) f(z) is analytic and (ii) f(z) is bounded, i.e. $|f(z)| < M$ for some constant M, then f(z) must be a constant.

Proof : Let a and b be two arbitrary distinct points in z-plane. Also let C be a circle of radius r having centre a which encloses the point b.



Now by Cauchy's Theorem ,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} \quad , f(b) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-b}$$

$$f(b)-f(a)=\frac{1}{2\pi i}\int_C\left[\frac{1}{z-b}-\frac{1}{z-a}\right]f(z)dz$$

$$=\frac{b-a}{2\pi i}\int_C\frac{f(z)dz}{(z-a)(z-b)}$$

Now, we have

$|z-a|=r$. Again, $|z-b|=|z-a+a-b|\geq|z-a|-|a-b|=r-|a-b|>r/2$

If we choose r so large that $|a-b|<\frac{r}{2}$

Thus $|f(b)-f(a)|=\left|\frac{b-a}{2\pi i}\int_C\frac{f(z)dz}{(z-b)(z-a)}\right|<\frac{|b-a|}{2\pi}\frac{M2\pi r}{\frac{1}{2}\cdot r}=\frac{2|b-a|M}{r}$

Taking $r\to\infty$ we see that

$$|f(b)-f(a)|=0 \qquad \text{or,} \quad f(b)=f(a)$$

which shows that f(z) must be constant.

Theorem 2.12: Fundamental Theorem of Algebra

Every polynomial of degree $n\geq 1$ has at least one zero.

Proof: Let $f(z)=a_0+a_1z+a_2z^2+\ldots+a_nz^n$, $a_n\neq 0$ be any polynomial.

Let, if possible f(z) has no zero i.e. $f(z)\neq 0\ \forall\ z\in C$, then $\frac{1}{f(z)}$ is analytic in every domain.
Now we have

$$\frac{1}{f(z)}=\frac{1}{a_0+a_1z+a_2z^2+\cdots+a_{n-1}z^{n-1}+a_nz^n}$$

$$=\frac{1}{z^n}\cdot\frac{1}{a_n+\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}}\to 0 \text{ as } z\to\infty$$

Thus to every $\varepsilon>0$, there corresponds a $R>0$ such that $\left|\frac{1}{f(z)}\right|<\varepsilon$ for $|z|>R$.

Also, $\frac{1}{f(z)}$ is continuous in the bounded closed domain $|z|\leq R$ and as such it is bounded therein and accordingly there exists a number M such that $\left|\frac{1}{f(z)}\right|<M$ for $|z|\leq R$.

Thus we see that $\left|\frac{1}{f(z)}\right|<\text{Max}(M,\in)$ for every z.

Hence by Liouville's Theorem $\frac{1}{f(z)}$ is constant so that we arrive at an absurd conclusion. Thus our assumption is wrong and the polynomial f(z) must have at least one zero.

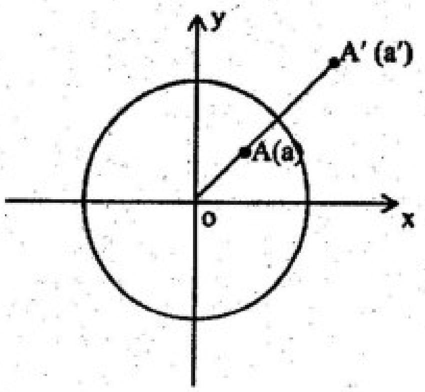
Corollary 2.1 : Every polynomial equation

$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n = 0$ where $n \geq 1$ and $a_n \neq 0$ has exactly n roots.

Theorem 2.13: Poission's Integral Formulae for a circle

If f(z) is analytic within and on a circle C defined by $|z| = R$ and if a is a point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{R^2 - a\bar{a}}{(z-a)(R^2 - z\bar{a})} f(z)dz$$



Proof: Let f(z) be analytic within and on the circle C defined by $|z| = R$. Let A be any point inside C so that $a = re^{i\theta}$, $0 < r < R$. Now the inverse point of A w.r.t. the circle C is given by $a' = \frac{R^2}{\bar{a}}$ which lies outside the circle C and is denoted by the point A′.

Now by Cauchy integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a} \quad \dots(2.17)$$

Since f(z) is analytic within and on the circle C therefore $\frac{f(z)}{z-a'}$ is also analytic within and on the circle and hence by Cauchy's Theorem ,

$$\int_C \frac{f(z)dz}{z-a'} = 0 \quad \dots(2.18)$$

$\therefore$ from (2.17) and (2.18),

$$f(a)=\frac{1}{2\pi i}\left[\int_C\frac{f(z)dz}{z-a}-\int_C\frac{f(z)dz}{z-a'}\right]=\frac{1}{2\pi i}\int_C\frac{a-a'}{(z-a)(z-a')}f(z)dz$$

$$=\frac{1}{2\pi i}\int_C\frac{a\bar{a}-R^2}{(z-a)(z\bar{a}-R^2)}f(z)dz$$

$$=\frac{1}{2\pi i}\int_C\frac{R^2-a\bar{a}}{(z-a)(R^2-z\bar{a})}f(z)dz \qquad \text{...(2.19)}$$

SOLVED PROBLEMS

Example 1. Find the value of the integral

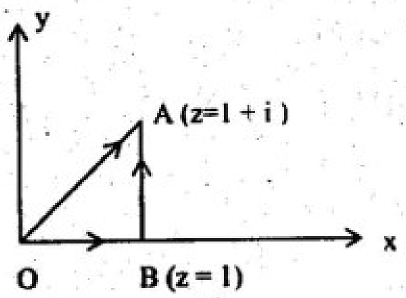
$$\int_0^{1+i}(x+y+ix^2)dz$$

(i) along the straight line from $z = 0$ to $z = 1 + i$

(ii) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + i$.

Solution: $\because z = x + iy$

$$\therefore dz = dx + idy$$



(i) OA is a straight line joining $z = 0$ to $z = 1 + i$.

Clearly $y = x$ on OA

$$\therefore dy = dx$$

and $$\int_{OA}(x-y+ix^2)dz=\int_0^1(x-x+ix^2)(dx+idx)$$

$$=i(1+i)\int_0^1 x^2 dx$$

$$=\frac{i(1+i)}{3}=\frac{i-1}{3} \qquad \text{Ans.}$$

(ii) The real axis from $z=0$ to $z=1$ is the line OB, $y=0$ on OB and so $z=x$, $dz=dx$.

Thus $\int_{OB}(x-y+ix^2)dz=\int_0^1(x-0+ix^2)dx$

$$=\int_0^1(x+ix^2)dx$$

$$=\frac{1}{2}+\frac{i}{3}$$

Again, BA is the line parallel to imaginary axis from $z=1$ to $z=1+i$.

Now, $x=1$ on BA so that $dx=0$, $dz=idy$ on BA.

$$\therefore \int_{BA}(x-y+ix^2)dx=\int_0^1(1-y+i)dy$$

$$=-1+\frac{i}{2}$$

Thus $\therefore \int_{OBA}(x-y+ix^2)dz=\int_{OB}(x-y+ix^2)dz+\int_{BA}(x-y+ix^2)dz$

$$=\left(\frac{1}{2}+\frac{i}{3}\right)\left(\frac{i}{2}-1\right)$$

$$=-\frac{1}{2}+\frac{5i}{6} \qquad \text{Ans.}$$

Example 2 Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}dz$ if C is

(i) the circle $|z|=3$

(ii) the circle $|z|=1$

Solution : Cauchy's integral formula is $f(a)=\frac{1}{2\pi i}\oint_C \frac{f(z)dz}{z-a}$ when f(z) is analytic within and on a closed contour C and a is any point within C.

If we consider the contour C as $|z| = 3$ then clearly 2 is a point inside C.

In this case, $f(z) = e^z$ therefore $f(2) = e^2$

Hence, $\frac{1}{2\pi i}\oint_C \frac{e^z dz}{z-a} = f(2) = e^2$

If we consider the circle $|z| = 1$, then clearly 2 is a point which is outside the circle $|z|=1$. If we take $f(z) = \frac{e^z}{z-2}$, then f(z) is analytic inside and on the circle $|z| = 1$. Therefore by Cauchy's Theorem

$$\frac{1}{2\pi i}\oint_C \frac{e^z dz}{z-a} = 0.$$

Example 3 If $f(z) \equiv a_k z^k + a_{k-1} z^{k-1} + \ldots + a_0$, $a_k \neq 0$, $k < 0$, be an arbitrary polynomial, then the equation

$f(z) = 0$ has a root.

Solution : If $f(z) \neq 0$ for any z, then $F(z) = \frac{1}{f(z)}$ is analytic for all finite z. Now $|F(z)| \to 0$ as $|z| \to \infty$, so that $|F(z)|$ is bounded and therefore F(z) is constant, by Liouville's Theorem.

Hence f(z) is also a constant and we thus arrive at a contradiction.

Example 4. Prove that every polynomial equation

$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0$ where the degree $n \geq 1$ and $a_n \neq 0$ has exactly n roots.

Solution : By the Fundamental Theorem of Algebra, P(z) has at least one root. Denoting this root by α we can write $P(\alpha) = 0$. Hence

$$P(z) - P(\alpha) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n - (a_0 + a_1 \alpha + a_2 \alpha^2 + \ldots + a_n \alpha^n)$$

$$= a_1 (z - \alpha) + a_2 (z^2 - \alpha^2) + \ldots + a_n (z^n - \alpha^n)$$

$$= (z - \alpha) Q(z)$$

where Q(z) is a polynomial of degree (n – 1). Applying the fundamental Theorem of algebra again, we see that Q(z) has at least one zero which we can denote by β (which may equal α) and so $P(z) = (z - \alpha)(z - \beta)R(z)$. Continuing in this manner we see that P(z) has exactly n roots.

Example 5 Give an example to show that if f(z) is analytic inside and on a simple closed curve C and f(z) = 0 at some point inside C, then $|f(z)|$ need not assume its minimum value on C.

Solution : Let f(z) = z for $|z| \le 1$, so that C is a circle with centre at the origin and radius on C. We have f(z) = 0 at z = 0. If $z = re^{i\theta}$, then $|f(z)| = r$ and it is clear that the minimum value of $|f(z)|$ does not occur on C but occurs inside C where r = 0, i.e. at z = 0.

Supplementary Problems

1. Prove that if f(z) is integrable along a curve C having finite length L and if there exists a positive number M such that $|f(z)| \le M$ on C, then

$$\left|\int_C f(z)dz\right| \le ML$$

2. If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1,1) and (2,3), find the value of

$$\int_C (12z^2 - 4iz)dz$$

3. Evaluate $\oint_C \frac{dz}{z-a}$ where C is any simple closed curve C and z = a is (i) outside C (ii) inside C.

4. Let $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$. Evaluate $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

5. If f(z) is analytic within and on a circle C defined by $|z| = R$ and if a is a point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{R^2 - a\bar{a}}{(z-a)(R^2 - z\bar{a})} f(z)dz$$ and hence deduce that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(Rr^{i\phi})d\phi$$ where $a = re^{i\theta}$ is any point inside the circle $|z| = R$.

6. The function of a real variable defined by f(x) = sinx is (a) analytic everywhere and (b) bounded i.e. $|\sin x| \le 1$ for all x but it is certainly not a constant. Does this contradict Liouville's Theorem? Explain.

7. If F(z) is analytic inside and on a simple closed curve C except for a pole of order m at z = a inside C, prove that

$$\frac{1}{2\pi i}\oint_C F(z) = \lim_{z\to a}\frac{1}{(m-i)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m F(z)\}.$$

8. If f(z) is analytic inside and on a simple closed curve C, prove that

(i) $$f'(a) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} f(a+e^{i\theta})d\theta$$

(ii) $$\frac{f^n(a)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} e^{-ni\theta} f(a+e^{i\theta})d\theta$$

9. If f(z) is analytic in a region R, prove that f'(z), f''(z), ... are analytic in R i.e. all higher derivatives exist in R.

10. Let f(z) be analytic in a region included in the circle $|z| \le R$, and let $u(r,\theta)$ be its real part. Then for $0 \le r \le R$, show that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}u(R,\phi)d\phi.$$

UNIT 3

POWER SERIES

Introduction : This unit is devoted to a consideration of functions which are analytic at all points in a bounded domain except at a finite numbers. Such exceptional points are known as singular points. In this context, a study of isolated singular points has been made and the same includes also application to the theory of algebraic equations.

Definition 3.1 Power series : A series about a is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n$.

One of the easiest example of power series is the geometric series $\sum_{n=0}^{\infty} z^n$

It is easy to see that

$$1 - z^{n+1} = (1-z)(1 + z + \ldots + z^n)$$

i.e. $1 + z + \ldots + z^n = \frac{1-z^n}{1-z}$

If $|z| < 1$ then $0 = \lim z^n$ and so the geometric series is convergent with

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $|z| > 1$ then $|z|^n = \infty$ and the series diverges . Not only is this result an archetype for what happens to a general power series, but it can be used to explore the convergence properties of power series.

Taylor's Theorem

If a function f(z) is analytic within a circle C with its centre a and radius r then at every point z inside C,

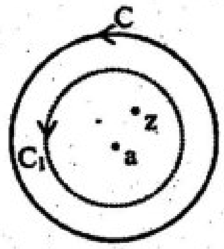
$$f(z) = f(a) + (z-a)f'(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

or, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ where $a_n \frac{f^n(a)}{n!}$

i.e. f(z) can be expressed as a power series about a.

Proof: Let z be any point inside the circle C of centre a and radius r. Construct a circle C_1 with centre at a and enclosing z(fig 3.1)



Then by Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}dw \qquad \ldots(3.1)$$

we have

$$\frac{1}{w-z}=\frac{1}{(w-a)-(z-a)}=\frac{1}{(w-a)}\left\{\frac{1}{1-{(z-a)}/{(w-a)}}\right\}$$

$$=\frac{1}{w-a}\left\{1+\left(\frac{z-a}{w-a}\right)+\left(\frac{z-a}{w-a}\right)^2+\cdots+\left(\frac{z-a}{w-a}\right)^{n-1}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{1-{(z-a)}/{(w-a)}}\right\}$$

$$\frac{1}{w-z}=\frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\cdots+\frac{(z-a)^{n-1}}{(w-a)^n}+\left(\frac{z-a}{w-a}\right)^n\frac{1}{w-z} \qquad ..(3.2)$$

Multiplying both sides of (3.2) by f(w) and using (3.1), we have

$$f(z)=\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}dw+\frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}+\cdots+\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}+U_n \qquad ..(3.3)$$

where $U_n=\frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n\frac{f(w)}{w-z}dw$

Using Cauchy's integral formulae

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw\ ,\ n=0,1,2,3,\ldots$$

(3.3) becomes $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+U_n$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this we note that since w is on C_1,

$$\left|\frac{z-a}{w-a}\right| = \gamma < 1$$

where γ is a constant. Also, we have $|f(z)| \le M$ where M is a constant, and

$$|w-z| = |(w-a)-(z-a)| \ge r_1 - |z-a|$$

where r_1 is the radius of C_1. Hence, we have

$$|U_n| = \frac{1}{2\pi}\left|\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}dw\right| \le \frac{1}{2\pi}\frac{\gamma^n M}{r_1-|z-a|}\cdot 2\pi r_1 = \frac{\gamma^n M}{r_1-|z-a|}$$

Since $0 < \gamma < 1$, we see that $U_n \to 0$ as $n \to \infty$

Hence from (3.3) we have,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + \cdots$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!} f^n(a)$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{f^n(a)}{n!}.$$

Definition 3.2 A series of the form $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is called a Laurent series. The part a_0 + $a_1(z-a) + a_2(z-a)^2 + \ldots$ is called the analytic part of the Laurent series while the remainder part $a_{-1}(z-a)^{-1} + a_2(z-a)^{-2} + \ldots$is called the principal part. If the principal part is zero then Laurent series reduces to a Taylor series.

Theorem 3.2: Laurent's Theorem :

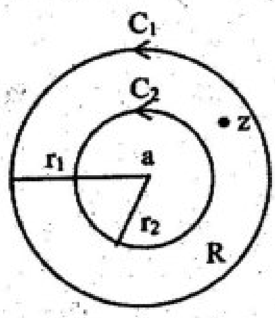
If f(z) is analytic inside and on the boundary of the ring shaped region R bounded by two concentric circles C_1 and C_2 with centre at a and respective radii r_1 and r_2 ($r_1 > r_2$), then for all z in R,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty}\frac{a_{-n}}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^{n+1}}dw$, n=0, 1, 2,...

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{-n+1}}\,dw\,,\ n=1,2,3,\ldots$$

Proof: By Cauchy's integral formula



$$f(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}\,dw \qquad \ldots(3.4)$$

Consider the first integral in (3.4), and we write

$$\frac{1}{w-z} = \frac{1}{(w-a)\left\{1-{}^{(z-a)}\!/\!_{(w-a)}\right\}}$$

$$= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \cdots + \frac{(z-a)^{n-1}}{(w-a)^n} + \left(\frac{z-a}{w-a}\right)^n \frac{1}{w-z} \qquad \ldots(3.5)$$

so that $\frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw + \frac{z-a}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw + \cdots +$

$$\frac{(z-a)^{n-1}}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw + U_n$$

$$= a_0 + a_1(z-a) + \ldots + a_{n-1}(z-a)^{n-1} + U_n \qquad \ldots(3.6)$$

where $a_0 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{w-a}\,dw\,,\ a_1 = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^2}\,dw,\ldots,a_{n-1} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(w)}{(w-a)^n}\,dw$ and

$$U_n = \frac{1}{2\pi i}\oint_{C_1}\left(\frac{z-a}{w-a}\right)^n \frac{f(w)}{w-z}\,dw$$

Let now consider the second integral in (3.4). We have on interchanging w and z in (3.5),

$$-\frac{1}{w-z} = \frac{1}{(z-a)\left\{1-{}^{(w-a)}\!/\!_{(z-a)}\right\}} = \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \cdots + \frac{(w-a)^{n-1}}{(z-a)^n} + \left(\frac{w-a}{z-a}\right)^n \frac{1}{z-w}$$

so that

$$-\frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\oint_{C_2}\frac{f(w)}{w-a}dw + \frac{1}{2\pi i}\oint_{C_2}\frac{w-a}{(z-a)^2}f(w)dw + \cdots +$$

$$\frac{1}{2\pi i}\oint_{C_2}\frac{(w-a)^{n-1}}{(z-a)^n}f(w)dw + V_n$$

$$= \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + V_n \qquad \ldots(3.7)$$

where,

$$a_{-1} = \frac{1}{2\pi i}\oint_{C_2} f(w)dw,\ a_{-2} = \frac{1}{2\pi i}\oint_{C_2}(w-a)f(w)dw, \ldots,$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2}(w-a)^{n-1}f(w)dw$$

and $V_n = \frac{1}{2\pi i}\oint_{C_2}\left(\frac{w-a}{z-a}\right)^n \frac{f(w)}{z-w}dw$

From (3.4), (3.6) and (3.7) we have

$$f(z) = \{a_0 + a_1(z-a) + \ldots + a_{n-1}(z-a)^{n-1}\} +$$

$$\left\{\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}\right\} + U_n + V_n \quad \ldots(3.8)$$

The required result follows if we can show that

$$\lim_{n\to\infty} U_n = 0 \text{ and } \lim_{n\to\infty} V_n = 0$$

Now, we have $$|U_n| = \left|\frac{1}{2\pi i}\oint_C \left(\frac{z-a}{w-a}\right)^n \frac{f(w)dw}{w-z}\right| \qquad \ldots(3.9)$$

For every point w on C_1, $\left|\frac{z-a}{w-a}\right| = m_1 < 1$

where m is a constant. Also we have $|f(w)| \le M$ where M is a constant and

$$|w-z| = |(w-a)-(z-a)| \ge |w-a| - |z-a| = r_1 - |z-a|$$

$\therefore$ From (3.9) we have $$|U_n| \le \frac{1}{2\pi}\frac{m_1{}^n M}{r_1 - |z-a|}\cdot 2\pi r_1 = \frac{m_1{}^n M r_1}{r_1 - |z-a|}$$

Since $0<m_1<1$ therefore $U_n \to 0$ as $n \to \infty$. Again, for every point w on C_2,

$$\left|\frac{w-a}{z-a}\right| = m_2 < d$$

where m_2 is a constant and

$$|z-w| = |(z-a)-(w-a)| \geq |z-a| - |w-a| = |z-a| - r_2$$

Thus $|V_n| \leq \frac{1}{2\pi}\frac{m_2{}^n M}{|z-a|-r_2} 2\pi r_2 = \frac{m_2{}^n M}{|z-a|-r_2} r_2$

Since $0< r_2 <1$ therefore $V_n \to 0$ as $n \to \infty$, Thus from (3.8) we have

$$f(z) = a_0 + a_1(z-a) + \ldots + a_{n-1}(z-a)^{n-1} + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw \quad n = 0, 1, 2, \ldots$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, \; n = 1, 2, 3, \ldots$$

Definition 3.3 A zero of an analytic function f(z) is a value of z such that f(z) = 0. An analytic function f(z) is said to have a zero of order m if f(z) can be expressed in the form $f(z) = (z-a)^m \phi(z)$, where $\phi(z)$ is analytic and $\phi(a) \neq 0$. The analytic function f(z) is said to have a zero of order z = a if z = a is a zero of order one.

Definition 3.4 A point where a given function is not analytic is called a singular point of that function. Considering the set of singular points of a function, we may see that a singular point will be an isolated point of the set , if the function is analytic at each point in some deleted neighbourhood of the point. Clearly a limiting point of the set, of singular points is itself a singular point so that the set of singular points is closed. We shall, in the following, consider isolated singular points only.

A point z = a is said to be an isolated singularity of a function f(z) if

(i) f(z) is not analytic at z = a ,

(ii) f(z) is analytic in the deleted neighbourhood of z = a.

Hence, if z be any point of this neighbourhood, then by Laurent's Theorem,

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}a_{-n}(z-a)^{-n}$$

The part $\sum_{n=0}^{\infty}a_n(z-a)^n$ is called analytic part of Laurent's series and $\sum_{n=1}^{\infty}a_{-n}(z-a)^{-n}$ is called the principal part of the expansion of f(z) at the isolated singularity z = a.

Relatively to the principal part, we have three possibilities:

(a) no term.

(b) a finite number of terms with non-zero coefficients

(c) an infinite number of terms with non-zero coefficients.

These three possibilities may be considered as follows:

(a) **Removable singularity** : If the principal part of f(z) contains no part, then the singularity z = a is called removable singularity i.e. if a single-valued function f(z) is not defined at z = a is called a removable singularity

Example: If $f(z)=\frac{\sin z}{z}$, then z = 0 is called a removable singularity.

(b) **Pole :** If the principal part of f(z) consists of a finite number of terms say m, then the singularity z = a is called a pole of order m. A pole of order one is called a simple pole i.e. If there exists a positive integer n such that $\lim_{z\to a}(z-a)^{-n}f(z)=k\neq 0$, then z = a is called a pole of order n.

Example: If $f(z)=\frac{1}{(z-2)^4(z-3)}$, then z = 2 is called a pole of order 4 and z = 3 is called a pole order one.

(c) **Essential singularity** : If the principal part of f(z) consists of an infinite number of terms, then the singularity z = a is called an essential singularity i.e. If there exists no finite value of n such that $\lim_{z\to a}(z-a)^n f(z)=C$ (finite non-zero constant), then z = a is an essential singularity.

Example: If $f(z)=e^{\frac{1}{z}}$, then z = 0 is called an essential singularity.

Definition 3.5 Meromorphic function : A function f(z) is said to be moromorphic in a finite plane if it is analytic in that plane except at a finite number of poles.

Definition 3.6 Entire function : A function which is analytic everywhere in the finite plane (i.e. everywhere except at ∞) is called an entire function or Integral function.

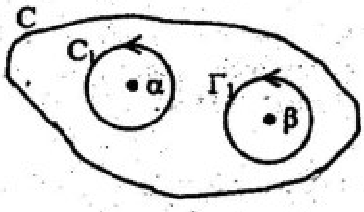
Theorem 3.3: The Argument Theorem :

Let f(z) be analytic inside and on a simple closed curve C except for a finite number of poles inside C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P$$

where N and P respectively the number of zeros and poles of f(z) inside C.

Proof : Let f(z) be analytic inside and on a simple closed curve C except for a pole $z = \alpha$ of order p inside C. Suppose also that inside C f(z) has a zero $z = \beta$ of order n.



Let C_1 and Γ_1 be non-overlapping circles lying inside C and enclosing $z = \alpha$ and $z = \beta$ respectively. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)}dz \qquad \text{...(3.10)}$$

since f(z) has a pole of order p at $z = \alpha$, we have

$$f(z) = \frac{F(z)}{(z-\alpha)^p} \qquad \text{...(3.11)}$$

where F(z) is analytic and different from zero inside and on C_1. Then taking logarithms in (3.10) and differentiating, we find

$$\frac{f'(z)}{f(z)} = \frac{F'(z)}{F(z)} - \frac{p}{z-\alpha}$$

so that $$\frac{1}{2\pi i}\oint_{C_1} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\oint_{C_1} \frac{F'(z)}{F(z)}dz - \frac{p}{2\pi i}\oint_{C_1} \frac{dz}{z-\alpha}dz = 0 - p = -p \qquad \text{...(3.12)}$$

Again, since f(z) has a zero of order n at $z = \beta$, we have

$f(z)=(z-\beta)^n G(z)$

Where G(z) is analytic and different from zero inside and on C.

Then by logarithmic differentiation we have

$$\frac{f'(z)}{f(z)} = \frac{n}{z-\beta} + \frac{G'(z)}{G(z)}$$

so that

$$\frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)} dz = \frac{n}{2\pi i}\oint_{\Gamma_1} \frac{dz}{z-\beta} dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{G'(z)}{G(z)} dz = n \qquad ...(3.13)$$

Hence from (3.10), (3.11) and (3.12) we have

$$\frac{1}{2\pi i}\oint_{C} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\oint_{C} \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f'(z)}{f(z)} dz = -p + n \qquad ...(3.14)$$

Now, we suppose that f(z) has poles of order p_m at $z = \alpha_m$ for $m = 1, 2, \ldots, r$ and f(z) has zero of order n_m at $z = \beta_m$ for m=1, 2, …,s inside C. We enclose each pole and zero by non-overlapping circles $C_1, C_2, \ldots, C_r$ and $\Gamma_1, \Gamma_2, \ldots, \Gamma_s$. Then (5.14) becomes

$$\frac{1}{2\pi i}\oint_{C} \frac{f'(z)}{f(z)} dz = -\sum_{m=1}^{r} p_m + \sum_{m=1}^{s} n_m = N - P \text{ if } \sum_{m=1}^{s} n_m = N, \sum_{m=1}^{r} p_m = P$$

Theorem 3.4 Rouche's Theorem

If f(z) and g(z) are analytic inside and a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then f(z) + g(z) and f(z) have the same number of zeros inside C.

Proof: Since the function f(z) and f(z) + g(z) analytic inside and on C therefore they have no poles inside C. Let N and N′ be the number of zeros of f(z) and f(z) + g(z) respectively inside C. Therefore using the fact that they have no poles inside C we have

$$N = \frac{1}{2\pi i}\oint_{C} \frac{f'}{f} dz \qquad \text{and} \qquad N' = \frac{1}{2\pi i}\oint_{C} \frac{f'+g'}{f+g} dz$$

$$\therefore N' - N = \frac{1}{2\pi i}\oint_{C} \left[\frac{f'+g'}{f+g} - \frac{f'}{f}\right] dz$$

Let us consider $F(z) = \dfrac{g(z)}{f(z)}$

so that $\qquad g(z) = F(z)f(z)$

Again, we have

$$|g| < |f| \Rightarrow \left|\frac{g}{f}\right| < 1 \Rightarrow |F| < 1$$

$$\therefore N' - N = \frac{1}{2\pi i}\oint_C \frac{f' + F'f + Ff'}{f + fF}dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\oint_C \left[\frac{f'}{f} + \frac{F'}{1+F}\right]dz - \frac{1}{2\pi i}\oint_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{F'}{1+F}dz$$

$$= \frac{1}{2\pi i}\oint_C F'(1 - F + F^2 - \cdots)dz$$

using the fact that $|F| < 1$ on C so that the series is uniformly convergent on C and then using Cauchy's Theorem we have

$$N' - N = 0 \qquad \text{i.e. } N' = N$$

i.e. the number of zeros of f(z) + g(z) and f(z) are equal.

Solved Problems

Example 1. Prove that $\log z = (z-1) - \frac{(z-1)^2}{2!} + \cdots, |z-1| < 1.$

Solution: Let f(z) = logz. By Taylor's Theorem ,

$$f(z) = \sum_{n=0}^{\infty}(z-a)^n \frac{f^{(n)}(a)}{n!}$$

$$= f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots$$

Taking a = 1, we get $f(1) = \log 1 = 0$,

Now, $f'(z) = \frac{1}{z}, f''(z) = -\frac{1}{z^2}, \ldots$ etc.

Thus $f(1) = 0, f'(1) = 1, f''(1) = -1, \ldots$etc.

and $f(z) = f(1) + (z-1)f'(1) + \frac{(z-1)^2}{2!}f''(1) + \ldots$

$$= (z-1) - \frac{(z-1)^2}{2!} + \ldots$$

Example 2. Obtain the expression for $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid when (i) $|z|<1$, (ii) $1<|z|<4$, (iii) $|z|>4$.

Solution: Let $f(z)=\frac{(z-2)(z+2)}{(z+1)(z+4)}=\frac{z^2-4}{z^2+5z+4}$

$$=1-\frac{5z-1}{(z+4)(z+1)}$$

$$=1-\frac{1}{1+z}-\frac{4}{z+4} \qquad \ldots(i)$$

(i) when $|z|<1$,

$$f(z)=1-(1+z)^{-1}-\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-[1-z+z^2-\ldots+(-1)^n z^n+\ldots]-$$

$$\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\cdots+(-1)^n\left(\frac{z}{4}\right)^n-\cdots\right]$$

$$=-1+[-z+z^2+\ldots+(-1)^{n+1}z^n+\ldots]-$$

$$\left[\frac{z}{4}-\left(\frac{z}{4}\right)^2-\cdots+(-1)^{n+1}\left(\frac{z}{4}\right)^n-\cdots\right]$$

$$=-1+\sum_0^{\infty}(-1)^{n+1}\left[1+4^{-n}\right]z^n$$

This is Maclaurin's series.

(ii) when $1<|z|<4$ then $\frac{1}{|z|}<1, \frac{|z|}{4}<1$

Now (i) can be expressed as

$$f(z)=1-(1+z)^{-1}-\left(1+\frac{z}{4}\right)^{-1}$$

$$=1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right]-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\cdots\right]$$

$$=\left[-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\cdots\right]-\left[-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\cdots\right]$$

$$=\sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{z^n}-\left(\frac{z}{4}\right)^n\right]$$

This is Laurent's series.

(iii) when $|z| > 4$ then $\frac{4}{|z|} < 1$.

Now (i) can be expressed as

$$f(z) = 1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}-\frac{4}{z}\left(1+\frac{4}{z}\right)^{-1}$$

$$=1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\cdots\right]-\frac{4}{z}\left[1-\frac{4}{z}+\left(\frac{4}{z}\right)^2-\cdots\right]$$

$$=1-\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^n}-\frac{4}{z}\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{z}\right)^n$$

$$=1-\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{z^{n+1}}+\left(\frac{4}{z}\right)^{n+1}\right]$$

$$=1+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}}(1+4)^{n+1}$$

Example 3. If the function f(z) is analytic and single valued in $|z-a| < R$, prove that when $0 < r < R$,

$$f'(a)=\frac{1}{\pi r}\int_0^{2\pi}P(\theta)e^{-i\theta} \text{ where } p(\theta) \text{ is real part of } f(a+re^{i\theta}).$$

Also prove that

$$\frac{f^{(n)}(a)}{n!}=\frac{1}{\pi r^n}\int_0^{2\pi}p(\theta)e^{-in\theta}d\theta.$$

Solution : Suppose f(z) is analytic inside the circle C whose equation is $|z-a| = r$ such that $0 < r < R$ and so $z-a = re^{i\theta}$.

Then f(z) can be expressed by Taylor's Theorem about z = a as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \qquad \text{...(i)}$$

$$\therefore \overline{f(z)} = \sum_{n=0}^{\infty} \bar{a}_n r^n e^{-in\theta} \qquad \text{...(ii)}$$

We note that $\int_0^{2\pi} e^{ik\theta} d\theta = 0$ if $k \neq 0$ and k is integer.

$$\text{Now } \frac{1}{2\pi i}\int_C \frac{\overline{f(z)}}{(z-a)^{n+1}} dz = \frac{1}{2\pi i}\int_0^{2\pi}\left(\sum_{m=0}^{\infty} \bar{a}_m r^m e^{-im\theta}\right)\frac{ire^{i\theta}d\theta}{r^{n+1}e^{i(n+1)\theta}}$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} \bar{a}_m r^{m-n} \int_0^{2\pi} e^{-i(m+n)\theta} d\theta$$

$$= 0 \text{ as } \left[e^{-i(m+n)\theta}\right]_0^{2\pi} = 0 \qquad \text{...(iii)}$$

$$\text{Also,} \qquad \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} \qquad \text{...(iv)}$$

Adding (iii) and (iv),

$$\frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^{n+1}} dz = \frac{f^{(n)}(a)}{n!}$$

$$\text{or,} \qquad \frac{f^{(n)}(a)}{n!} = \frac{1}{\pi i}\int_C \frac{p(\theta)d\theta}{(z-a)^{n+1}}$$

$$= \frac{1}{\pi i}\int_0^{2\pi} \frac{p(\theta) ire^{i\theta} d\theta}{r^{n+1}e^{i(n+1)\theta}}$$

$$= \frac{1}{\pi r^n}\int_0^{2\pi} p(\theta) e^{-in\theta} d\theta$$

when n = 1, then we have

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} p(\theta) e^{-i\theta} d\theta.$$

Example 4. If $a > c$, prove that the equation $az^n = e^z$ has n roots inside $|z| = 1$.

Solution : Let us consider the circle C defined by $|z| = 1$

$\therefore z = e^{i\theta}$ on C.

Given equation is $az^n - e^z = 0$. Let $f(z) = az^n$, $g(z) = -e^z$

$\therefore$ Both the function f(z) and g(z) are analytic inside and on C.

Now, $\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-e^z}{az^n}\right| = \frac{|-1||e^z|}{|az^n|}$

$\leq \frac{|e^z|}{a|z|^n}$ since a is positive and real

$= \frac{|1 + z + \frac{1}{2!}z^2 + \cdots|}{a|z|^n} \leq \frac{1 + |z| + \frac{1}{2}|z|^2 + \cdots}{a|z|^n}$

$= \frac{1 + 1 + \frac{1}{2!} + \cdots}{a} = \frac{e}{a} < 1$ as $a > 1$.

$\therefore |g(z)| < |f(z)|$

Thus the conditions of Rouche's Theorem are satisfied. Hence $f(z) + g(z) = az^n - e^z$ has n number of zeros all located at the origin . Thus f(z)+g(z) has n number of zeros inside $|z| = 1$ i.e. the equation $az^n = e^z$ has n roots inside $|z| = 1$.

Example 5. Let $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ Evaluate $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

Solution : By Argument Theorem , $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P$ where N denotes the number of zeros of f(z) inside C and P denotes the number of poles of f(z) inside C.

Now, the zeros of f(z) are given by $(z^2 + 1)^2 = 0$ i.e. $z = \pm i$

$\therefore$ the number of zeros of f(z) is 4.

Again, the poles of f(z) are given by $(z^2 + 2z + 2)^3 = 0$ i.e. $z = -1 \pm i$

$\therefore$ the number of poles of f(z) is 6. Thus $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = 4 - 6 = -2$

Supplementary Problems

1. Expand $f(z) = \sin z$ in a Taylor series about $z = \frac{\pi}{4}$

2. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for (i) $1 < |z| < 3$ (ii) $|z| > 3$, (iii) $0 < |z+1| < 2$, (iv) $|z| < 1$.

3. Find the function f(z) which is analytic throughout the circle C and its interior, whose centre is at the origin and whose radius is unity and has the value

$$\frac{a - \cos\theta}{a^2 - 2a\cos\theta + 1} + \frac{i\sin\theta}{a^2 - 2a\cos\theta + 1}, a > 1$$

and θ is the vectorial angle, at points on the circumference of C.

4. Show that

$$e^{\frac{c}{2}\left(z - \frac{1}{z}\right)} = \sum_{-\infty}^{\infty} a_n z^n \text{ where } \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)d\theta$$

5. Prove that the function $\sin\left[c\left(z + \frac{1}{z}\right)\right]$ can be expanded in a series of the type $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ in which the coefficient of both of z^n and z^{-n} are

$$\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta d\theta.$$

6. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

7. Determine the number of roots of the equation $z^8 - 4z^5 + z^2 - 1 = 0$ inside the circle $|z| = 1$.

8. Using Rouche's Theorem, show that every polynomial of degree n has exactly n zeroes.

9. Prove that one root of the equation $z^4 + z^3 + 1 = 0$ lies in the first quadrant.

10. Show that the equation $z^4 - z^3 + 4z^2 + 2z + 3 = 0$ has no roots in the first quadrant.

UNIT 4

THE CALCULUS OF RESIDUES

Introduction : This unit is devoted to the technique of evaluating certain types of real definite integrals with the help of the notions of complex integration and of residue at a point. In each given case, the choice of a suitable curve along which integration to be effected will play an important role. This curve is usually known as a **contour** and integration along the same as **Contour Integration.**

Definition 4.1: Let f(z) be a single-valued function which has a pole of order m at z = a, then by definition of pole the principal part of Laurent's expansion of f(z) contains only m terms so that we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{m}b_n(z-a)^{-n} \qquad \ldots(4.1)$$

where $a_n=\frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-a)^{n+1}}$ and $b_n=\frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-a)^{-n+1}}$

C being a circle defined by $|z-a|=r$

Evidently, $$b_1=\frac{1}{2\pi i}\oint_C f(z)dz \qquad \ldots(4.2)$$

The coefficient b_1 is called the residue of f(z) at the pole z = a. Since the value of b_1 does not depend upon the order of the pole hence (4.2) represents a general definition of the residue at a pole.

In the case, where z = a is a pole of order m then there is a simple formula for b_1 given by

$$b_1=\lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} \qquad \ldots(4.3)$$

If m = 1 (simple pole) the result is especially simple and is given by

$$b_1=\lim_{z\to a}(z-a)f(z) \qquad \ldots(4.4)$$

Definition 4.2 The residue of f(z) at infinity may also be defined. If f(z) has an isolated singularity at infinity, or is analytic there, and if C is a large circle which encloses all the finite singularities of f(z), then the residue at z = ∞ is defined to be

$$\frac{1}{2\pi i}\int_C f(z)dz$$

taken round C in the negative sense(negative with respect to the origin), provided that this integral has a definite value.

If we apply the transformation $z = \frac{1}{\zeta}$ to the integral, it becomes

$$\frac{1}{2\pi i}\int_{\Gamma} -f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}$$

taken positively round as small circle, centre the origin. It follows that if

$$\lim_{\rho\to 0}\left\{ -f\left(\frac{1}{\zeta}\right)\Big/\zeta \right\} = \lim_{z\to\infty} -zf(z)$$

has a definite value, that value is the residue of f(z) at infinity.

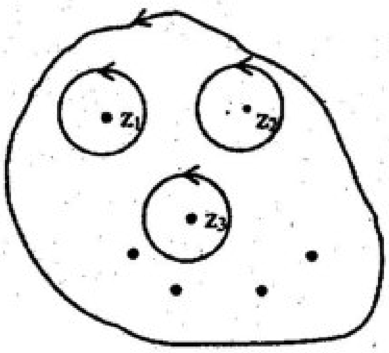
Theorem 4.1 Cauchy's Residue Theorem

If f(z) is analytic within and on a simple closed curve C except at a finite number of poles $z_1, z_2, \ldots, z_n$ within C, then

$\int_C f(z)dz = 2\pi i \sum R$ where ΣR is the sum of the residues of f(z) of the poles within C

Proof : Let $z_1, z_2, \ldots, z_n$ be the n poles of f(z) within C. We draw a set of circles C_r of radius ε and centre z_r, which do not intersect and all lie within C, provided that ε is sufficiently small. Then f(z) is analytic in the region between C and these small circles C_r. Now by Cauchy Theorem we have,

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_r} f(z)dz \qquad \ldots(4.5)$$



Again, the residue of f(z) at a pole $z = z_1$ is

$$\frac{1}{2\pi i}\int_{C_1} f(z)dz = R_1 \quad \text{i.e. } R_1 2\pi i = \int_{C_1} f(z)dz$$

The residue of f(z) at a pole $z = z_2$ is

$$\frac{1}{2\pi i}\int_{C_2} f(z)dz = R_2 \quad \text{i.e. } 2\pi i R_2 = \int_{C_2} f(z)dz$$

Similarly, the residue of f(z) at a pole $z = z_n$ is

$$\frac{1}{2\pi i}\int_{C_n} f(z)dz = R_n \quad \text{i.e. } 2\pi i R_n = \int_{C_n} f(z)dz$$

Therefore from (6.5) we have

$$\int_C f(z)dz = 2\pi i[R_1 + R_2 + \cdots + R_n]$$

$$= 2\pi i \sum R$$

where ΣR denotes the sum of residue of f(z) at its poles within C.

Theorem 4.2 If $\lim_{z\to a}\{(z-a)f(z)\} = b$ (a constant) and if C is the arc so that $\theta_1 < \arg(z - a) \leq \theta_2$, of the circle $|z-a| = r$ then

$$\lim_{r\to 0}\int_C f(z)dz = ib(\theta_2 - \theta_1)$$

Proof: Since $\lim_{z\to a}\{(z-a)f(z)\} = b$ therefore given $\varepsilon > 0$, $\exists\ \delta$ depending upon ε such that

$$|(z-a)f(z) - b| < \varepsilon \text{ for } |z-a| < \delta.$$

But $|z-a| = r$ therefore if one take $r < \delta$ then $|(z-a)f(z) - b| < \varepsilon$ on the arc C, which implies

$$(z-a)f(z) - b = \eta \quad \text{where } |\eta| < \varepsilon$$

$$\text{i.e. } f(z) = \frac{b+\eta}{z-a}$$

$$\therefore \int_C f(z)dz = \int_C \frac{b+\eta}{z-a}dz = \int_{\theta_1}^{\theta_2} \frac{b+\eta}{re^{i\theta}} ire^{i\theta} d\theta$$

$$= i\int_{\theta_1}^{\theta_2} (b+\eta)d\theta = ib(\theta_2 - \theta_1) + i\int_{\theta_1}^{\theta_2} \eta d\theta$$

$$\therefore \left|\int_C f(z)dz - ib(\theta_2-\theta_1)\right| = \left|i\int_{\theta_1}^{\theta_2}\eta d\theta\right| < \varepsilon\int_{\theta_1}^{\theta_2} d\theta = \varepsilon(\theta_2-\theta_1)$$

Taking limit as $\varepsilon\to 0$ and consequently $r\to 0$ we obtain

$$\lim_{r\to 0}\int_C f(z)dz = ib(\theta_2-\theta_1)$$

Theorem 4.3 If $\lim_{R\to\infty} z\int_C f(z)dz = b$ and if C is the arc, $\alpha\le\theta\le\beta$ of the circle $|z| = R$, then

$$\lim_{R\to\infty}\int_C f(z)dz = i(\beta-\alpha)b$$

Proof : Since $\lim_{R\to\infty} z\int_C f(z)dz = b$, hence for given $\varepsilon>0$, we can choose R so large that

$$|zf(z)-b| < \varepsilon \text{ on the arc C}$$

which implies

$$zf(z)-b=\eta \quad \text{where } |\eta|<\varepsilon$$

$$\text{or, } f(z)=\frac{b+\eta}{z}$$

$$\therefore \int_C f(z)dz = \int_C \frac{b+\eta}{z}dz = \int_\alpha^\beta \frac{b+\eta}{Re^{i\theta}} iRe^{i\theta}d\theta = i\int_\alpha^\beta (b+\eta)d\theta = ib(\beta-\alpha)+i\int_\alpha^\beta \eta d\theta$$

$$\therefore \left|\int_C f(z)dz - ib(\beta-\alpha)\right| = \left|i\int_\alpha^\beta \eta d\theta\right| < \varepsilon(\beta-\alpha)$$

Making $\varepsilon\to 0$ and consequently $R\to\infty$ we get $\lim_{R\to\infty}\int_C f(z)dz = i(\beta-\alpha)b$

Note 4.1 If $0\le\theta\le\pi/2$, then the inequality $\frac{2\theta}{\pi}\le\sin\theta\le\theta$ is known as Jordan's inequality.

Theorem 4.4 Jordan's lemma :

If C be a semi-circle, centre the origin, radius R and f(z) be subject to the conditions:

(i) f(z) is meromorphic in the upper half plane,

(ii) $f(z)\to 0$ uniformaty as $|z|\to\infty$ for $0\le \arg z\le\pi$,

(iii) m is positive, then

$$\int_C e^{miz} f(z)dz \to 0 \text{ as } R \to \infty$$

Proof : Since $f(z)\to 0$ uniformly as $|z| \to \infty$ therefore $\exists\ \varepsilon > 0$ such that

$$|f(z)| < \varepsilon\ \forall z \text{ on } C$$

If $z = Re^{i\theta}$ then $\left|e^{imz}\right| = e^{-mR\sin\theta}$ $\quad [\because \left|e^{ip}\right| = 1 \text{ for every real p}]$

$$\therefore \left|\int_C e^{imz} f(z)dz\right| < \int_0^{\pi} e^{-mR\sin\theta}\varepsilon R d\theta$$

$$\leq 2\int_0^{\pi/2} \varepsilon e^{-mR\left(2\theta/\pi\right)} R d\theta = 2R\varepsilon\left[\frac{e^{-mR\left(2\theta/\pi\right)}}{-mR\frac{2}{\pi}}\right]_0^{\pi/2} = \frac{\pi\varepsilon}{m}\left[1 - e^{-mR}\right] \to 0 \text{ as } R \to \infty$$

$$\therefore \lim_{R\to\infty} \int_C e^{miz} f(z)dz = 0, m > 0$$

Note 4.2 The Residue Theorem, although quite simple to prove, is a very useful tool which can be applied in many different situations. We now consider a number of techniques for the evaluation of various kinds of real integrals.

I Integration round the unit circle

We consider the integral of the type $\int_0^{2\pi} \phi(\cos\theta, \sin\theta)d\theta$

where $\phi(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$. If we write $z = e^{i\theta}$, then

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin\theta = \frac{1}{2\pi}\left(z - \frac{1}{z}\right), \frac{dz}{iz} = d\theta$$

and so $\int_0^{2\pi} \phi(\cos\theta, \sin\theta)d\theta = \int_0 \psi(z)dz$.

Where $\psi(z)$ is a rational function of z and C is the unit circle $|z| = 1$.

Hence $\int_C \psi(z)dz = 2\pi i \sum R$

where ΣR denotes the sum of residues of $\psi(z)$ at its poles inside C.

II Evaluation of a type of infinite integral

Let Q(z) be a function of z satisfying the conditions:

(i) Q(z) meromorphic in the upper half plane;

(ii) Q(z) has no poles on the real axis;

(iii) $zf(z) \to 0$ uniformly, as $|z| \to \infty$ for $0 \le \arg z \le \pi$;

(iv) $\int_0^{\infty} Q(x)dx$ and $\int_{-\infty}^{0} Q(x)dx$ both converge, then $\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum R^+$

where ΣR^+ denotes the sum of the residues of Q(z) at its poles in the upper half plane.

Choose as contour a semi-circle, centre the origin and radius R, in the upper half plane. Let the semicircle be denoted by Γ, and choose R large enough for the semi-circle to include all the poles of Q(z). Then by Residue's Theorem,

$$\int_{-R}^{R} Q(x)dx + \int_{\Gamma} Q(z)dz = 2\pi i \sum R^+$$

From (iii), if R be large enough, $|zQ(z)| < \varepsilon$ for all points on Γ, and so

$$\left|\int_{\Gamma} Q(z)dz\right| = \left|\int_0^{\pi} Q(Re^{i\theta})Re^{i\theta}\, i d\theta\right| < \varepsilon \int_0^{\pi} d\theta = \pi\varepsilon$$

Hence, as $R \to \infty$, the integral round Γ tends to zero. If (iv) is satisfied, it follows that

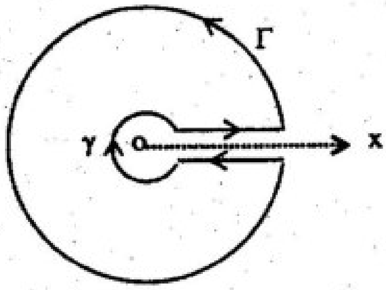
$$\int_{-\infty}^{\infty} Q(x)dx = 2\pi i \sum R^+$$

If Q(z) be a rational function of Z, it will be the ratio of two polynomials N(z)/D(z), and condition (iv) is satisfied if the degree of D(z) exceeds that of N(z) by at least two.

The theorem can be extended to the case in which D(z)=0 has non-repeated real roots, so that Q(z) has simple poles on the real axis. In this case we can exclude the poles on the real axis by enclosing them with semi-circles of small radii. This procedure is called "Indenting at a point"

III Integral involving many-valued functions.

A type of integral of the form $\int_0^{\infty} x^{a-1}Q(x)dx$, where a is not an integer, can also be evaluated by contour integration, but sine z^{a-1} is a many-valued function, it becomes necessary to use the cut plane. One method of dealing with integrals of this type is to use as contour a large circle Γ, centre the origin and radius R; but we must cut the plane along the real axis from 0 to ∞ and also enclose the branch point $z = 0$ in a small circle γ of radius R. The contour is illustrated in the fig below.



IV Use of quadrant, sector and rectangular contours.

The contours used so far have been either circles or semi-circles, and although a large semi-circle in the upper half plane is generally used for integrals of the type discussed in II, there is no special merit in a semi-circle. The rectangle with vertices $\pm R$, $\pm R+iR$ could also be used in these case. To evaluate the values of some useful integrals integrating a given function round a prescribed contour, we require sometimes a rectangle, sector or quadrant of a circle.

Note 4.3 Summation of series by the calculus of Residues:

The method of contour integration can be used with advantage for summing series of the type $\Sigma f(n)$, if $f(z)$ be a meromorphic function of a fairly simple kind,

Let C be a closed contour including the points $m, m + 1, \ldots, n$, and suppose that $f(z)$ has simple poles at the points $a_1, a_2, \ldots, a_k$ with residues $b_1, b_2, \ldots, b_k$. Consider the integral

$$\int_C \pi \cot \pi z f(z) dz$$

The function $\pi\cot\pi z$ has simple poles inside C at the points $z = m, m+1, \ldots, n$ with residue unity at each pole. The residues at these poles of $\pi\cot\pi z f(z)$ are accordingly $f(m), f(m+1), \ldots, f(n)$. Hence, by the residue Theorem

$$\int_C f(z)\pi \cot \pi z dz = 2\pi i\{f(m) + f(m+1) + \cdots + f(n) + b_1 \pi \cot \pi a_1 + \cdots + b_k \pi \cot \pi a_k\}$$

If conditions are satisfied which ensure that the contour integral tends to zero as $n\to\infty$, we can find the sum of the series $\Sigma f(n)$. Suppose that $f(z)$ is a rational function, none of whose zeros or poles are integers, such that $zf(z)\to 0$ on $|z| \to \infty$. Let C be the square with corners $\left(n+\frac{1}{2}\right)(\pm 1 \pm i)$ we have seen that $\cot\pi z$ is bounded on this square and so

$$\left|\int_C zf(z)\pi\cot\pi z\frac{dz}{z}\right| \le \frac{\pi ML\varepsilon}{R}$$ for n large enough, where M is the upper bound of $|\cot \pi z|$ on C, L is the length of C and R is the least distance of the origin from the contour. Since L=8R, the integral tends to zero as $n\to\infty$, and so

$$\lim_{n\to\infty}\sum_{m=-n}^{n} f(m) = -\pi\{b_1 \cot \pi a_1 + \cdots + b_k \cot \pi a_k\}$$

If we use πcoecπz instead of πcotπz, we can obtain similarly the sum of series of the type $\Sigma(-1)^m f(m)$.

Solved Problems

1. Find the zeros and poles of $f(z)=\frac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Solution: The zeros of f(z) are given by $z^2+4=0$ i.e. $z = \pm 2i$.

The poles of f(z) are given by

$$z^3+2z^2+2z=0 \quad \text{i.e. } z=0, -1\pm i.$$

therefore the residue of f(z) at z =0 is

$$\lim_{z\to 0}\left\{z\frac{z^2+4}{z^3+2z+2z}\right\} = \lim_{z\to 0}\frac{z^2+4}{z^2+2z+2} = 2$$

the residue of f(z)at z= -1+ i is

$$\lim_{z\to -1+i}\left[\{z-(-1+i\}\frac{z^2+4}{z^3+2z^2+2z}\right] = \lim_{z\to -1+i}\frac{z^2+4}{z\{z-(-1-i)\}} = -\frac{1}{2}(1-3i)$$

And the residue of f(z) at z =-1-i is

$$\lim_{z\to -1-i}\left[\{z-(-1-i)\}\frac{z^2+4}{z^3+2z^2+2z}\right] = \lim_{z\to -1-i}\frac{z+4}{2\{z-(-1-+i)\}} = -\frac{1}{2}(1+3i)$$

2. Let f(z)be analytic inside and on a simple closed curves C except at a pole a of order m inside C.

Prove that the residue of f(z) at a is given by

$$a_{-1} = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$$

Solution : If f(z) has a pole a of order m, then the Laurent series of f(z) is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \ldots + \frac{a_{-m}}{(z-a)^m}$$

The multiplying both sides by $(z-a)^m$, we have

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \ldots + a_{-1}(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + \ldots$$

Differentiating both sides (m-1) times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}\Big((z-a)^m f(z)\Big) = a_{-1}(m-1)! + m(m-1)\ldots 2.1 a_0 (z-a) + \ldots$$

Thus on letting $z \to a$,

$$\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}}\Big((z-a)^m f(z)\Big) = (m-1)! a_{-1}$$

Thus, $a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{(z-a)^m f(z)\right\}$

3. Prove that if a >b> 0, then

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left\{a - \sqrt{a^2 - b^2}\right\}$$

Solution : Let $z = e^{i\theta}$

$$\therefore \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \text{ and } \frac{dz}{iz} = d\theta$$

$$\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a + b\cos\theta} = \int_C \frac{\frac{1}{4i^2}\left(z - \frac{1}{z}\right)^2}{a + b \cdot \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$ where C is the unit circle $|z| = 1$

$$= \frac{i}{2b}\int_C \frac{(z^2-1)^2 dz}{z^2\left(z^2 + 2az/_{b+1}\right)} = \frac{i}{2b}\int_C \frac{(z^2-1)^2 dz}{z^2(z-\alpha)(z-\beta)} = \frac{i}{2b}\int_C F(z)dz$$

where $\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}, \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$

are the roots of the quadratic $z^2 + \frac{2az}{b} + 1 = 0$.

Since the product of the roots α, β is unity, we have $|\alpha||\beta| = 1$ where $|\beta| > |\alpha|$, and so $z = \alpha$ is the only simple pole inside C. The origin is a pole of order two. Therefore, we calculate the residues at $z = \alpha$ and $z = 0$.

The residue of f(z) at $z = \alpha$ (of order 1) is given by

$$\lim_{z\to\alpha}(z-\alpha)F(z) = \lim_{z\to\alpha}\frac{(z^2-1)^2}{z^2(z-\beta)} = \frac{(\alpha^2-1)^2}{\alpha^2(\alpha-\beta)}$$

$$= \frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{\alpha-\beta} = \frac{(\alpha-\beta)^2}{\alpha-\beta} = \alpha-\beta = \frac{2\sqrt{a^2-b^2}}{b}$$

and the residue of f(z) at $z = 0$ (of order 2) is the coefficient of $\frac{1}{z}$ in $\dfrac{(z^2-1)^2}{z^2\left(z^2+\frac{2az}{b}+1\right)}$, where

z is small.

Now $$\frac{(z^2-1)^2}{z^2\left(z^2+\frac{2az}{b}+1\right)} = \frac{\left(1-\frac{1}{z^2}\right)^2}{1+\frac{2a}{b}\frac{1}{z}+\frac{1}{z^2}}$$

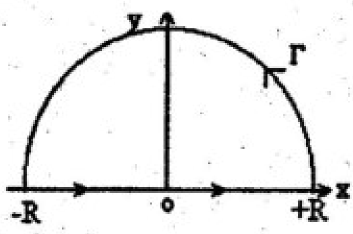
$$= \left(1-\frac{2}{z^2}+\frac{1}{z^4}\right)\left(1+\frac{2a}{b}\frac{1}{z}+\frac{1}{z^2}\right)^{-1} = \left(1-\frac{2}{z^2}+\frac{1}{z^4}\right)\left\{1-\left(\frac{2a}{b}\frac{1}{z}+\frac{1}{z^2}\right)+\cdots\right\}$$

Clearly the coefficient of $\frac{1}{z}$ is $-\frac{2a}{b}$. Hence By Cauchy residue Theorem

$$I = \frac{i}{2b}2\pi i\left[-\frac{2a}{b}+\frac{2\sqrt{a^2-b^2}}{b}\right] = \frac{2\pi}{b^2}\left[a = \sqrt{a^2-b^2}\right]$$

4. Prove that, if $a > 0$ then $$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3}$$

Solution: Let $f(z) = \dfrac{1}{x^4+a^4}$. We consider the integral $I = \int_C f(z)dz$ where C is the closed contour consisting of the real axis from –R to +R and the semi-circle Γ which is the upper



half of the large circle $|z| = R$, traversed in the positive sense as shown in fig.

Now the poles of f(z) are given by

$$z^4 + a^4 = 0$$

i.e. $z = ae^{i(2n+1)\pi/4}$, $n = 0, 1, 2, \ldots$

i.e. $z = ae^{i\pi/4}, ae^{\frac{i3\pi}{4}}, \ldots$

But the only pole lying within C are

$$\alpha = ae^{i\pi/4} \text{ and } \beta = ae^{i3\pi/4}$$

$\therefore$ The residue of f(z) at $z = \alpha$ is given by

$$\lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{(z-\alpha)}{z^4+a^4} = \lim_{z\to\alpha}\frac{1}{4z^3} = -\frac{1}{4a^3}e^{i\pi/4}$$

and the residue of f(z) at $z = \beta$ is given by

$$\lim_{z\to\beta}(z-\beta)f(z) = \lim_{z\to\beta}\frac{(z-\beta)}{z^4+a^4} = -\frac{1}{4a^3}e^{\frac{i3\pi}{4}}$$

$\therefore$ By Cauchy residue Theorem ,

$$\int_C f(z)dz = 2\pi i\left[-\frac{1}{4a^3}(e^{i\pi/4} + e^{i3\pi/4}\right] = \frac{\pi}{\sqrt{2}a^3}$$

or, $\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{\sqrt{2}a^3}$

Making $R\to\infty$, we have

$$\lim_{R\to\infty}\int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{\sqrt{2}a^3} \qquad \ldots(i)$$

Now, $\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty}\frac{z}{z^4+a^4} = 0$

$$\therefore \lim_{R\to\infty}\int_\Gamma f(z)dz = i(\pi - 0)\cdot 0 = 0$$

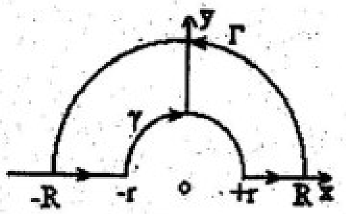
Thus from (i) we have $\int_{-\infty}^{\infty}\frac{1}{x^4+a^4}dx = \frac{\pi}{\sqrt{2}a^3}$ i.e. $\int_0^{\infty}\frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3}$

5. Prove that $\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$ if $m > 0$

and hence evaluate $\int_0^\infty \frac{\sin x}{x}dx$

Solution: Let $f(z) = \frac{e^{imz}}{z}$. We consider the integral $\int_C f(z)dz$ where the closed contour C consists of Γ, the upper half of the large circle $|z| = R$ and the real axis from $-R$ to $+R$, indented at $z = 0$ by a small circle γ of radius r. Evidently, f(z) has no singularity within C and hence by Cauchy residue Theorem

$$\int_C f(z)dz = 0$$



$$\text{i.e. } \int_r^R f(x)dx + \int_\Gamma f(z)dz + \int_{-R}^{-r} f(x)dx + \int_\gamma f(z)dz = 0 \quad \ldots(i)$$

Now, since

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} z.\frac{e^{imz}}{z} = 1$$

$$\therefore \lim_{r\to 0}\int_\gamma f(z)dz = -i(\pi - 0)\cdot 1 = -i\pi$$

Again $\lim_{z\to\infty} \frac{1}{z} = 0$

$\therefore$ By applying Jordan's lemma we have $\lim_{R\to\infty}\int_\Gamma e^{imz}\frac{dz}{z} = 0$

Now making $r\to 0$ and $R\to\infty$, we have from (i)

$$\int_0^\infty f(x)dx + \int_{-\infty}^0 f(x)dx = \pi i$$

i.e. $\int_{-\infty}^\infty f(x)dx = \pi i$ i.e. $\int_0^\infty f(x)dx = i\frac{\pi}{2}$ i.e. $\int_0^\infty \frac{e^{imx}}{x}dx = i\frac{\pi}{2}$

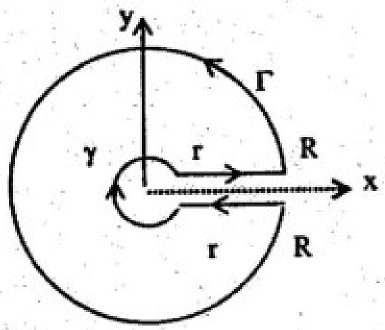
i.e. $\int_0^\infty \frac{\cos mx}{x}dx + \int_0^\infty \frac{\sin mx}{x}dx = i\frac{\pi}{2}$

Comparing the imaginary parts we have,

$\int_0^\infty \frac{\sin mx}{x}dx = \frac{\pi}{2}$, when m = 1, we have $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

6. Prove that $\int_0^\infty \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a}$ if $0 < a < 1$.

Solution : Let $f(z) = \frac{z^{a-1}}{1+z}$, $0 < a < 1$ and $I = \int_C f(z)dz$ where C is a closed contour consisting of the large circle $|z| = R$, a small circle $|z| = r$ such that a cut along the positive real axis join their ends as shown in fig.



Since $1 - a > 0$ $\therefore f(z) = \frac{1}{(1+z)z^{1-a}}$

And the poles of f(z) are given by

$$(1+z)z^{1-a} = 0 \qquad \text{i.e. } z = 0, -1.$$

Hence the only pole of f(z) within C is -1.

Residue of f(z) at z = -1 is given by $\lim_{z\to -1}\{(z+1)f(z)\} = \lim_{z\to -1}\left\{(z+1)\cdot\frac{z^{a-1}}{z+1}\right\} = (-1)^{a-1} = -e^{i\pi a}$

$\therefore$ By Cauchy's residue Theorem, $\int_C f(z)dz = -2\pi i e^{i\pi a}$

i.e. $\int_r^R f(x)dx + \int_\Gamma f(z)dz + \int_R^r f(xe^{2\pi i})d(xe^{2\pi i}) + \int_\gamma f(z)dz = -2\pi i e^{\pi i a}$ ...(i)

Now, $\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} z\frac{z^{a-1}}{1+z} = 0$ $\quad\therefore \lim_{R\to\infty}\int_\Gamma f(z)dz = i(2\pi - 0)\cdot 0 = 0$

Again, $\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z(z^{a-1})}{1+z} = 0$ $\quad\therefore \lim_{r\to 0}\int_r f(z)dz = 0$

Thus making $r\to 0, R\to\infty$, we have from (i)

$$\int_0^\infty f(x)dx + \int_\infty^0 f(xe^{2\pi i})d(xe^{2\pi i}) = -2\pi i e^{\pi i a}$$

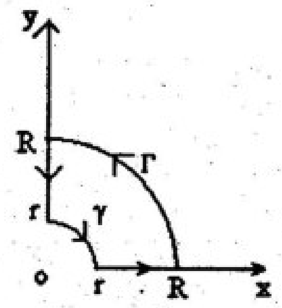
i.e. $\int_0^\infty \frac{x^{a-1}}{x+1}dx + \int_\infty^0 \frac{x^{a-1}e^{2\pi i a}}{x+1}dx = -2\pi i e^{\pi i a}$ $\quad$ i.e. $(1 - e^{2ia\pi})\left[\int_0^\infty \frac{x^{a-1}}{x+1}dx\right] = -2\pi i e^{\pi i a}$

i.e. $(e^{-ia\pi} - e^{ia\pi})\int_0^\infty \frac{x^{a-1}}{x+1}dx = -2\pi i$ $\quad$ i.e. $-2i\sin\pi a\int_0^\infty \frac{x^{a-1}}{x+1}dx = -2\pi i$ $\quad$ i.e. $\int_0^\infty \frac{x^{a-1}}{x+1}dx = \frac{\pi}{\sin\pi a}$

7. By contour integration, prove that $\int_0^\infty \frac{\sin x^2}{x}dx = \frac{\pi}{4}$

and hence deduce that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$

Solution : Let $f(z) = \frac{e^{iz^2}}{z}$ and $I = \int_C f(z)dz$, where C is the closed contour consisting of the positive quadrant Γ of a large circle $|z| = R$, and the positive real and imaginary axes indented



at $z = 0$, as shown in figure.

The pole of $f(z)$ is given by $z = 0$, but this pole is avoided by indentation.

$\therefore$ By Cauchy residue Theorem,

$$\int_C f(z)dz = 0$$

i.e. $\int_r^R f(x)dx + \int_\Gamma f(z)dz + \int_R^r f(iy)d(iy) + \int_\gamma f(z)dz = 0$ ...(i)

Now, $\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\frac{e^{iz^2}}{z} = 1$

$$\therefore \lim_{r\to 0}\int_\gamma f(z)dz = -i(\frac{\pi}{2} - 0)\cdot 1 = -i\frac{\pi}{2}$$

Again, $\lim_{R\to\infty}\int_\Gamma f(z)dz = \lim_{z\to\infty}\int_\Gamma \frac{e^{iz^2}}{z}dz$

$$= \lim_{u\to\infty}\int_\Gamma \frac{e^{iu}du}{2u} \quad \text{[letting } z^2 = u]$$

Now, since $\lim_{u\to\infty}\frac{1}{2u} = 0$ therefore by Jordan's lemma $\lim_{R\to\infty}\int_\Gamma f(z)dz = \lim_{R\to\infty}\int_\Gamma \frac{1}{2u}e^{iu}du = 0$

Thus making $r\to 0, R\to\infty$, we have from (i)

$\int_0^\infty f(x)dx + \int_\infty^0 f(iy)d(iy) = \frac{i\pi}{2}$ i.e. $\int_0^\infty \frac{e^{ix^2}}{x}dx + \int_\infty^0 \frac{e^{i(iy)^2}}{iy}idy = i\pi/2$

i.e. $\int_0^\infty \frac{e^{ix^2}}{x}dx - \int_0^\infty \frac{e^{-ix^2}}{x}dx = i\pi/2$ i.e. $\int_0^\infty \frac{e^{ix^2} - e^{-ix^2}}{x}dx = i\pi/2$

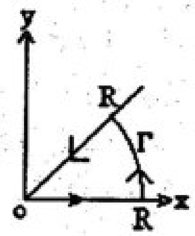
i.e. $\int_0^\infty \frac{2i\sin x^2}{x}dx = i\pi/2$ i.e. $\int_0^\infty \frac{\sin x^2}{x}dx = \pi/4$

Substituting $x^2 = t$, we have

$$\int_0^\infty \frac{\sin t}{\sqrt{t}}\frac{dt}{2\sqrt{t}} = \pi/4 \quad \text{i.e. } \int_0^\infty \frac{\sin x}{x}dx = \pi/2$$

8. Prove that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\pi/2}$

Solution: Let $f(z) = e^{iz^2}$ and $I = \int_C f(z)dz$, where C is the closed contour consisting of an arc Γ of a large circle $|z| = R$, where $0 \le \arg z \le \pi/4$ as shown in fig.



Since f(z) has no pole within C.

$\therefore$ By Cauchy's Theorem,

$$\int_C f(z)dz = 0$$

i.e. $\int_0^R f(x)dx + \int_\Gamma f(z)dz + \int_R^0 f(re^{i\pi/4})d(re^{i\pi/4}) = 0$ ...(i)

Now, $\lim_{R\to\infty} \int_\Gamma f(z)dz = \lim_{z\to\infty} \int_\Gamma e^{iz^2}dz = \lim_{u\to\infty} \int_\Gamma \frac{e^{u}du}{2\sqrt{u}}$

Now, $\lim_{u\to\infty} \frac{1}{2\sqrt{u}} = 0$ therefore by Jordan's lemma

$$\lim_{R\to\infty} \int_\Gamma f(z)dz$$

Thus making $R\to\infty$, we have from (i)

$\int_0^\infty f(x)dx + \int_\infty^0 f(re^{i\pi/4})d(re^{i\pi/4}) = 0$ i.e. $\int_0^\infty e^{ix^2}dx - \int_0^\infty e^{i\left(re^{i\pi/4}\right)^2} e^{i\pi/4}dr = 0$

i.e. $\int_0^\infty e^{ix^2}dx - \frac{1+i}{\sqrt{2}}\int_0^\infty e^{-r^2}dr = 0$ i.e. $\int_0^\infty e^{ix^2}dx = \frac{1+i}{\sqrt{2}}\int_0^\infty e^{-r^2}dr = \frac{1+i}{2\sqrt{2}}\Gamma(1/2)$

i.e. $\int_0^\infty (\sin x^2 + \cos x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i)$

Hence comparing the real and imaginary parts we have

$$\int_0^\infty \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}, \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

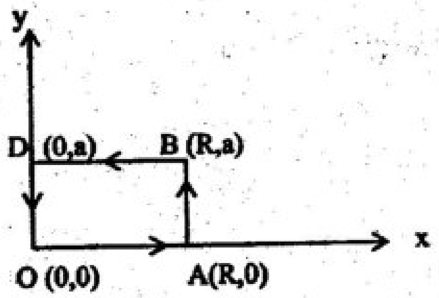
9. Prove that (i) $\int_0^\infty e^{-x^2} \cos 2ax dx = \frac{1}{2}\sqrt{\pi} e^{-x^2}$

$$\text{(ii) } \int_0^\infty e^{-x^2} \sin 2ax dx = e^{-a^2} \int_a^b e^{-x^2} dx$$

by integrating e^{-x^2} round the rectangle whose vertices are 0, R, R + ia, ia.

Solution:

Let $f(z) = e^{-z^2}$ and $I = \int_C f(z)dz$, where C is the perimeter of the rectangle whose vertices are 0, R, R + ia, ia respectively as shown in the fig.



Since f(z) has singularity within C, therefore by Cauchy's Theorem,

$$\int_C f(z)dz = 0$$

$$\text{i.e. } \int_{OA} f(z)dz + \int_{AB} f(z)dz + \int_{BD} f(z)dz + \int_{DC} f(z)dz = 0$$

$$\int_0^R f(x)dx + \int_0^a f(R+iy)d(R+iy) + \int_R^0 f(x+ia)d(x+ia) + \int_a^0 f(iy)idy = 0 \quad \text{...(i)}$$

Thus making $R \to \infty$, we have from (i)

$$\int_0^\infty e^{-x^2} dx + \lim_{R\to\infty} \int_0^a e^{-(R+iy)^2} idy + \int_\infty^0 e^{-(x+iy)^2} dx + i\int_a^0 e^{y^2} dy = 0 \quad \text{...(ii)}$$

Now, $\int_0^\infty e^{-x^2}dx = \int_0^\infty e^{-u}\frac{du}{2\sqrt{u}} = \frac{1}{2}\int_0^\infty e^{-u}u^{\frac{1}{2}-1}du = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$

Again $\int_0^a e^{-(R+iy)^2}idy = \int_0^a e^{-(R^2+2iy-y^2)}idy$

$$\left|\int_0^a e^{-(R+iy)^2}idy\right| = \left|\int_0^a e^{-(R^2+2iy-y^2)}idy\right| \le e^{-R^2}\int_0^a e^{a^2}dy \text{ for } y \le a$$

$$= ae^{-R^2}e^{a^2} \to 0 \text{ as } R \to \infty$$

and hence we have

$$\lim_{R\to\infty}\int_0^a e^{-(R+iy)^2}idy = 0$$

Again, $\int_0^\infty e^{-(x+ia)^2}dx = e^{a^2}\int_0^\infty e^{-x^2}\cdot e^{-2iax}dx$

and $\int_0^a e^{-(iy)^2}idy = i\int_0^a e^{x^2}dx$

Then we have from (ii),

$$\frac{\sqrt{\pi}}{2} - e^{a^2}\int_0^\infty e^{-x^2}\cdot e^{-2iax}dx - i\int_0^a e^{x^2}dx = 0$$

$$e^{a^2}\int_0^\infty e^{-x^2}(\cos 2axdx - i\sin 2ax)dx = \frac{\sqrt{\pi}}{2} - i\int_0^a e^{x^2}dx$$

Comparing the real and imaginary parts we have

$$e^{a^2}\int_0^\infty e^{-x^2}\cos 2axdx = \frac{1}{2}\sqrt{\pi} \qquad \text{and } e^{a^2}\int_0^\infty e^{-x^2}\sin 2axdx = \int_a^b e^{-x^2}dx.$$

10. Find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$ where $a > 0$.

Solution: Let $f(z) = \frac{1}{z^2+a^2}$

And so $zf(z) \to 0$ as $|z| \to \infty$.

The poles of f(z) are given by $z^2 + a^2 = 0$ i.e. $z = \pm ia$ and the residues at these poles are $\pm\frac{1}{2ai}$.

Hence, $$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = -\pi\left\{\frac{1}{2ai}\cot\pi ai - \frac{1}{2ai}\cot(-\pi ai)\right\}$$

$$= \frac{\pi}{2ai}\{-i\coth\pi a - i\coth\pi a\} = \frac{\pi}{a}\coth\pi a$$

i.e. $$\frac{1}{a^2} + 2\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth\pi a$$

i.e. $$\sum_{n=1}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth\pi a - \frac{1}{2a^2}$$

Supplementary Problems

1. Find the residues of $\dfrac{z^2}{(z-1)(z-2)(z-3)}$ at 1, 2, 3 and ∞ and show that their sum is zero.

2. Evaluate $\dfrac{1}{2\pi i}\oint_C \dfrac{e^{zt}}{z^2(z^2+2z+2)}dz$ around the circle C with equation $|z| = 3$.

3. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)(x^2+2x+2)} = \frac{7\pi}{50}$

4. Prove that $\displaystyle\int_0^{2\pi}\frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2} = \pi\frac{1-p+p^2}{1-p}, 0 < p < 1.$

5. Prove that if $a > 0$, $m > 0$ then

$$\int_0^{\infty}\frac{\cos mx}{(a^2+x^2)^2}dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}.$$

6. Prove that $\displaystyle\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dx = -\frac{\pi}{4}$, using as contour a large semi-circle in the upper half plane indented at $z = 0$.

7. Show that if $0 < a < 1$ then $\displaystyle\int_0^{\infty}\frac{x^{a-1}}{1-x}dx = \pi\cot\pi a$

8. By integrating $e^{iz}z^{a-1}$ round a quadrant of a circle of radius R, prove that if $0 < a < 1$ then

$$\int_0^\infty x^{a-1}\cos x\,dx = \Gamma_{(a)}\cos\frac{\pi a}{2}$$

9. By contour integration, prove that

$$\int_0^\infty \frac{\cosh ax}{\cosh x}dx = \frac{\pi}{2}\sec\left(\frac{\pi a}{2}\right) \text{if } |a| < 1.$$

10. Show that

(i) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a}\operatorname{cosech}\pi a - \frac{1}{2a^2}$ where $a > 0$.

(ii) $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots = \frac{\pi^3}{32}$

(iv) $\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth\pi a$ where $a > 0$.

11. Prove that $\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(n+a)^2} = \frac{\pi^2\cos\pi a}{2\sin^2\pi a}$

Where a is real and different from $0, \pm1, \pm2, \ldots$

12. Use the method of contour integration to prove the following results:

(i) $\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+c^2)} = \frac{\pi(b+2c)}{2bc^3(b+c)^3}, b > 0, c > 0.$

(ii) $\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right), a > b > 0.$

(iii) $\int_0^\infty\frac{x^6\,dx}{(a^4+x^4)^2} = \frac{3\sqrt{2}\pi}{16a}, a > 0.$

(iv) $\int_0^\infty\frac{\sin^2 mx\,dx}{x^2(x^2+a^2)} = \frac{\pi}{4a^3}(e^{-2ma}-1+2ma), m > 0, a > 0$

University Questions

1996

1. (a) Construct a Reimann surface for the function $z^{-\frac{1}{3}}$. Show that the Reimann surface for the function

$$f(z) = z^{\frac{1}{2}} + z^{\frac{1}{3}} \text{ has 6 sheets.}$$

(b) If u and v are harmonic in a Region R, prove that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \text{ is analytic in R.}$$

(c) If f(z) = u + iv in an analytic function of z, and if

$$u = \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x} \quad \text{find v.}$$

2. (a) Discuss the transformation

$$w = \frac{i(1-z)}{1+z}.$$

and show that it transforms the circle $|z|=1$ into the real axis of the w-plane and the interior of the circle $|z|<1$ into the upper half of the w-plane.

(b) Find the bilinear transformation which maps 1, -i, 2, on 0, 2, -i respectively.

3. (a)Find the orthogonal trajectories of the family of curves in the xy-plane defined by $e^{-x}(x\sin y - y\cos y) = \alpha$ where α is a real constant.

(b) Evaluate $\int_c (z^2 + 3z)dz$

along the circle $|z|=2$ from (2, 0) to (0, 2) in a counterclockwise direction.

(c) If c is the circle $|z|=r$, show that

$$\lim_{r\to\infty} \int_c \frac{z^2 + 2z - 5}{(z^2 + 4)\ (z^2 + 2z + 2)} dz = 0$$

4. (a) State and prove Rouche's theorem about the zeros of analytic function.

(b) Evaluate $\frac{1}{2\pi i}\int_c \frac{\cos \pi z}{z^2 - 1} dz$ around a rectangle with vertices at –i, 2-i, 2+i, i.

5. (a) If a function f(z) is analytic in the closed ring bounded by two concentric circles c_1 and c_2 of centre a and radii R_1 and R_2 ($R_2<R_1$) and if z is any point of the annulus, prove that

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

where $a_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(t)dt}{(t-a)^{n+1}}$, and $b_n=\frac{1}{2\pi i}\int_{c_2}\frac{f(t)dt}{(t-a)^{-n+1}}$.

(b) Expand $\frac{1}{z^2-3z+2}$ for

(i) $0<|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$

6 (a) Distinguish between a pole and essential singularity of a function of a complex variable z. Prove that the zeros of an analytic function are isolated points.

(b) Apply the calculus of residues to evaluate any two of the following integrals:

(i) $\int_0^{\infty}\frac{(\log x)^2}{1+x^2}dx$

(ii) $\int_0^{\infty}\frac{x^{p-1}}{1+x}dx \quad (0<p<1)$

(iii) $\int_0^{2\pi}\frac{d\theta}{(5-\sin\theta)^2}$

1997

1. (a) Let (X,d) be a complete metric space and let d_1 be another metric on X defined by

$$d_1(x,y)=\frac{d(x,y)}{1+d(x,y)} \quad \text{for } x,y\in X.$$

Prove that (X, d_1) is also a complete metric space.

(b) Let (X_1, d_1) and (X_2, d_2) be metric spaces. If $f:X_1\to X_2$ is continuous function and X_1 is compact, prove that f is uniformly continuous.

(c) If f(z) is analytic in a region R, Prove that

$f'(z), f''(z), f'''(z)\cdots$ are analytic in R.

2. (a) Prove that the function

$$v = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

is harmonic. Determine the corresponding analytic function u+iv in terms of z.

(b) Find the bilinear transformation which transforms $R(z) \geq 0$ into the unit circle $|w| \leq 1$.

(c) Prove that the mapping given by $w = f(z)$ from the z-plane to w-plane is conformal at z_0 if f(z) is analytic at z_0 and $f'(z_0) \neq 0$

3. (a) If an entire function is bounded for all values of z, then show that it is constant.

(b) Evaluate $\oint_c \frac{e^{iz}}{z^3} dz$ where C is the circle $|z| = 2$.

(c) Evaluate $\frac{1}{2\pi i} \oint_c \frac{f'(z)}{f(z)} dz$, where $f(z) = \frac{(z^2+1)^2}{(z^2+2z+2)^3}$ and C is the circle $|z| = 4$.

4. (a) If f(z) is analytic inside and on a simple closed curve C, then prove that the maximum value of $|f(z)|$ occurs on C, unless f(z) is a constant.

(b) Prove that $\exp\left\{\frac{1}{2}c\left(z - z^{-1}\right)\right\} = \sum_{-\infty}^{\infty} a_n z^n$, $z \neq 0$ where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)d\theta$

5 (a) For the function $f(z) = \frac{2z^3+1}{z^2+z}$, find a Taylor series valid in the neighbourhood of the point $z = i$ and a Laurent's series valid within the annulus of which the center is the origin.

(b) State Rouche's theorem and apply it to determine the number of roots of the equation

$$z^8 - 4z^5 + z^2 - 1 = 0 \ .$$

6 (a) The function of a real variable defined by $f(x) = \sin x$ is analytic everywhere and bounded, so that $|\sin x| \leq 1$ for all x but it is certainly not a constant. Does it contradict Liouville's theorem ? Explain.

(b) Apply the calculus of residues to evaluate any two of the following integrals:

(i) $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx \quad a > b > 0$

(ii) $\int_0^\infty \frac{\log(x^2+1)}{(x^2+1)}dx$

(iii) $\int_0^\infty \frac{\sin^2 x}{x^2}dx$.

1998

1. (a) Find the distance formula in extended complex plane.

(b) Prove that the range of a function continuous in compact set is itself compact.

(c) Discuss the application of the transformation $\omega = \frac{iz+1}{z+i}$ to the areas in the z-plane which are respectively inside and outside the unit circle with its centre at origin.

2. (a) Define conformal mapping with example. Find the image of $\{z : \text{Re}<0, \text{Im}<\pi\}$ under the exponential function.

(b) Find the Mobius transformation which transforms the half plane $\text{Im } z \geq 0$ into the circle $|w| \leq 1$.

3. (a) If f(z) is regular within and on a closed contour C and if a is a point within C, then

prove that $$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Deduce that $$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz$$

(b) Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}dz$ if C is the circle $|z-1| = 4$.

4. (a) If the function f(z) is analytic and single valued in $|z-a|<R$, prove that for $0 < r < R$

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta$$ where $P(\theta)$ is the real part of $(a + re^{i\theta})$

(b) Find the value of

$$\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz \qquad \text{if C is the circle } |z| = 1.$$

5. (a) The only singularities of a single valued function f(z) are poles of order 2 and 1 at z = 1 and z = 2 with residues of these poles, 1 and 3 respectively. If f(0) = 3/2, f(-1)=1, determine the function.

(b) Find the function f(z) which is analytic within and on a circle C with centre at origin and with radius unity, has the values

$$\frac{a-\cos\theta}{a^2-2a\cos\theta+1} + \frac{i\sin\theta}{a^2-2a\cos\theta+1}, \quad a > 1,$$

θ being the vectorial angle, at points on the circumference of C.

6. (a) Let f(z) be analytic inside and on a simple closed curve C except at a pole a of order m inside C. Prove that the residue of f(z) at a is given by

$$a_{-1} = \lim_{z\to a} \frac{1}{(m-1)!} \frac{d^m - 1}{dx^m - 1} \{(z-a)^m f(z)\}$$

(b) By integrating a suitable function round a rectangle whose vertices are the points ±X, ±X + 2πi, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin a\pi}, \quad \text{if } 0 < a < 1.$$

(b) Prove that

$$\sum_{i=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \cot h(\pi a), \text{ where } a>0$$

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2006

MATHEMATICS

Paper : 201

Full Marks : 80

Time : 4 hours

The figures in the margin indicate full marks for the questions

(New Syllabus)

(**Complex Analysis**)

1. Answer any *four* parts : 4×4=16

(a) If a function $f(z)$ is continuous on a contour C of finite length L and if M is the upper bound of $|f(z)|$ on C, then prove that

$$\left|\int_C f(z)\,dz\right| \le ML$$

(b) Evaluate

$$\frac{1}{2\pi i}\int_C \frac{\cos \pi z}{z^2-1}\,dz$$

around a rectangle with vertices at $-i$, $2-i$, $2+i$, i.

(c) If f is entire analytic and bounded in the complex plane, then prove that $f(z)$ is constant throughout the plane.

(d) Prove that every polynomial equation

$$P(z) = \sum_{i=0}^{n} a_i z^i = 0$$

where the degree $n \ge 1$ and $a_n \ne 0$, has exactly n roots.

(e) If $f(z)$ is analytic inside and on a circle C of radius r and centre at $z = a$, prove that

$$\left|f^{(n)}(a)\right| \le \frac{Mn!}{r^n},\ n = 0, 1, 2, 3, \cdots$$

where M is constant such that

$$|f(z)| < M$$

2. Answer any *four* parts : $4\times4=16$

(a) Prove that both the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

and the corresponding series of derivatives

$$\sum_{n=0}^{\infty} n a_n z^{n-1}$$

have the same radius of convergence.

(b) Find the Laurent series of the function

$$\frac{1}{z^2(z-3)^2}$$

at $z=3$ and specify the region of convergence and nature of singularity at $z=3$.

(c) If $f(z)$ is analytic within and on a simple closed curve C, except at a finite number of poles inside C, then prove that

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N-P$$

where N and P denote respectively the number of zeros and poles inside C.

(d) State Rouche's theorem and use it to show that if $a>e$, the equation $az^n = e^z$ has n roots inside $|z|=1$.

(e) Prove that all the roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$.

3. Answer any *four* parts : $4\times4=16$

(a) Let $f(z)$ be analytic inside a simple closed path C and on C, except for a finitely many singular points $z_1, z_2, \cdots, z_k$ inside C. Prove that

$$\oint_C f(z)dz = 2\pi i\sum_{j=1}^{k} \text{Res}[f(z), z_j]$$

(b) Find the residues of the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2 (z^2+4)}$$

at all its poles in the finite plane.

(c) Applying calculus of residues show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{\pi}{12}$$

(d) By contour integration prove that

$$\int_0^{\infty} \sin x^2 dx = \int_0^{\infty} \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

(e) By contour integration prove that

$$\int_0^{\infty} \frac{\log(x^2+1)}{1+x^2} dx = \pi \log 2$$

4. (a) Define a conformal transformation. Prove that the angle between two curves C_1 and C_2 passing through the point z_0 in the z-plane is preserved in magnitude and sense under the transformation $w = f(z)$ if $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$. 6

(b) Find a bilinear transformation which maps the upper half of the z-plane into the unit circle in the w-plane in such a way that $z = i$ is mapped into $w = 0$ while the point at infinity is mapped into $w = -1$. Also, give a graphical description of the situation.

(c) Prove (with a graphical explanation) that the semi-plane $R(z) > d_1 > 0$ is mapped conformally on the exterior of the parabola $v^2 = 4d_1^2 (d_1^2 - u)$ by the transformation $w = z^2$. 6

Or

(d) Prove that the bilinear transformations transform two points which are inverse w.r.t. a circle into two points which are inverse with respect to the transformed circle.

5. (a) Define analytic continuation with graphical explanation. Prove that the series

$$1+z+z^2+z^4+z^8+\cdots=1+\sum_{n=0}^{\infty} z^{2^n}$$

cannot be continued analytically beyond $|z|=1$. 6

(b) If $F_1(z)$ and $F_2(z)$ are analytic in a region R and $F_1(z)=F_2(z)$ on an arc PQ in R, then prove that $F_1(z)=F_2(z)$ in R. 4

(c) Prove that the power series

$$z+\frac{1}{2}z^2+\frac{1}{3}z^3+\cdots$$

and $$i\pi-(z-2)+\frac{1}{2}(z-2)^2-\cdots$$

have no common region of convergence yet they are analytic continuations of the same functions. 6

Or

(d) Show that the function

$$f(z)=\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\frac{z^3}{a^4}+\cdots$$

can be continued analytically outside the circle of convergence.

2007

MATHEMATICS

Paper : 201

Full Marks **: 80**

Time **: 3 hours**

The figures in the margin indicate full marks for the questions

(New Syllabus)

(Complex Analysis)

1. Answer any *four* parts : **4×4=16**

(a) $f(z)$ is analytic inside and on triangle ABC with derivative $f'(z)$ which is continuous at all points inside and on it. If $\Delta, \Delta_1, \Delta_2, \ldots$ is a sequence of triangles each of which is contained in the preceding and there exists a point z_0 which lies in every triangle of the sequence, then for any $\varepsilon > 0$ there is a δ such that $|\eta| < \varepsilon$ whenever $|z - z_0| < \delta$, show that

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z - z_0)$$

where $\eta = \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$.

(b) Prove Cauchy-Goursat theorem for multiply-connected regions.

(c) If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points $(1, 1)$ and $(2, 3)$, find the value of $\int_C (12z^2 - 4iz)\, dz$.

(d) Prove Cauchy's integral formula for simply connected region R.

(e) State and prove Gauss' mean value theorem.

2. Answer any *four* parts : 4×4=16

(a) State and prove Laurent's theorem.

(b) Prove that an absolutely convergent series is convergent.

(c) Find the Laurent's series about the indicated singularity for each of the following :

(i) $\frac{e^{2z}}{(z-1)^3}$; about $z=1$

(ii) $\frac{z-\sin z}{z^3}$; about $z=0$

(d) Define a meromorphic function. Explain this notion with the help of two examples.

(e) If $f(z)=z^5-3iz^2+2z-1+i$, then evaluate

$$\oint_C \frac{f'(z)}{f(z)}dz$$

where C encloses all the zeros of $f(z)$.

3. Answer any *four* parts : 4×4=16

(a) Explain the terms :

(i) Residue of a pole of order greater than unity

(ii) Residue at infinity

(b) If $\lim_{z\to a}(z-a)f(z)=A$ and if C is the arc $\theta_1\leq\theta\leq\theta_2$ of the circle $|z-a|=r$, then show that $\lim_{r\to 0}\int_C f(z)dz=iA(\theta_2-\theta_1)$.

(c) Prove that

$$\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx=\frac{5\pi}{12}$$

(d) Prove that

$$\int_0^{\infty}\frac{x^b}{1+x^2}=\frac{\pi}{2}\sec\frac{\pi b}{2}$$

where $(-1<b<1)$.

(j) State (only) Mitlag-Leffler expansion theorem completely and explain with the help of an example. 3+1=4

4. Answer any *four* parts : 4×4=16

(a) Prove that the bilinear transformation can be considered as a combination of the transformations of translation, rotation, stretching and inversion.

(b) Find a bilinear transformation which maps points z_1, z_2, z_3 of the z-plane into points w_1, w_2, w_3 of the w-plane respectively.

(c) Let $w = F(z)$ be a bilinear transformation. Show that the most general linear transformation for which $F(F(z)) = z$ is given by $\frac{w-p}{w-q} = k\frac{z-p}{z-q}$, where $k^2 = 1$.

(d) Prove that a bilinear transformation which has only one fixed point α can be put in the form

$$\frac{1}{w-z} = \frac{1}{z-\alpha} + \lambda$$

(e) Discuss the transformation $w = z^2$ in detail.

5. Answer any *four* parts : 4×4=16

(a) Discuss the notion of analytic continuation with examples.

(b) Show that the series

(i) $\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

(ii) $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$

are analytic continuation of each other.

(c) Find a function which represents all possible analytic continuation of

$$F_1(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n$$

(d) By the use of analytic continuation, show that $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$.

(e) Given that the identity $\sin^2 z + \cos^2 z = 1$ holds for real values of z. Prove that it also holds for all complex values of z.

2009

MATHEMATICS

Paper : 201

(Complex Analysis)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. *(a)* State and prove complex form of Green's theorem. Hence deduce the Cauchy's theorem. 6

Or

$f(z)$ is analytic in a simply-connected region R and a, z are points in R.

(i) Prove $F(z) = \int_a^z f(u)\,du$ is analytic in R.

(ii) Prove $F'(z) = f(z)$.

(b) Assuming the Cauchy's integral formula and the result

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\,dz$$

for the analytic function $f(z)$ inside and on a curve C, of a simply-connected region R, prove that, if $f(z)$ is analytic in a region R, then $f'(z), f''(z), \ldots$ are analytic in R. 6

Or

State and prove the minimum modulus theorem. Justify the necessity of non-vanishing of the complex function for its modulus to assume its minimum value on C (with usual meaning).

(c) Determine whether

$$G(z) = \int_1^z \frac{d\zeta}{\zeta}$$

is independent of the path joining 1 and z, and indicate the relationship with Morera's theorem. 4

2\. (a) State and prove Laurent's theorem. 6

Or

Classify all possible singularities of a function $f(z)$ by examination of the Laurent series.

(b) Find Laurent series about $z=1$ and $z=-2$ respectively for each of the following functions. Also name the singularities in each case and give the region of convergence of each : 6

(i) $f(z) = \dfrac{e^{2z}}{(z-1)^2}$

(ii) $f(z) = (z-3)\sin\dfrac{1}{z+2}$

Or

If $f(z) = z^5 - 3iz^2 + 2z - 1 + i$, evaluate

$$\oint_C \frac{f'(z)}{f(z)}\, dz$$

where C encloses all the zeros of $f(z)$.

(c) Prove that $\tan z = az$, $a > 0$, has infinitely many real roots. 4

3. (a) Define a residue of a complex function $f(z)$. Explain the technique for obtaining the residues of a function with the help of the example

$$f(z) = \frac{2z+1}{z^2 - z - 2}$$

at its poles. 6

Or

Prove the residue theorem.

(b) Evaluate

$$\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)} dz$$

around the circle C with equation $|z| = 3$. 5

Or

Show that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{\pi}{12}$$

(c) State and prove Mittag-Leffler expansion theorem. 5

4. (a) Find the Jacobian of the transformation in

(i) $w = \sqrt{2} e^{\pi i/4} z + (1 - 2i)$

(ii) $w = z^2$

and interpret geometrically. 6

Or

If $w = f(z) = (z^2 + 1)^{1/2}$, then—

(i) show that $z = \pm i$ are branch points of $f(z)$;

(ii) show that a complete circuit around both branch points produces no change in the branches of $f(z)$.

(b) Discuss in detail that the bilinear transformation can be considered as a combination of the transformation of translation, rotation, stretching and inversion. 6

Or

Discuss the Riemann surfaces for the function $w = f(z) = (z^2+1)^{1/2}$. Also show its branch line.

(c) Find a bilinear transformation which maps points z_1, z_2, z_3 of the z-plane into points w_1, w_2, w_3 of the w-plane respectively. 4

5. *(a)* Show that the series

(i) $\sum \frac{z^n}{2^{n+1}}$

(ii) $\sum \frac{(z-i)^n}{(2-i)^{n+1}}$

are analytic continuation of each other. 6

Or

Prove that if an analytic function $f(z)$ vanishes at all points on an arc PQ inside its region R, then $F(z)$ vanishes throughout R. Hence deduce that if the identity $\sin^2 z + \cos^2 z = 1$ holds for real values of z, then it also holds for all complex values of z.

(b) Prove the following giving emphasis on the notion of analytic continuity : 5

(i) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

(ii) $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Or

Evaluate :

(i) $\int_0^{\sqrt{2}} t^2 \sqrt{2-t^2}\, dt$

(ii) $\int_0^2 t^4 (16-t^4)^{1/2}\, dt$

(c) Prove that the zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$$

is analytic in the region of the z-plane for which $\operatorname{Re}(z) \geq 1 + \delta$ for any fixed positive δ.

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