M202

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Gauhati University

M.A./M.Sc. in Mathematics
Semester 2

Paper II
Functional Analysis

Contents:

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Unit -1

Introduction :

Functional Analysis is the study of certain structures arising out of the marriage of topology and algebraic structures, and application of these structures to analytical problems. The origin of the systematic study of such abstract structures goes back to M.Fréchet (1906). It was S. Banach who, in 1932, in his book gave an elegant account of the work of many mathematicians involved in the development of functional analysis approach. Mathematicians observed that problems from different fields often enjoy related features and properties. This fact was exploited for effective unification with the omission of unessential details of individual cases. For instance, absolute value of difference between two reals, modulus of difference between two complex numbers, maximum absolute difference between functional values of two functions and ,two sequences have been unified as metric notion . Similarly. normed spaces, Banach spaces , Hilbert spaces etc. came into existence. In this connection, the concept of a 'space' is used in a very wide and surprisingly general sense.

An 'abstract space' is a set of elements satisfying certain axioms. With the choice of different sets of axioms we shall come across different spaces.

It has been widely accepted that functional analysis plays a pivotal role in the growth of Pure and Applied Sciences . It plays key role in solving problems of mathematics, physics, mechanics, engineering, operational research and many other branches. Functional analysis is now regarded as queen of applied Sciences.

The study material in functional analysis is broken into three sets.

SET I : Banach spaces : definition and examples, continuous linear functions, finite dimensional normed spaces, bounded linear functions.

SET II :The Hahn Banach theorem, the natural embedding of N in N^{**} , weak and weak* topologies, the open mapping theorem , the closed graph theorem, the Banach Steinhaus theorem and the conjugate of an operator.

SET III: Hilbert Spaces and finite dimensional spectral theory : Definition and Examples, orthogonal complements and orthogonal sets, the conjugate space H^* , the adjoined of an operator, self-adjoined, normal, unitary, positive and projection operators, the spectral theorem.

Symbols :

K	:	Scalar field of real or complex numbers
R	:	field of reals
C	:	field of complex numbers

Span E	:	linear span of a subset E of a linear space
dim X	:	dimension of linear space X
X/Y	:	quotient space of a linear space X by a subspace Y.
R(F)	:	range space of linear map F.
N(F)	:	null space of linear map F
l^p	:	the set of p-summable scalar sequences , $1 \le p \le \infty$
l^∞	:	the set of bounded scalar sequences
E^0	:	the interior of a subset E of a metric space
$\overline{E}$	:	the closure of a subset E of a metric space
$\{x_n\}$	:	sequence whose nth term is x_n
$x_n \to x$	:	Sequence $\{x_n\}$ converges to x
dist(x,E)	:	distance of x from a subset E of a metric space
c[a,b]	:	the set of real valued continuous functions on [a,b].
C	:	space of convergent scalar sequences
C_0	:	space of scalar sequences converging to zero
C_{00}	:	scalar sequences having only finitely many non-zero terms.
$\lVert x + Y \rVert$	:	quotient norm of x+Y.
L(X,Y)	:	space of linear mapping from a linear space X into linear space Y.
B(X,Y)	:	space of bounded linear mappings from a normed linear space X into a normed linear space Y.
X^*	:	dual space of normed linear space X.
$\langle x,y \rangle$	:	inner product of x with y.
H	:	Hilbert Space
$x \perp y$	:	x is orthogonal to y.
$E \perp F$	:	E is orthogonal to F.
$E^\perp$	:	elements orthogonal to set E.
T^*	:	adjoint operator of T.
$\forall$	:	for all.
$\exists$	:	there exists.

1. Prerequisites : For easy understanding of the meterial in the sequel a freshman requires the knowledge of metric space and linear space . Some essentials on metric space and linear space theory have been introduced in this section without proof before the start of actual syllabus.

Definitions 1.1 :

Let X be a non empty set and d: X×X→R be a mapping on $X \times X$. 'd' is called a metric on X and (X,d) is called a **metric space** if d satisfies :

(M1) $d(x,y) \geq 0 \ \forall \ x,y \in X$

(M2) $d(x,y) = 0$ if and only if $x=y$

(M3) $d(x,y)= d(x,y)$ [symmetry]

and (M4) $d(x,y) \leq d(x,z)+ d(z,y) \ \forall \ x,y,z \in X$ [Triangular inequality]

If $Y \subset Z$ then $\bar{d}(y_1, y_2) = d(y_1, y_2) \ \forall (y_1, y_2) \in Y \times Y$ induces a metric on Y. In fact, $(Y, \bar{d})$ is a metric space. So every subset of a metric space is a metric space with the induced metric.

Examples of metric spaces :

(1) (R,d), $d(x_1, x_2) = |x_1 - x_2|, (x_1, x_2) \in X \times X$

(2) (R^n,d), $d((x_1, \cdots x_n),(y_1, \cdots y_n)) = \sum_{i=1}^{n} (x_i - y_i)^2$

(3) $(C[a, b], d], d(f, g) = \sup\{|f(x) - g(x)| : a \leq x \leq b\}$

Let (X,d) be a metric space, $x_0 \in X$ and r>0. Then

(a) $B(x_0, r) = \{x| \ d(x, x_0) < r\}$

(b) $\bar{B}(x_0, r) = \{x| \ d(x, x_0) \leq r\}$

(c) $S(x_0, r) = \{x| \ d(x, x_0) = r\}$

are called **open ball, closed ball** and **open sphere** respectively. x_0 is called centre and r is called radius. A subset M of a metric space (X,d) is called an open set if for $x \in M, \exists r > 0$ such that $B(x, r) \subset M$. That is every point of M is an interior point of M. A subset F of (X,d) is said to be closed if X\F is an open set. An open ball $B(x_0,r)$ is called a r-neighborhood of x_0. A subset N of (X,d) which contains a r-neighborhood of x_0 is called a neighborhood of x_0 and x_0 is called interior point of N. The interior of subset M is the set of all interior points of M and is denoted by M^0 or int(M). In fact, M^0 is the largest open set contained in M. If $\Im_d$ be the collection of all open subsets of a metric space (X,d) then

(T_1) $X \in \Im_d, \phi \in \Im_d$

(T_2) $\Im_d$ is closed with respect to arbitrary union

and (T_3) $\Im_d$ is closed with respect to finite intersection.

In general, a non empty set X together with a collection $\Im$ of subsets X satisfying (T_1), (T_2), (T_3) is called a topological space and $\Im$ is called a topology on X. $\Im_d$ is called a topology induced by metric 'd'.

A mapping $F:(X,d)\to(Y,\bar{d})$ is said to be continuous at $x_0 \in X$ if for $\varepsilon>0$ there exists a $\delta>0$ such that

$$\bar{d}(F(x), F(x_0)) < \varepsilon \text{ when } d(x, x_0) < \delta$$

If F is continuous, at every point of X, then F is said to **continuous** in X or simply continuous.

Theorem 1.2 A mapping $F : (X, d) \to (Y, \bar{d})$ is continuous if and only if inverse image of every open (closed) subset of Y is an open (closed) subset of X.

Definitions 1.3 : Let M be a subset of a metric space (X,d). A point $x_0 \in X$ is called an **accumulation point or limit point** of M if every neighborhood of x_0 contains atleast one point $y \in M$ distinct from x_0. The set consisting of the points of M and the accumulation points of M is called the **closure** of M and it is denoted by $\bar{M}$. $\bar{M}$ is the smallest closed set containing M. A subset M of a metric space X is said to be **dense** in X if $\bar{M} = M$. A metric space X is said to be **separable** if it has a countable dense subset . For subsets A and B of (X,d),

(a) $A \subset \bar{A}$ (b) $(\bar{A})^- = A$ (c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and (d) $\bar{A} \cap \bar{B} \subset \overline{A \cap B}$

A sequence $\{x_n\}$ in (X,d) is said to converge to a point $x \in X$ if for $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$d(x_n, x) < \varepsilon \; \forall n \geq n_0$$

Equivalently, $\lim_{n\to\infty} d(x_n, x) = 0$. We denote it by $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ which is not convergent is said to be **divergent**.

A non-empty subset M of a metric space (X,d) is said to be bounded if its diameter

$$d(M) = \sup\{d(x, y) | x, y \in M\} < \infty$$

A sequence $\{x_n\}$ is **bounded** if the corresponding point set is bounded, that is, $\{x_n\} \subset B(x_0, r)$ for some $x_0 \in X$ and $r>0$.

A sequence $\{x_n\}$ is a **Cauchy sequence** if for $\varepsilon>0$ there exists $n_0 \in R$ such that

$$d(x_n, x_m) < \varepsilon \quad \forall n, m \geq n_0$$

A metric space X is said to be **complete** if every cauchy sequence in X converges to a point of X.

Theorem 1.4 In a metric space (X,d),

(a) every convergent sequence is cauchy sequence

(b) every cauchy sequence is bounded

(c) the limit of convergent sequence is unique

(d) $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$

Theorem 1.5 : For an infinite subset M of a metric space (X,d)

(a) $x \in \overline{M}$ if and only if there is a sequence $\{x_n\}$ in M such that $x_n \to x$

(b) M is closed if and only if $x_n \in M, x_n \to x$ implies $x \in M$

Theorem 1.6 : A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Theorem 1.7 A mapping $F : X \to Y$ of a metric space (X,d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies $F(x_n) \to F(x_0)$

Definition 1.8 : If $S:N \to X \equiv (X, d)$ be a sequence and $g:N \to N$ be a strictly increasing function then $S \circ g : N \to X$ is called a subsequence of the sequence S.

A metric space (X,d) is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact when considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

2. **Linear space or Vector space:** Vector space or Linear space plays a very important role in many branches of mathematics. The disadvantage of metric spaces of not being capable to add, subtract or multiply an element by a scalar are overcome in a linear space structure. Normed linear spaces of which Banach space and Hilbert space are special cases are defined for linear spaces.

Definition 2.1 : Let X be a non empty set and K be a field. Define internal composition

$+ : X \times X \to X$ and external composition

$\bullet : K \times X \to X$, $(x_1, x_2) \to x_1 + x_2$ and $(\alpha, x) \to \alpha x$

X is called a linear space over the field K and denoted by X(K) if

A. (i) $x_1 + x_2 \in X \ \forall x_1, x_2 \in X$

(ii) $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) \ \forall x_1, x_2, x_3 \in X$

(iii) there exists $0 \in X$ such that $x + 0 = x = 0 + x$ for all $x \in X$.

(iv) for $x \in X$ there exists $-x \in X$ such that $x + (-x) = 0 = (-x) + x$

(v) $x_1 + x_2 = x_2 + x_1 \quad \forall x_1, x_2 \in X$

B. (i) $\alpha x \in X \ \forall \alpha \in K$ and $x \in X$

(ii) $\alpha(x_1+x_2) = \alpha x_1 + \alpha x_2 \quad \forall \alpha \in K$ and $x_1 \in X, x_2 \in X$

(iii) $(\alpha+\beta)x = \alpha x + \beta x \quad \forall \alpha, \beta \in K$ and $x \in X$

(iv) $(\alpha\beta)x = \alpha(\beta x) \quad \forall \alpha, \beta \in K$ and $x \in X$

(v) $1.x = x$ for all $x \in X$and $1 \in K$ $(1.\alpha = \alpha = \alpha.1 \ \forall \alpha \in K)$

As easy consequences of definition , we have

$0.x = 0, \alpha.0 = 0, (-1)x = -x, -(x-y) = y-x$

A linear space X(K) is real linear space or complex linear space according as K is R or C.

Examples of linear spaces

(i) R^n (R) with $(x_1 \ldots\ldots, x_2) + (y_1 \ldots\ldots, y_2) = (x_1+y_1, \ldots\ldots\ldots, x_n+y_n)$

and $\alpha(x_1, \ldots\ldots x_n) = (\alpha x_1 \ldots\ldots\ldots \alpha x_n)$

(ii) C^n (C) with some operations as in R^n (R)

(iii) ℓ^p or $\ell^p(K) = \left\{ \{x_n\} \subset K : \sum_{n=1}^{\infty} |x_n|^p \langle \infty, 1 \le p \le \infty \right\}$ with operations

$\{x_n+y_n\} = \{x_n+y_n\}$ and $\alpha\{x_n\} = \{\alpha x_n\}$

(iv) ℓ^∞ or $m = \{\{x_n\} \subset K : \sup_{n \ge 1} |x_n| < \infty\}$ with operations same as in ℓ^p

(v) C[a,b](R) with operations $(f + g)x = f(x) + g(x)$

$(\alpha f)x = \alpha f(x)$

A non empty subset Y of a vector spaee X(K) is called a linear space of X if Y(K) is a subspace of X(K) if and only if $y_1 \in Y, y_2 \in Y, \alpha, \beta \in K$ implies $\alpha y_1 + \beta y_2 \in Y$. X and {0} are trivial subspaces of X(K). A subspace other than these two subspaces is called a proper subspace of X . A subset C of a linear space X is said to be convex if $tC+(1-t)C \subset C$ $\forall$ t, $0 \le t \le 1$.

An expression of the form $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n$ where $\alpha_i \in K$ and $x_i \in X \equiv X(K)$ is called **linear combinations** of $x_1, x_2, \ldots\ldots, x_n$.

For any non empty subset M of a vector space X(K) the set of all linear combinations of vectors of M is called the **span** of M, written as span M or <M>. It is the smallest subspace of X containing M. <M> is also called a subspace **generated** or **spanned** by M.

A finite subset $M = \{x_1, x_2, \ldots\ldots, x_n\}$ is called **linearly independent (L.I)** set if

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0 \text{ implies } \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0.$$

An infinite subset is called L.I. set if every finite subset of it is L.I. A subset which is not L.I. is called linearly dependent set. In this case, there exists a set of scalars $r_1, r_2, \ldots\ldots, r_n$ not all zero but

$$r_1 x_1 + r_2 x_2 + \ldots\ldots + r_n x_n = 0$$

A linear space X(K) is said to be **finite dimensional** and dimension of X is said to be n if X contains a linearly independent set of n vectors but all sets having more than n vectors are linearly dependent. In this case, we write $\dim X = n$. $X = \{0\}$ is a zero dimensional space. If X is not finite dimensional then it is said to be **infinite dimensional** space. R^n and C^n are n dimensional space but ℓ^∞ and C[a,b] have infinite dimensional space.

If $\dim X = n$, then a linearly independent n-tuples of a vector space X is called a basis for X. If $B = \{e_1, \ldots\ldots, e_n\}$ is a basis for X then $x \in X$ can be uniquely expressible as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n \quad (\alpha_i \in K)$$

$B = \{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots 0), \ldots e_n = (0, 0, \ldots, 1)\}$ is a basis for R^n and C^n.

More generally, if X is any vector space, not necessarily finite dimensional, a subset B of X is called a basis (Hamel basis) for X if B is linearly independent and B spans X.

Theorem 2.2 : Every vector space $X \neq \{0\}$ has a basis.

Theorem 2.3 : All bases for finite dimensional or infinite dimensional vector space X have the same cardinal number. This means that two bases have the same number of elements or one-one correspondence. This number is called dimension of X.

Theorem 2.4 : If X is an n-dimensional vector space then any proper subspace Y of X has dimension less than n.

Definition 2.5 : Let Y be a subspace of vector space X. The coset of an element $x \in X$ with respect to Y is denoted by x+Y and is defined by

$$x + Y = \{x + y \mid y \in Y\}$$

Clearly, $x_1 + Y = x_2 + Y$ if and only if $x_1 - x_2 \in Y$ and in particular $y + Y = Y$ iff $y \in Y$

The set X/Y of distinct cosets in a vector space under algebraic operations defined by

$$(x_1 + Y) + (x_2 + Y) = (x_1 + x_2) + Y$$

$$\alpha(x + Y) = \alpha x + Y$$

This space is called **quotient space** and is denoted by X/Y.

The dimension of X/Y is called codimension of Y and is denoted by codim Y, that is,

$$\text{codim } Y = \dim X / Y$$

3. Normed linear spaces and Banach Spaces:

Definition 3.1: Let X be a linear space over the field K. A norm on X is a map $\| \| : X \to R_+$ from the linear space X to the set of non negative reals which satisfies :

(N_1) $\|x\| = 0$ if and only if $x=0$.

(N_2) $\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in K$ and $x \in X$ (homogenity)

(N_3) $\|x + y\| \le \|x\| + \|y\| \quad \forall x, y \in X$ (triangle inequality)

The linear space X equipped with a norm is defined as a normed linear space (nls) or simply a normed space.

Note : 1. $\|-x\| = \|(-1)x\| = |-1|\|x\| = \|x\|$

2. $\|0\| = \|0.x\| = |0|\|x\| = 0$

3. $0 = \|0\| = \|x + (-x)\| \le \|x\| + \|-x\| = 2\|x\|$. Hence $0 \le \|x\|$

4. $\|x\| = \|(x - y) + y\| \le \|x - y\| + \|y\|$. So, $\|x\| - \|y\| \le \|x - y\|$

Also, $\|y\| - \|x\| \le \|y - x\| = \|x - y\|$. Hence $\big|\|x\| - \|y\|\big| \le \|x - y\|$

Proposition 3.2 A norm in a linear space X induces a metric on X.

Proof: Let $d : X \times X \to R_+$ be defined by

$$d(x_1, x_2) = \|x_1 - x_2\|$$

Then (M1) $d(x_1, x_2) = \|x_1 - x_2\| \ge 0 \quad \forall x_1 \in X, x_2 \in X$

(M2) $d(x_1, x_2) = \|x_1 - x_2\| = 0$ if and only if $x_1 = x_2$

(M3) $d(x_1, x_2) = \|x_1 - x_2\| = \|x_2 - x_1\| = d(x_2, x_1)$

(M4) $d(x_1, x_3) = \|x_1 - x_3\| = \|(x_1 - x_2) + (x_2 - x_3)\|$

$$\le \|(x_1 - x_2)\| + \|(x_2 - x_3)\|$$
$$= d(x_1, x_2) + d(x_2, x_3)$$

This shows that (X,d) is a metric space .

Definition 3.3 A nls $(X, \|\cdot\|)$ is said to be a Banach space if (X,d) is a complete metric space with the metric d, induced by the norm, that is, $d(x,y)=\|x - y\|$.

Since a nls is a metric space , all the notions of metric spaces can be redefined in a nls. For instance,

1. a sequence $\{x_n\}$ in a nls $(X, \|\cdot\|)$ converges to $x \in X$ if for $\varepsilon > 0$ there exists $n_0 \in \angle$ such that

$$\|x_n - x\| < \varepsilon \quad \forall n \geq n_0$$

2. $F : (X, \|\cdot\|_x) \to (Y, \|\cdot\|_y)$ is continuous at $x_0 \in X$ if for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|F(x) - F(x_0)\|_Y < \varepsilon \text{ when } \|x - x_0\|_x < \delta.$$

Similarly any notion of a metric space can be redefined simply replacing d(x,y) by $\|x - y\|$

Exercises : If $(X_i, \|\cdot\|_i)$ $(i = 1,2,\ldots\ldots, n)$ be a finite collection of nlss then

$X = \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n$ is a nls with norm defined by

$$\|(x_1, x_2, \ldots, x_n)\| = \|x_1\|_1 + \|x_2\|_2 + \ldots\ldots + \|x_n\|_n$$

Proposition 3.4 A map $F : \left(\prod_{i=1}^{n} X_i, \|\cdot\|\right) \to (Y, \|\cdot\|)$ is continuous at $(a_1,a_2,\ldots\ldots,a_n)$ if for every neighbourhood V of $F(a_1,a_2,\ldots\ldots,a_n)$ there exists neighborhoods V_i of $a_i \in X_i$ such that

$$F\left(\prod_{i=1}^{n} V_i\right) \subset V$$

Proof: Let F be continuous at $(a_1,a_2,\ldots\ldots,a_n) \in \prod_{i=1}^{n} X_i$. For $\varepsilon>0$, there exists $\delta>0$ such that

$$\|F(x_1, x_2, \ldots, x_n) - F(a_1, a_2, \ldots, a_n)\| < \varepsilon \text{ if}$$

$$\|(x_1, x_2, \ldots, x_n) - (a_1, a_2, \ldots, a_n)\| < \delta$$

$$\text{i.e. if } \|x_1 - a_1\|_1 + \ldots\ldots + \|x_n - a_n\|_n < \delta$$

Let $V = \{y : \|y - F(a_1, \ldots a_n)\| < \varepsilon\}$ and $V_i = \{x : \|x - x_i\| < \delta/n\}$

Hence, $F(x_1, \ldots x_n) \in V$ if $x_i \in V_i (i = 1,2,\ldots, n)$. So, $F\left(\prod_{i=1}^{n} V_i\right) \subset V$.

The proof of the converse is left as an exercise for the readers.

Proposition 3.5 Let $(X, \|\ \|)$ be a normed linear space.

Then (1) the map $(x,y) \to x+y$ from $X \times X$ into X

(2) the map $(\alpha, x) \to \alpha x$ from $K \times X$ into X

(3) the map $x \to \|x\|$ from X into K are continuous.

Proof:(1) Let $(a, b) \in X \times X$

$$\|(x+y)-(a+b)\| = \|(x-a)+(y-b)\| \le \|x-a\| + \|y-b\|$$

So, for $\varepsilon > 0$, $\|(x + y) - (a + b)\| < \varepsilon$ if $\|x - a\| < \varepsilon/2$ and $\|y - b\| < \varepsilon/2$

That is, $F(V_1 \times V_2) \subset V$ where $V_1 = \{x : \|x - a\| < \varepsilon/2\}$, $V_2 = \{y : \|y - b\| < \varepsilon/2\}$

and $V = \{y : \|y - (a + b)\| < \varepsilon\}$

So, by proposition 3.4, F(x+y)=x+y is continuous at (a,b). Since (a,b) is an arbitrary point of $X \times X$, F is continuous in X.

(2) Let $G(\alpha, x) = \alpha x$. According to theorem 1.7, it is enough to show that

$$(\alpha_n, x_n) \to (\alpha, x) \text{ in } X \times X \Rightarrow \alpha_n x_n \to \alpha x \text{ in } X.$$

Given $\varepsilon > 0, \exists\, n_0 \in N$ such that,

$$\|(\alpha_n, x_n) - (\alpha, x)\| = |\alpha_n - \alpha| + \|x_n - x\| < \varepsilon \quad \forall n \ge n_0$$

So, $\alpha_n \to \alpha$ and $x_n \to x$. Since $\{\alpha_n\}$ is bounded, $\sup_{n \ge 1} |\alpha_n| = M < \infty$

$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \le |\alpha_n|\ \|x_n - x\| + |\alpha_n - \alpha|\ \|x\| < M\|x_n - x\| + |\alpha_n - \alpha|\ \|x\|$$

Since, $x_n \to x$ and $\alpha_n \to \alpha$, there exists $n_0 \in N$

Such that $\|x_n - x\| < \dfrac{\varepsilon}{2M}$ and $|\alpha_n - \alpha| < \dfrac{\varepsilon}{2\|x\|}$ (provided $x \ne 0$)

Hence $\|\alpha_n x_n - \alpha x\| < \varepsilon \forall n \ge n_0$.

(3) Let $a \in X$, We have already seen that $\big|\|x\| - \|a\|\big| < \|x - a\|$

Let $h(x)=\|x\|$. Then $|h(x) - h(a)| < \varepsilon$ if $\|x - a\| < \delta$. Hence norm is a continuous function

Note: It follows from the continuity of norm that

$$x_n \to x \Rightarrow \|x_n\| \to \|x\| \quad \text{That is, } \|x\| = \lim_{n\to\infty} \|x_n\|$$

$$\text{or,} \quad \left\| \lim_{n\to\infty} x_n \right\| = \lim_{n\to\infty} \|x_n\|. \qquad \text{In particular, } \left| \lim_{n\to\infty} \alpha_n \right| = \lim_{n\to\infty} |\alpha_n|.$$

Definition 3.6 A map $F : (X, \|\|) \to (Y, \|\|)$ is called an isometry if $\|x_1 - x_2\| = \|F(x_1) - F(x_2)\|$ and F is said to be linear if $F(\alpha x_1 + \beta x_2) = \alpha F(x_1) + \beta F(x_2) \ \forall x_1, x_2 \in X$ and $\alpha, \beta \in K$. Clearly a linear map F is isometry if and only if $\|F(x)\| = \|x\| \ \ \forall x \in X$.

$F : (X, \|\|) \to (Y, \|\|)$ is an **isometric isomorphism** if F is linear , one-one, onto, and isometry. F is called a homeomorphism if it is one-one, onto, and F and F^{-1} are continuous.

Proposition 3.7

(a) The translation $T_a : X \to X, T_a(x) = x + a$ is an isometry and a homeomorphism

(b) The multiplicative operator $M_\lambda : X \to X, M_\lambda(x) = \lambda x (\lambda \neq 0)$ is a homeomorphism

Proof (a): It is immediate that $\|T_a(x) - T_a(y)\| = \|x - y\|$ and $T_a(x-a) = x$

Since, $T_a(x) = T_a(y) \Rightarrow x + a = y + a \Rightarrow x = y$, T_a is one-one and onto.

Continuity of T_a follows from the equality $\|T_a(x) - T_a(y)\| = \|x - y\|$.

$$T_a^{-1}(x) = (x - a) = T_{-a}(x).$$

So, $T_a^{-1} = T_{(-a)}$ is also continuous . Hence T_a is a homeomorphism.

(b): $M_\lambda(x) = M_\lambda(y) \Rightarrow \lambda x = \lambda y \Rightarrow x = y$ if $\lambda \neq 0$

Also, $M_\lambda^{-1}(x) = \lambda^{-1} x \ \ \forall x \in X$ and $M_\lambda^{-1} = M_{\lambda^{-1}}$.

So, M_λ is one-one and onto.

$$\|M_\lambda(x) - M_\lambda(x_0)\| = \|\lambda x - \lambda x_0\| = |\lambda| \, \|x - x_0\|$$

This shows that M_λ is continuous for all λ.So, $M_\lambda^{-1} = M_{\lambda^{-1}}$ is also continuous and M_λ is a homeamorphism.

Definitions 3.8 For subsets A and B of a linear space X,

$A + B = \{a + b | a \in A \text{ and } b \in B\}$. In particular,

$a + A = \{a + x | x \in A\}$

$\lambda A = \{\lambda x | x \in A\}$

Clearly, $A+B= \bigcup_{a \in A} (a + B)$.

Proposition 3.9 If A is a subset of a nls $(X, \|\|)$, then

(i) $\overline{A} + a = \overline{A + a}$ (ii) $\lambda\overline{A} = \overline{\lambda A}$ $\forall \lambda \in K$

proof: If $F : (X, \|\|) \to (Y, \|\|)$ is a homeomorphism and $A \subset X$,

then $F(\overline{A}) = \overline{F(A)}$. Since T_a and M_λ are homeomorphisms,

$$T_a(\overline{A}) = \overline{T_a(A)} \text{ and } M_\lambda(\overline{A}) = \overline{M_\lambda(A)}$$

and the results follow. (2) holds when $\lambda=0$.

Exercise : (1). Show that $B(a, \varepsilon) = B(0, \varepsilon) + a$

(2) Show that a bijective isometry is a homeomorphism.

(3) $x_n \to x$ and $y_n \to y \Rightarrow x_n + y_n \Rightarrow x + y$

(4) $\alpha_n \to \alpha$, $x_n \to x$ $\Rightarrow \alpha_n x_n \to \alpha x$.

Proposition 3.10 Let X be a normed linear space. Then

(a) if G is open in X and $A \subset X$, then G+A is open in X;

(b) if C is convex subset of X then the closure $\overline{C}$ and the interior C^0 are also convex. If $C^0 \neq \phi$, then $\overline{C} = \overline{C^0}$

(c) if Y be a subspace of X then $Y^o \neq \phi$ if and only if Y=X.

Proof:

(a) The translation map $T_{x_0} = x + x_0$ is a homeomorphism.

So, if G is open in X then $T_{x_0} = G + x_0$ is open in X and $G+A= \bigcup_{a \in A} (G + a)$ being union of open sets is open in X.

(b) Let C be a convex subset of X. Let $x,y \in \overline{C}$ and $0 \le t \le 1$. Then there exists sequences $\{x_n\}$ and $\{y_n\}$ in C such that $x_n \to x$ and $y_n \to y$. Now $tx_n + (1-t)y_n \to tx + (1-t)y$. Since C is convex, $tx_n + (1-t)y_n \in C$. Hence $tx + (1-t)y \in \overline{C}$ and $t\overline{C} + (1-t)\overline{C} \subset \overline{C}$. This proves that $\overline{C}$ is a convex set.

Again $tC^0 + (1-t)C^0 \subset tC + (1-t)C \subseteq C$.

Since $M_t(x) = tx$ is a homeomorphism, tC^0 and $(1-t)C^0$ are open subsets and by part(a), $tC^0 + (1-t)C^0$ is an open subset of C. Hence $tC^0 + (1-t)C^0 \subset C^0$ and C^0 is a convex subset of X.

Let $C^0 \neq \phi$. To show that $\bar{C} = \overline{C^0}$ if C is a convex set. $C^0 \subset C$ implies $\overline{C^0} \subset \bar{C}$. It is enough to how that $C \subset \overline{C^0}$. Since $C^0 \neq \phi$, there exists $a \in X$ such that $a \in C^0$. So, there exists $r>0$ such that $B(a, r) \subset C$. Then $a + ry \in C \forall y \in X$ with $\|y\| < 1$. In fact,

$$\|(a + ry) - a\| = \|ry\| = r\|y\| < r$$

Let $x \in C$ and $x_t = ta + (1-t)x \quad (0<t<1)$

Next, we show that

$$B(x_t, tr) = \{z \in X : z = x_t + try, \|y\| < 1\} \text{ with } \|y\| \leq 1,$$

$$\|z - x_t\| = \|try\| < tr \Rightarrow z \in B(x_t, tr)$$

Conversely, $\quad z \in B(x_t, tr) \Rightarrow \|z - x_t\| < tr \Rightarrow \|y_o\| < tr$

where $y_o = z - x_t$ or $z = x_t + y_o = x_t + tr.y, y = \frac{1}{tr} y_o$ with $\|y\| < 1$

Now we show that $B(x_t, tr) \subset C$ and $x_t \in C^o$.

Also, $x_t = ta + (1-t)x \to x$ as $t \to 0$ (because 'a' is fixed).

This shows that $x \in \overline{C^o}$. Thus $C \subset \overline{C^o}$ and hence $\bar{C} \subset \overline{C^o}$.

(c) If Y is a subspace of a non zero space X, then to show that $Y^o \neq \phi$ if and only if $Y=X$.

If Y=X then $Y^o = X \neq \phi$. Conversely, let $Y^o \neq \phi$.

Consider $a \in Y^o$. Then there exists $r>0$ such that $B(a, r) \subset Y$. For $x \in X$ if $x=0$ then $x \in Y$. If $x \neq 0$ then for $\varepsilon < r$

$$\left\|\frac{(r-\varepsilon)x}{\|x\|} - 0\right\| = r - \varepsilon < r$$

$$\text{So, } \frac{(r-\varepsilon)x}{\|x\|} \in B(0, r) = B(a, r) - a \Rightarrow a + \frac{(r-\varepsilon)x}{\|x\|} \in B(a, r) \subset Y \quad \Rightarrow x \in Y$$

Hence Y = X.

4. Examples of Banach Spaces :

Example 1. R is a Banach Space with norm $\|x\| = |x|$, the absolute value of x.

R(R) is a vector space .

Further

(N1) $|x| \geq 0$

(N2) $|x| = 0$ if and only if $x=0$

(N3) $|\alpha x| = |\alpha||x|$ and

(N4) $|x + y| \leq |x| + |y|$

So, $(R,|\,|)$ is a normed linear space .

The completeness of R follows from the following two lemmas.

Lemma 1: A Cauchy sequence is bounded.

Let $\{\alpha_n\}$ be a Cauchy sequence in R. For $\varepsilon = 1, \exists n_o \in N$ such that $|\alpha_n - \alpha_m| < 1 \forall n, m \geq n_o$. Then $|\alpha_n| \leq |\alpha_n - \alpha_{n_o}| + |\alpha_{n_o}| < 1 + |\alpha_{n_o}| \quad \forall n \geq n_o$

Let M= $\max^m \{1 + |\alpha_{n_o}|, |\alpha_1|, \ldots, |\alpha_{n_o - 1}|\}$. Hence $|\alpha_n| \leq M \forall n \geq 1$ and $\{\alpha_n\}$ is bounded.

Lemma 2: If a subsequence of a Cauchy sequence $\{\alpha_n\}$ converges to a point α_o then the sequence $\{\alpha_n\}$ also converges to the same limit α_o.

Suppose the subsequence $\{\alpha_{n_k}\}$ of the Cauchy sequence $\{\alpha_n\}$ converges to the point α_o. There exists $k_o \in N$ such that

$$|\alpha_{n_k} - \alpha_o| < \varepsilon/2 \quad \forall n_k \geq k_o.$$

Also, $|\alpha_n - \alpha_m| < \varepsilon/2 \quad \forall n, m \geq k_1, k_1 \in .N$

Let k_2 be the smallest integer in the set $\{n_k | k = 1,2,3,......\}$ which is greater than a equal to $\max\{k_0, k_1\}$. Then,

$|\alpha_n - \alpha_0| \leq |\alpha_n - \alpha_{k_2}| + |\alpha_{k_2} - \alpha_0| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. The result follows.

Now we show the completeness of R. If $\{\alpha_n\}$ be a Cauchy sequence in R then $\{\alpha_n\}$ is bounded. By Bolzano-weierstrass Theorem this bounded sequence has a convergent subsequence $\{\alpha_{n_k}\}$ which converges to $\alpha_0 \in R$. By lemma 2, $\alpha_n \to \alpha_0$. Hence R is a Banach Space .

Example 2. The linear space C is a Banach Space with normed defined by $\|z\| = |z|$ for $z \in C$. Clearly $(C, |\,|)$ is a nls. Let $\{z_n = x_n + iy_n\}$ be a Cauchy sequence of complex numbers. Then there exists $n_0 \in N$, such that,

$|z_n - z_m|^2 = (x_n - x_m)^2 + (y_n - y_m)^2 < \varepsilon^2 \;\; \forall n, m \geq n_0$. Consequently, $|x_n - x_m| < \varepsilon$ and $|y_n - y_m| < \varepsilon \;\; \forall n, m \geq n_0$. This shows that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences of reals. Since R is complete, there exist x_0 and y_0 in R such that

$$\|x_n - x_0\| < \frac{\varepsilon}{\sqrt{2}} \quad \text{and} \quad \|y_n - y_0\| < \frac{\varepsilon}{\sqrt{2}} \quad \forall n \geq n_1.$$

Let $z_0 = x_0 + iy_0$. Then $\|z_n - z_0\|^2 = (x_n - x_0)^2 + (y_n - y_0)^2 < \varepsilon^2 \;\; \forall n \geq n_1$

Hence $z_n \to z_0$ and C is a Banach Space.

Example 3. The n-dimensional Euclidean space R^n is a Banach Space with norm, defined by

$$\|(x_1, \ldots x_n)\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$ $R^n(R)$ is a vector space with operations

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

$$\alpha(x_1, \ldots, x_n) = (\alpha x_1, \cdots, \alpha x_n) \quad \alpha \in R$$

For $\bar{x} = (x_1, \cdots x_n), \bar{y} = (y_1, \cdots y_n)$ in R^n,

$$\|\bar{x}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \geq 0$$

(N1) $\bar{x} = (0,0,\ldots.0)$ if and only if $\|\bar{x}\| = 0$

$$(N2) \quad \|\alpha\bar{x}\| = \|(\alpha x_1, \ldots \alpha x_n)\| = \sqrt{\sum_{i=1}^{n} (\alpha x_i)^2} = |\alpha| \sqrt{\sum_{i=1}^{n} x_i^2} = |\alpha| \|\bar{x}\|$$

$$(N3) \quad \|\bar{x} + \bar{y}\|^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + y_i^2 + 2x_i y_i = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + 2\sum_{i=1}^{n} x_i y_i$$

$$\leq \|\bar{x}\|^2 + 2\|\bar{x}\|\|\bar{y}\| + \|\bar{y}\|^2 \quad \text{(Cauchy Schwartz inequality)}$$

$$= \left(\|\bar{x}\| + \|\bar{y}\|\right)^2$$

Hence $\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$.

Hence $(R^n, \| \|)$ is a normal linear space . To prove R^n is a Banach Space , consider Cauchy sequence $\{\overline{x_m}\}$ in R^n where $\overline{x_m} = (x_{m,1}, x_{m,2}, ... x_{m,n})$

$$\|\overline{x_m} - \overline{x_n}\|^2 = \sum_{i=1}^{n}(x_{m,i} - x_{n,i})^2 < \varepsilon^2 \quad \forall m, n \geq n_0$$

$$\text{So,} |x_{m,i} - x_{n,i}| < \varepsilon \qquad \forall m, n \geq n_0$$

This shows that $\{x_{m,i}\}_{n\geq 1}$ is a Cauchy sequence of reals for each i=1,2,…n. Since R is complete, $\exists x_i \in R$ such that $\lim_{m\to\infty} x_{m,i} = x_i (1 \leq i \leq n)$. For $\varepsilon > 0, \exists n_i \in N$ such that

$$|x_{m,i} - x_i| < \frac{\varepsilon}{\sqrt{n}} \quad \forall m \geq n_i$$

Let $n_0 = \max^m \{n_1, n_2, ..., n_m\}$ and $\overline{x} = (x_1, x_2, ..., x_n)$.

Then $\|\overline{x_m} - \overline{x}\|^2 = \sum_{i=1}^{n} (x_{mi} - x_i)^2 < \varepsilon^2 \quad \forall m \geq n_0$

Hence, $\|\overline{x_m} - \overline{x}\| < \varepsilon \quad \forall m \geq n_0$. That is, $\lim_{m\to\infty} \overline{x_m} = \overline{x} \in R^n$.

Example 4. C^n is a complex Banach Space with norm,

$$\|(z_1, \cdots z_n)\| = \left(\sum_{i=1}^{n} |z_i|^2\right)^{1/2}.$$

Proof : Similar to that of R^n.

Example 5. C[a,b] is a Banach Space with norm defined by $\|f\| = \sup_{a\leq x\leq b} |f(x)|$.

Proof :

Since a continuous real valued function in [a,b], is bounded . $\|f\|$ is a finite real number.

Also $\|f\| = \sup_{a\leq x\leq b} |f(x)| \geq 0$

(N1) f = 0 implies $\|f\| = 0$ and conversly $\|f\| = 0$ implies $|f(x)| = 0 \ \forall x \in [a, b]$. Hence f=0, the zero function .

(N2) For $\alpha \in R$, $f \in C[a, b]$,

$$\|\alpha f\| = \sup\{|(\alpha f)(x)| : x \in [a,b]\} = |\alpha| \sup\{|f(x)| : x \in [a,b]\} = |\alpha| \ \|f\|$$

(N3) For $f \in C[a, b]$, $g \in C[a, b]$ and $x \in [a, b]$

$$|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\| + \|g\|.$$

So, $\|f + g\| = \sup\{|f(x) + g(x)| : a \le x \le b\} \le \|f\| + \|g\|$

Thus C[a,b] is a nls. It can be seen that C[a,b] is a Banach Space . Let $\{f_n\}$ be a Cauchy sequence in C[a,b]. For $\varepsilon > 0$, there exists $n_0 \in N$ such that

(*) $\quad \|f_n - f_m\| = \sup_{a \le x \le b} |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \ge n_0$

This shows that $\{f_n(x)\}_{n \ge 1}$ is a cauchy sequence of reals for each $x \in [a, b]$. Since R is complete, $\lim_{n \to \infty} f_n(x) \in R$.

Define $f : [a, b] \to R$ by, $\quad f(x) = \lim_{n \to \infty} f_n(x)$.

We can show that $\lim_{n \to \infty} f_n = f$ and $f \in [a, b]$

From (*) $\quad |f_n(x) - f_m(x)| < \varepsilon \quad \forall n, m \ge n_0$ and $\forall x \in [a, b]$

Keeping m fixed and letting $n \to \infty$, we have,

$$|f(x) - f_m(x)| < \varepsilon \quad \forall m \ge n_0 \text{ and } \forall x \in [a, b]$$

Hence $\|f - f_m\| = \sup_{a \le x \le b} |f(x) - f_m(x)| < \varepsilon \quad \forall m \ge n_0$

Thus we have, $\lim_{m \to \infty} f_m = f$. As in (*), we have

$$|f(x) - f_m(x)| < \varepsilon/3 \quad \forall m \ge n_1 \text{ and } \forall x \in [a, b]$$

Also $\quad |f_n(x) - f_m(x)| < \varepsilon/3 \quad \forall n, m \ge n_1$ and $\forall x \in [a, b]$

From the continuity of f_n at x_0, we have

$$|f_n(x) - f_n(x_0)| < \varepsilon/3 \text{ when } |x - x_0| < \delta$$

Now,

$$|f(x) - f(x_0)| \le |f(x) - f_{n_1}(x)| + |f_{n_1}(x) - f_{n_1}(x_0)| + |f_{n_1}(x_0) - f(x_0)|$$
$$< \varepsilon \text{ when } |x - x_0| < \delta$$

This shows that f is continuous at x_0 and hence in [a,b].

Hence C[a,b] is a Banach Space .

Theorem 4.1 Let M be a non empty subset of a normed linear space $(X, \|\|)$ and $\overline{M}$ its closure. Then

(a) $x \in \overline{M}$ if and only if there is a sequence $\{x_n\}$ in M such that $x_n \rightarrow x$

(b) M is closed if and only if $x_n \in M$, $x_n \rightarrow x$ implies $x \in M$.

Proof. **(a)** Let $x \in \overline{M}$. Either $x \in M$ or x is an accumulation point of M. If $x \in M$, $\{x, x, ...\}$ is a sequence in M which converges to x. If $x \notin M$, x is an accumulation point of M. Ball $B\left(x, \frac{1}{n}\right)$ contains a point $x_n \in M, x_n \neq x$. Taking n=1,2,3,..., $\{x_n\}$ is a sequence in M with $x_n \in B\left(x, \frac{1}{n}\right)$. Hence $\|x_n - x\| < \frac{1}{n}$. This shows $x_n \rightarrow x$.

Conversely let $\{x_n\} \subset M$ and $x_n \rightarrow x$. Either $x \in M$ or, $x \notin M$. Then $x_n \in B\left(x, \frac{1}{n}\right)$ and $x_n \neq x$. So, x is an accumulation point of M. In either case, $x \in \overline{M}$.

(b) M is closed if and only if $M = \overline{M}$. Then (b) follows from (a).

Theorem 4.2: A subspace M of Banach Space X is a Banach Space if and only if M is closed in X.

Proof : Let the subspace M of the Banach Space X be a Banach Space. If $x \in \overline{M}$ then there exists a sequence $\{x_n\}$ in M such that $x_n \rightarrow x$. The sequence $\{x_n\}$ then being a cauchy in Banach Space M, converges to a point of M. From the uniqueness of limit of convergent sequence $\{x_n\}$, it follows that $x \in M$. Hence M is closed by Theorem 4.1(b).

Assume M is a closed linear subspace of the Banach Space X. If $\{x_n\}$ is a cauchy sequence in M, then it is also a cauchy sequence in X. By the completeness of X, $x_n \rightarrow x \in X$. Since M is closed in X, by Theorem 4.1(b), $x \in M$. Hence M is a Banach Space.

Theorem 4.3 :

A map $T : X \rightarrow Y$ from a nls X into a nls Y is continuous at a point x_0 if and only if $x_n \rightarrow x_0$ implies $Tx_n \rightarrow Tx_0$.

Proof: Let $T : X \rightarrow Y$ be continuous at $x_0 \in X$. For given $\varepsilon > 0$ there exists $\delta > 0$ such that

(*) $\|Tx - Tx_0\| < \varepsilon$ whenever $\|x - x_0\| < \delta$

If $x_n \rightarrow x_0$, then there exists $n_0 \in R$ such that

$$\|x_n - x_0\| < \delta \quad \forall n \geq n_0$$

So, from (*), $\|Tx_n - Tx_0\| < \varepsilon \ \forall n \geq n_0$. Hence $Tx_n \to Tx_0$.

Conversely, let us assume that $x_n \to x_0$ implies $Tx_n \to Tx_0$.

Suppose T is not continuous at x_0 then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there is an $x \neq x_0$ satisfying

$$\|x - x_0\| < \delta \quad \text{but} \quad \|Tx - Tx_0\| \geq \varepsilon.$$

In particular, for $\delta = \frac{1}{n}$ there is an x_n satisfying

$$\|x_n - x_0\| < \frac{1}{n} \text{ but } \|Tx_n - Tx_0\| \geq \varepsilon.$$

This shows that $x_n \to x_0$ but $Tx_n \not\to Tx_0$. This is a contradiction and the result is proved.

Theorem 4.4: The closure $\overline{F}$ of a linear subspace F of a Banach Space X is a Banach Space.

We observe that every subspace of a normed linear space is a normed linear space and closed subset of a complete metric space is complete. It is enough to show that the closure $\overline{F}$ of a linear subspace is a linear subspace. Let x,y be in $\overline{F}$ and $\alpha, \beta \in K$. Then there exist $\{x_n\} \subset F$ and $\{y_n\} \subset F$ such that $x_n \to x$ and $y_n \to y$. Since F is a subspace, $\{\alpha x_n + \beta y_n\} \subset F$ and $\alpha x_n + \beta y_n \to \alpha x + \beta y$.

Hence $\alpha x + \beta y \in \overline{F}$. So, $\overline{F}$ is a closed linear subspace of the Banach Space X and consequently $\overline{F}$ is a Banach Space.

Example 6: l^∞ or m is a Banach Space.

$$l^\infty \text{ or } m = \left\{ \{\alpha_n\} \subset K \,\middle|\, \sup_n |\alpha_n| < \infty \right\} \text{ and let } \|\{\alpha_n\}\| = \sup_n |\alpha_n|$$

l^∞ is a linear space with operations

$$\{\alpha_n\} + \{\beta_n\} = \{\alpha_n + \beta_n\} \text{ and } \lambda\{\alpha_n\} = \{\lambda\alpha_n\}$$

where $0=\{0,0,0,\dots\}$ is the zero element of l^∞

Clearly $\|\{\alpha_n\}\| = \sup_n |\alpha_n| \geq 0$

(N1) $\{\alpha_n\} = 0$ if and only if $\sup_n |\alpha_n| = 0$ or $\|\{\alpha_n\}\| = 0$

(N2) $\|\lambda\{\alpha_n\}\| = \|\{\lambda\alpha_n\}\| = \sup_{n\geq 1} |\lambda\alpha_n| = |\lambda| \sup_{n\geq 1} |\alpha_n| = |\lambda| \ \|\{\alpha_n\}\|$ *for* $\lambda \in K$ *and* $\{\alpha_n\} \in l^\infty$

(N3) $\|\{\alpha_n\}+\{\beta_n\}\|=\|\{\alpha_n+\beta_n\}\|=\sup_{n\geq 1}|\alpha_n+\beta_n||\alpha_n+\beta_n|\leq|\alpha_n|+|\beta_n|\leq\|\{\alpha_n\}\|+\|\{\beta_n\}\|$

Hence, $\|\{\alpha_n\} + \{\beta_n\}\| \leq \|\{\alpha_n\}\| + \|\{\beta_n\}\|$

Let $\{\alpha^{(m)}\}$ be a cauchy sequence in l^∞ where

$\alpha^{(m)} = \{\alpha_{m,1}, \alpha_{m,2}, \ldots\}$. Then for $\varepsilon > 0$, there exists $n_0 \in N$ such that

(*) $\sup_{i\geq 1}|\alpha_{m,i} - \alpha_{n,i}| = \|\alpha^{(m)} - \alpha^{(n)}\| < \varepsilon \quad \forall m, n \geq n_0$.

So, $\{\alpha_{m,i}\}_{m\geq 1}$ is a cauchy sequence in K for each fixed $i \geq 1$.

Since K is a complete metric space, $\alpha_{m,i} \to \alpha_i \in K$

Let $\bar{\alpha} = \{\alpha_i\}_{i\geq 1}$. We can complete the proof showing that $\bar{\alpha} \in l^\infty$ and $\lim_{m\to\infty} \alpha^{(m)} = \bar{\alpha}$

From (*) $|\alpha_{m,i} - \alpha_{n,i}| < \varepsilon \quad \forall m, n \geq n_0$ and $\forall i \geq 1$

Taking limit as $n \to \infty$ and keeping m fixed, we have

(**) $|\alpha_{m,i} - \alpha_i| \leq \varepsilon \quad \forall m \geq n_0$ and $\forall i \geq 1$

Since $\alpha^{(n_0)} \in l^\infty$ we have $\sup_{i\geq 1}|\alpha_{n_0,i}| = K_{n_0} < \infty$

Hence, $|\alpha_i| \leq |\alpha_i - \alpha_{n_0,i}| + |\alpha_{n_0,i}| \leq \varepsilon + K_{n_0}$.

Since this inequality holds for all $i \geq 1$, $\sup_i|\alpha_i| < \infty$

Also from (**), $\|\alpha^{(m)} - \bar{\alpha}\| = \sup_{i\geq 1}|\alpha_{mi} - \alpha_i| \leq \varepsilon \quad \forall m \geq n_0$

Hence $\lim_{m\to\infty} \alpha^{(m)} = \bar{\alpha} \in l^\infty$. This proves that l^∞ is a Banach Space.

Identifying C and C_0 as closed linear subspaces of l^∞, we have two more examples of Banach Spaces.

Example 7: The linear space C consisting of all convergent sequences of complex numbers, with the norm induced from l^∞ is a Banach Space.

Since a convergent sequence is bounded, we have $C \subset l^\infty$. C is a linear subspace of l^∞ since, $\alpha_n \to \alpha, \beta_n \to \beta$ implies

$k\alpha_n + l\beta_n \to k\alpha + l\beta \ \ \forall k, l \in \forall$ That is $\{k\alpha_n + l\beta_n\} \in C \ \ \forall \alpha = \{\alpha_n\} \in C, \beta = \{\beta_n\} \in C$ and $k, l \in C$.

It suffices to show that C is a closed subspaces of l^∞. Let $\alpha = \{\alpha_n\} \in \overline{C}$, the closure of C in l^∞. There exists a sequence $\{\alpha^{(m)}\}$ in C such that $\alpha^{(m)} \to \alpha$, where $\alpha^{(m)} = \{\alpha_{m,i}\}_{i \geq 1}$. For given $\varepsilon > 0$ there exists $n_0 \in N$ such that $\sup_{i \geq 1} |\alpha_{m,i} - \alpha_i| < \varepsilon/3 \ \ \forall m \geq n_0$

In particular $|\alpha_{n_0,i} - \alpha_i| < \varepsilon/3 \ \forall i \geq 1$

Also, $\alpha^{(n_0)} = \{\alpha_{n_0,i}\}$ is a Cauchy sequence. So, $\exists n_1 \in N$ such that

$$|\alpha_{n_0,i} - \alpha_{n_0,i}| < \varepsilon/3 \ \ \forall i, j \geq n_1$$

Now, $|\alpha_i - \alpha_j| \leq |\alpha_i - \alpha_{n_0,i}| + |\alpha_{n_0,i} - \alpha_{n_0,j}| + |\alpha_{n_0,j} - \alpha_j| < \varepsilon \ \forall i, j \geq n_1$

This shows that $\alpha = \{\alpha_n\}$ is a Cauchy sequence in C. Since C is complete, $\{\alpha_n\}$ is a convergent sequence and $\alpha \in C$. Thus C is a closed linear subspace of l^∞ and hence C is a Banach Space.

Example 8: The set C_0 of all sequences of complex numbers converging to zero is a Banach Space with induced norm of l^∞.

Clearly $C_0 \subset C \subset l^\infty$. Also if $\alpha = \{\alpha_i\} \in C_0$ and $\beta = \{\beta_i\} \in C_0$ then

$k\alpha + l\beta = \{k\alpha_i + l\beta_i\}$ and $k\alpha_i + l\beta_i \to 0$ and hence $k\alpha + l\beta \in C_0 \ \ \forall k, l \in C$. Thus C_0 is a linear subspace of C and l^∞. It is enough to show that C_0 is a closed linear subspace of l^∞. Let $\alpha = \{\alpha_i\} \in \overline{C_0}$. Then there exists a sequence $\{\alpha^{(m)}\}$ in C_0 such that $\alpha^{(m)} \to \alpha$ where $\alpha^{(m)} = \{\alpha_{m,1}, \alpha_{m,2}, \cdots\}$. For given $\varepsilon > 0$ there exists $n_0 \in \angle$ such that

$$\sup_{i \geq 1} |\alpha_{m,i} - \alpha_i| = \|\alpha^{(m)} - \alpha\| < \varepsilon/2 \ \ \forall m \geq n_0$$

Also, $\alpha_{n_0,i} \to 0$. Hence $|\alpha_{n_0,i}| < \varepsilon/2 \quad \forall i \geq n_1$ for some $n_1 \in \angle$. So,

$$|\alpha_i| \leq |\alpha_{n_0,i} - \alpha_i| + |\alpha_{n_0,i}| < \varepsilon \ \ \forall i \geq n_1$$

This shows that $\alpha = \{\alpha_i\} \in C_0$. Consequently, C_0 is a closed linear subspace of l^∞ and hence C_0 is a Banach Space.

Example 9: For $1 \leq p < \infty$ the set l^p of all sequences $\{\alpha_n\}$ of scalars for which the series $\sum_{n \geq 1} |\alpha_n|^p$ converges, in a Banach Space with norm defined by

$$\|\{\alpha_n\}\| = \left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{\frac{1}{p}}$$

The closure property of l^p under addition and the triangle inequality follow from the Minkowskii's Inequality which is established as a cosequence of the following lemma and Holder's Inequality.

Lemma: If $\alpha > 0, \beta > 0$ and $\alpha + \beta = 1$ then for u>0 , v>0, $u^\alpha v^\beta \leq \alpha u + \beta v$

Proof of the lemma:

If t>0, $f(t) = t^\alpha (0 < \alpha < 1)$ is curve concave downwards [for $\alpha = \frac{1}{2}$, fig(i). So, the curve is below the tangent at (1,1). The equation of tangent at (1,1) is $y = \alpha t + (1 - \alpha) = \alpha t + \beta$

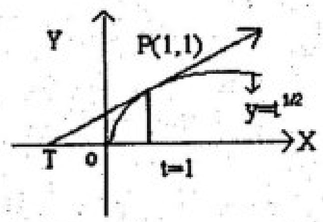


Fig. 1

So, $t^\alpha \leq \alpha t + \beta$. Putting $t = u/v$, we obtain $u^\alpha \leq \alpha u v^{\alpha-1} + \beta v^\alpha$

multiplying by $v^{1-\alpha}$, we have $u^\alpha v^\beta \leq \alpha u + \beta v$.

Holder's inequality : For $(1 < p < \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{i=1}^{n}|x_i y_i| \leq \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^q\right)^{\frac{1}{q}}$$

Proof : $\frac{1}{p} + \frac{1}{q} = 1$ implies $p=q(p-1)$

Put $p = \frac{1}{\alpha}$ and $q = \frac{1}{\beta}$. Let $u_i = \frac{|x_i|^p}{\sum_{j=1}^{n}|x_j|^p}$ and $v_i = \frac{|y_i|^q}{\sum_{j=1}^{n}|y_j|^q}$

From the lemma,

$u_i^{\alpha} v_i^{\beta} \le \alpha u_i + \beta v_i$, we have $\dfrac{|x_i|}{\left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}}} \cdot \dfrac{|y_i|}{\left(\sum_{j=1}^{n}|y_j|^q\right)^{\frac{1}{q}}} \le \alpha u_i + \beta v_i$

Therefore, $$\frac{\sum_{i=1}^{n}|x_i y_i|}{\left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|y_j|^q\right)^{\frac{1}{q}}} \le \alpha\sum_{j=1}^{n} u_i + \beta\sum_{j=1}^{n} v_i = \alpha + \beta = 1$$

So, the Holders inequality $\sum_{i=1}^{n}|x_i y_i| \le \left(\sum_{j=1}^{n}|x_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|y_j|^q\right)^{\frac{1}{q}}$ follows.

For $1 \le p < \infty$, we have

$$\sum_{i=1}^{n}|x_i+y_i|^p \le \sum_{i=1}^{n}|x_i+y_i|^{p-1}(|x_i|+|y_i|) = \sum_{i=1}^{n}|x_i+y_i|^{p-1}|x_i| + \sum_{i=1}^{n}|x_i+y_i|^{p-1}|y_i|$$

$$\le \left(\sum_{i=1}^{n}|x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|x_i+y_i|^{(p-1)q}\right)^{\frac{1}{q}}\left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}$$

(Holder's Inequality)

$$= \left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{\frac{1}{q}}\left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}\right).$$

Dividing by $\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{\frac{1}{q}}$

$$\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{1-\frac{1}{q}} \le \left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}|y_i|^p\right)^{\frac{1}{p}}$$

$$(*) \le \|\{x_i\}\| + \|\{y_i\}\| \text{ if } \{x_i\} \in l^p \text{ and } \{y_i\} \in l^p$$

Since this inequality holds for all $n \ge 1$, the sequence $\left\{\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{\frac{1}{p}}\right\}_{n\ge 1}$ being monotonic increasing and bounded above is convergent.

Hence the series $\sum_{i=1}^{\infty}|x_i + y_i|^p$ is convergent , i.e. $\{x_i\} + \{y_i\} \in l^p$. Thus l^p is closed under addition . The verification that l^p is a linear space is a routine work. Also from (*)

$$\left(\sum_{i=1}^{\infty}|x_i + y_i|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty}\left(\sum_{i=1}^{n}|x_i + y_i|^p\right)^{\frac{1}{p}} \le \|\{x_i\}\| + \|\{y_i\}\|$$

Hence $\|\{x_i\} + \{y_i\}\| \le \|\{x_i\}\| + \|\{y_i\}\|$. It is now easy to check that $(l^p, \|\ \|)$ is a normed linear space.

To prove the completeness of l^p , consider a Cauchy sequence $\{x^{(m)}\}$ in l^p where $x^{(m)} = \{x_{m,1}, x_{m,2}, \ldots\}$

(**) $|x_{m,i} - x_{n,i}|^p \le \sum_{i=1}^{\infty}|x_{m,i} - x_{n,i}|^p = \|x^{(m)} - x^{(n)}\|^p < \varepsilon^p \ \forall m, n \ge n_0$. This shows that $\{x_{m,i}\}_{m\ge1}$ is a cauchy sequence of scalars which converges to a point x_i.

Let $\bar{x} = \{x_i\}_{i\ge1}$. We are left to show that $x \in l^p$ and $\lim_{m\to\infty} x^{(m)} = \bar{x}$. From (**), we have

$$\sum_{i=1}^{k}|x_{m,i} - x_{n,i}|^p \le \|x^{(m)} - x^{(n)}\|^p < \varepsilon^p \ \forall m, n \ge n_0$$

Taking limit as $n \to \infty$, keeping m fixed, $\sum_{i=1}^{k}|x_{m,i} - x_i|^p \le \varepsilon^p \ \forall m \ge n_0$ and $\forall k \ge 1$

Taking limit as $k \to \infty$,

(***) $$\left(\sum_{i=1}^{\infty}|x_{m,i} - x_i|^p\right)^{\frac{1}{p}} \le \varepsilon \ \forall m \ge n_0$$

This shows that $\left(x^{(n_0)} - \bar{x}\right) \in l^p$. Also $x^{(n_0)} \in l^p$

Hence $\bar{x} = x^{(n_0)} - \left(x^{(n_0)} - \bar{x}\right) \in l^p$

Also from (***), $\|x^{(m)} - \bar{x}\| \le \varepsilon \ \forall m \ge n_0$

Hence $\lim_{m\to\infty} x^{(m)} = \bar{x}$. Thus $l^p (1 \le p < \infty)$ is a Banach Space .

Some examples of normed linear spaces which are not Banach Space

Example 10. The set P of all polynomials considered as functions of t on some finite closed interval [a,b] with norm defined by

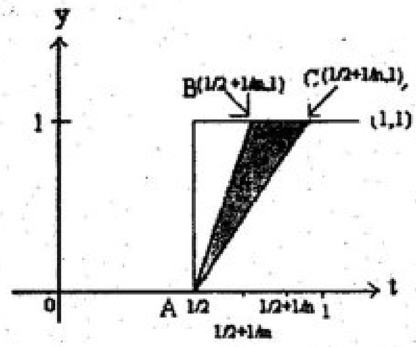
$\|x\| = \max_{t\in[a,b]} |x(t)|$, where $x(t) = \sum a_i t^i$, $(a_i \in R)$ finitely many a_i's being non-zero.

Then $(P, \|\ \|)$ is a normed linear space which is not complete.

Example 11. If X be the set of all continuous , real valued functions x(t) on [0,1] and $\|x\| = \int_0^1 |x(t)|dt$. Then X is a normed linear space but not a Banach Space .

Verification that $(X, \|\ \|)$ is a normed linear space is routine. Consider the cauchy sequence $\{x_m\}$ defined by

$$y = x_m(t) = 0 \text{ if } t \in \left[0, \frac{1}{2}\right]$$
$$= m\left(t - \frac{1}{2}\right) \text{ if } t \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{m}\right]$$
$$= 1 \text{ if } t \in \left[\frac{1}{2} + \frac{1}{m}, 1\right]$$



$$\|x_m - x_n\| = \int_0^1 |x_m(t) - x_n(t)|\,dt = area \ \ of \ \ \Delta ABC = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{m}\right) \ if \ \ n < m$$

$$= \frac{1}{2}\left|\frac{1}{n} - \frac{1}{m}\right| \le \left(\frac{1}{n} + \frac{1}{m}\right) < \frac{1}{n}.$$

This shows that $\{x_m\}$ is a Cauchy sequence in X. But this sequence does not converge to any $x \in X$. For every $x \in X$,

$$\|x_m - x\| = \int_0^1 |x_m(t) - x(t)|\ dt = \int_0^{1/2} |x(t)|\,dt + \int_{1/2}^{1/2+1/m} |x_m(t) - x(t)|\,dt + \int_{1/2+1/m}^{1} |1 - x(t)|\,dt$$

If $x_m \to x$ then each integrand approaches to zero. Since x is continuous and the integrands are non-negative,

$$x(t) = 0 \qquad \text{if } t \in [0, \tfrac{1}{2})$$
$$= m\left(t - \frac{1}{2}\right) \quad \text{if } t \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{m}\right]$$
$$= 1 \qquad \text{if } t \in \left[\frac{1}{2} + \frac{1}{m}, 1\right]$$

$$\text{As } m \to \infty,\ x(t) = 0 \quad \text{if } t \in \left[0, \frac{1}{2}\right)$$
$$= 1 \quad \text{if } t \in \left[\frac{1}{2}, 1\right]$$

Clearly x(t) is a discontinuous function . Hence the Cauchy sequence $\{x_m\}$ does not converge to any $x \in X$. This shows that X is not a Banach Space .

Series Convergence in Banach Spaces :

Definitions 5.1: Let X be a normed linear space and $x_n \in X$. The series $\sum_{n\geq 1} x_n$ is said to be summable in X if the sequence $\{S_m\}$ of its partial value sums $S_m = \sum_{n=1}^{m} x_n$ converges in X. If $\{S_m\}$ converges to S in X, then we write $S = \sum_{n=1}^{\infty} x_n$ and say that S is the sum of the series . A series $\sum_{n\geq 1} x_n$ is said to be absolutely **summable** if the series $\sum_{n\geq 1} \|x_n\|$ is convergent.

Theorem 5.1 : A normed space X is a Banach Space if and only if every absolutely summable series of elements in X is summable in X.

Proof: Let X be a Banach Space and $\sum_{n\geq 1} \|x_n\| < \infty$. If $S_m = \sum_{n=1}^{m} x_n$, then

$$\|S_{m+j} - S_m\| = \|x_{m+1} + ... + x_{m+j}\| \leq \|x_{m+1}\| + \ ...+ \|x_{m+j}\| \quad \forall m, j = 1,2,3...$$

Since $\sum_{m\geq 1} \|x_m\|$ is convergent, by cauchy criteria , for all $\varepsilon > 0\ \exists\ n_0 \in N$ such that

$\|S_{m+j} - S_m\| \leq \|x_{m+1}\| + ... + \|x_{m+j}\| < \varepsilon \quad \forall m, n \geq n_0$. Hence $\{S_m\}_{m\geq 1}$ is a cauchy sequence in the Banach Space X. Consequently $\sum_{n\geq 1} x_n$ is summable in X.

Conversely , suppose every absolutely summable series be summable in X. Let $\{S_n\}_{n\geq 1}$ be a cauchy sequence in X. There exists a positive integer m_1 such that

$$\|S_m - S_{m_1}\| < 1 \quad \forall m \geq m_1.$$

Define inductively $m_2, m_3, \cdots$ such that $m_n < m_{n+1}$ *and* $\|s_m - s_{m_n}\| < \frac{1}{n2}$ $\forall m \geq m_n$.

Let $x_n = s_{m_{n+1}} - s_{m_n}$ for $n = 1,2,3\cdots$. Then $\|x_n\| < \frac{1}{n^2}$ and $\sum_{n\geq 1}\|x_n\| < \sum_{n\geq 1}\frac{1}{n^2} < \infty$

Hence $\sum_{n\geq 1} x_n$ is summable in X. But $S_{m_n} = S_{m_1} + \sum_{j=1}^{n-1} x_j$. So, the subsequence $\{S_{m_n}\}_{n\geq 1}$ of the cauchy sequence $\{S_m\}$converges in X. Hence $\{s_m\}$converses in X and X is a Banach Space.

Theorem 5.2: Let Y be a closed subspace of a normed linear space X. Then

a) $X/Y = \{x + Y | x \in X\}$is a normed linear space with norm defined by

$$\||x + Y|\| = \inf\{\|x + y\| : y \in Y\}, \text{ called } \textbf{quotient} \text{ norm}$$

b) A sequence $\{x_n + Y\}$ converges to x+Y in X/Y iff there is a sequence $\{y_n\}$in Y such that $\{x_n + y_n\}$ converses to x in X.

c) X is a Banach space iff Y and X /Y are Banach spaces.

Proof: For $x \in X$, $\||x + Y|\| = \inf\{\|x + y\| : y \in Y\} \geq 0$

N1) If $x + Y = Y$, then $\||x + Y|\| = \inf\{\|y\| : y \in Y\} = 0$ (since $0 \in Y$)

Conversely, if $\|x + Y\| = 0$, there is a sequence $\{y_n\}$in Y such that $x + y_n \to 0$, that is, $y_n \to -x$. Since Y is closed, $-x \in Y$. So $x + Y = Y$.

N2) For $x \in X$ and $k(\neq 0) \in K$,

$$\begin{aligned}\|k(x+Y)\| &= \|kx+Y\| = \inf\ \{\|kx+y\| : y \in Y\} = |k| \inf\ \{\|x+k^{-1}y\| : y \in Y\} \\ &= |k| \inf\ \{\|x+u\| : u \in Y\} \quad (\textit{Since } k^{-1}Y = Y) \\ &= |k| \|x+Y\|\end{aligned}$$

It also holds for $k = 0$.

N3) For $\varepsilon > 0$ there exist y_1 and y_2 in Y such that

$$\|x_1 + y_1\| < \|x_1 + Y\| + \frac{\varepsilon}{2} \quad \textit{and} \quad \|x_2 + y_2\| < \|x_2 + Y\| + \frac{\varepsilon}{2}$$

So, $\|x_1 + Y + x_2 + Y\| \leq \|(x_1 + x_2) + (y_1 + y_2)\| \leq \|x_1 + Y\| + \|x_2 + Y\| + \varepsilon$

for every $\varepsilon > 0$. Hence the triangular inequality follows.

(b) Let $\{x_n + Y\}$ be a sequence in X/Y and $\{y_n\}$ be a sequence in Y such that

$x_n + y_n \to x$ in X. Then

$$\|(x_n+Y)-(x+Y)\| = \|(x_n-x)+Y\| \le \|x_n-x+y_n\| \to 0 \ \text{as} \ n\to\infty. \ \text{So,} \ x_n+Y \to x+Y$$

Conversely assume that $x_n + Y \to x + Y$ in X/Y

Then there exist $y_n \in Y$ such that

$$\|x_n-x+y_n\| < \|(x_n+Y)-(x+Y)\| + \frac{1}{n}, (n=1,2,....) \to 0 \ \text{as} \ n\to\infty$$

Hence, $x_n + y_n \to x$.

(c) Let X be a Banach Space . Y being a closed subspace of X, is also a Banach Space . Then to show that X/Y is a Banach Space , by Theorem 5.1, it is enough to show that absolutely convergent series $\sum_{n=1}^{\infty}(x_n + Y)$ is also convergent.

For each positive integer n, there exist $y_n \in Y$ such that

$$\|x_n + y_n\| \le \|x_n + Y\| + \frac{1}{n^2} (n = 1,2,3...)$$

Hence $\sum_{n=1}^{\infty}\|x_n + Y\|$ is convergent implies $\sum_{n=1}^{\infty}\|x_n + y_n\|$ is convergent.

Since X is a Banach Space , by Theorem 5.1 $\sum_{n=1}^{\infty}(x_n + y_n) = s \in X$.

Now,

$$\left\|\sum_{n=1}^{m}(x_n+Y)-(s+Y)\right\| = \left\|\sum_{n=1}^{m}x_n+Y-s+Y\right\| = \left\|\sum_{n=1}^{m}x_n-s+Y\right\| (\because Y+Y=Y)$$

$$= \left\|\sum_{n=1}^{m}(x_n+y_n)-s+Y\right\| \left(\because \sum_{n=1}^{m}y_n+Y=Y\right)$$

$$\le \left\|\sum_{n=1}^{m}(x_n+y_n)-s\right\| \to 0 \ \text{as} \ m\to\infty$$

Hence the absolutely convergent series $\sum_{n=1}^{\infty} x_n + Y$ is also convergent. This proves that X/Y is a Banach Space .

Conversely, assume that Y and X/Y are Banach Spaces. Let $\{x_n\}$ be a cauchy sequence in X. Since,

$$\|(x_n + Y) - (x_m + Y)\| = \|(x_n - x_m) + Y\| \le \|x_n - x_m\|,$$

$\{x_n + Y\}$ is a cauchy sequence in X/Y. If $x_n + Y \to x + Y$ in X/Y, by (b) part, there is a sequence $\{y_n\}$ in Y such that $x_n + y_n \to x$ in X.

Then $\{y_n\}$ is a cauchy sequence in Y, since

$$\|y_n - y_m\| \le \|y_n + x_n - x\| + \|x_m - x_n\| + \|x_m + y_m - x\|$$

Since Y is a Banach Space, $y_n \to y \in Y$.

Then $x_n = (x_n + y_n) - y_n \to x - y$ in X. Hence X is a Banach Space.

Theorem 5.3 Let $\|\|_j$ be a norm on a linear space X_j $(1 \le j \le m)$ and let $1 \le p < \infty$. For $x = (x_1, \ldots, x_n) \in X_1 x \ldots x X_m = X$,

$$\text{Let } \|x\|_p = \left(\sum_{j=1}^{m} \|x_j\|_j^p\right)^{\frac{1}{p}}, \text{if } 1 \le p < \infty \quad = \max_{1\le j\le m} \|x_j\|_j, \text{ if } p = \infty$$

Then (a) $\|\|_p$ is a norm on X

(b) If $x^{(n)} = (x_{n,1}, \ldots, x_{n,m}) \in X$, then $x^{(n)} \to x$ if and only if $x_{n,j} \to x_j$ as $n \to \infty$ $(1 \le j \le m)$

(c) X is a Banach Space if and only if X_j is a Banach Space.

Proof: (a) All conditions except the triangular inequality are obvious. If $(1 \le p \le \infty)$

$$\|x+y\|_p = \left(\sum_{j=1}^{m} \|x_j + y_j\|_j^p\right)^{\frac{1}{p}} \le \left(\sum_{j=1}^{m} \left(\|x_j\|_j + \|y_j\|_j\right)^p\right)^{\frac{1}{p}}$$

$$\le \left(\sum_{j=1}^{m} \|x_j\|_j^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{m} \|y_j\|_j^p\right)^{\frac{1}{p}} (\textit{Holder's inequality}) = \|x_j\|_p + \|y_j\|_p$$

If $p = \infty$,

$$\|x+y\|_\infty = \max_{1\le j\le m} \|x_j + y_j\|_j \le \max_{1\le j\le m} \left(\|x_j\|_j + \|y_j\|_j\right) \le \max_{1\le j\le m} \|x_j\|_j + \max_{1\le j\le m} \|y_j\|_j = \|x\|_\infty + \|y\|_\infty$$

Hence $(X, \|\|_p)$ is a normed linear space $(1 \le p \le \infty)$

(c) Let $\lim_{n\to\infty} x^{(n)} = x$. For $\varepsilon > 0$ there exists $n_0 \in N$ such that $\|x^{(n)} - x\|_p < \varepsilon \ \forall n \geq n_0$

Then for $1 \leq p < \infty$, $$\|x_{n,j} - x_j\|_j^p \leq \sum_{j=1}^{m} \|x_{n,j} - x_j\|_j^p < \varepsilon^p \quad \forall n \geq n_0$$

If $p = \infty$ $$\|x_{n,j} - x_j\|_j \leq \|x^{(n)} - x\|_\infty < \varepsilon \quad \forall n \geq n_0$$

Hence, $x_{n,j} \to x_j$ as $n \to \infty (1 \leq p \leq \infty)$

Conversely, let $x_{n,j} \to x_j$ in $X_j (1 \leq j \leq m)$

Then $\|x_{n,j} - x_j\|_j < \dfrac{\varepsilon}{\sqrt[p]{m}} \quad \forall n \geq n_0$

Hence $\|x^{(n)} - x\|_p^p < \varepsilon^p \ \forall n \geq n_0$ and $x^{(n)} \to x$ in X.

(c) Let X_j be a Banach Space $(1 \leq j \leq m)$ and $\{x^{(n)}\}$ be a Cauchy sequence in X. Then each $\{x_{n,j}\}_{n\geq 1} (1 \leq j \leq m)$ is a Cauchy sequence. Hence $x_{n,j} \to x_j \in X_j (1 \leq j \leq m)$. By 5.3(b), $x^{(n)} \to x$. This proves that X is a Banach Space.

Conversely, suppose X is a Banach Space. If $\{x_{n,j}\}$ is a Cauchy sequence in X_j, then $\{x^{(n)}\}$ is a Cauchy sequence in X where $x^{(n)} = (0,0,...,0, x_{n,j}, 0,...0)$

So, $x^{(n)} \to (x_1, x_2,..., x_j,..., x_m) \in X$. Then by 5.3 (b) $x_{n,j} \to x_j$ as $n \to \infty$ and $x_j \in X_j$. So, X_j is a Banach Space. This proves the theorem.

5. Continuity of Linear Maps:

Suppose X and Y are normed linear spaces and $x_0 \in X$. A linear map $F : X \to Y$ is **continuous at** $x_0 \in X$ if for $\varepsilon > 0$ there is a $\delta > 0$ such that $\|F(x - x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$. F is said to be continuous in X if it is continuous at every point of X. F is said to be **uniformly continuous** in X if if for $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|F(x - y)\| < \varepsilon \text{ whenever } \|x - y\| < \delta.$$

Theorem 6.1

(a) Every linear map from a finite dimensional normed linear space X to a normed linear space Y is continuous.

(b) If X is infinite dimensional and $Y \neq \{0\}$ then there is a discontinuous linear map from X to Y.

Proof(a): Suppose dim $X=n<\infty$ and $f : X \to Y$ is a linear map. Let

$B=\{x_1, x_2,...x_n\}$ be a basis for X and $\{u_m\}$ be a sequence in X where

$u_m = a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n$.

Suppose $u_m \to u = a_1x_1 + a_2x_2 + ... + a_nx_n$. Then by 5.3 (b),

$a_{m,j} \to a_j (1 \le j \le n)$. Then by the continuity of addition and scalar multiplication

$$F(u_m) = \sum_{j=1}^{n} a_{m,j}F(x_j) \to \sum_{j=1}^{n} a_jF(x_j) = F\left(\sum_{j=1}^{n} a_jx_j\right) = F(u)$$

Thus $u_m \to u \Rightarrow F(u_m) \to F(u)$. Hence F is continuous.

(b) Let X be an infinite dimensional normed linear space and let $\{e_1, e_2,...\}$ be an infinitely linearly independent subset of X. If $x_n = \frac{e_n}{n\|e_n\|}$, then $L=\{x_1, x_2,...\}$ is an infinite linearly independent subset of X and $\|x_n\| = \frac{1}{n} \to 0$ as $n \to \infty$. Extend L to a basis B such that $L \subset B$. Let Y be a non-zero normed linear space and $b(\neq 0) \in Y$.

Define $F : X \to Y$ by,

$$F(x) = b, \text{ if } x \in L$$
$$= 0, \text{ if } x \in B \setminus L$$

Extend F from B to the whole space X linearly. $x_n \to 0$ but $F(x_n) = b$ does not tend to 0. So, F is a linear discontinuous. Thus there exists a linear map from X to Y which is not continuous.

Theorem 6.2 Let X and Y be normed linear spaces and $F : X \to Y$ be a linear map. The following conditions are equivalent.

(i) F is bounded on $\overline{B}_r(0) = \{x : \|x\| \le r\}$ for some $r>0$.

(ii) F is continuous at 0

(iii) F is continuous on X

(iv) F is uniformly continuous on X

(v) $\|F(x)\| \le \alpha\|x\| \quad \forall x \in X$ and some $\alpha > 0$.

(vi) The null space $N(F) = \{x \in X | F(x) = 0\}$ of F is closed in X and the linear map $\overline{F} : X/N(F) \to Y$ defined by $\overline{F}(x + N(F)) = F(x), x \in X$, is continuous.

[$F:X\to Y$ is bounded on $A \subseteq X$ means $\|F(x)\| \le K < \infty \quad \forall x \in A$]

Proof: We complete the proof by showing that

$$(i) \to (v) \to (iv) \to \underset{\substack{\updownarrow \\ (vi)}}{(iii)} \to (ii) \to (i)$$

(i)→ (v): Let F be bounded on $\overline{B}_r(0)$ for some r>0. So, there exists M>0 such that $\|F(x)\| \le M$ for $x \in \overline{B}_r(0)$. Then for all $x \in X, x \ne 0$,

$$\|F(x)\| = \left\|F\left(\frac{\|x\|}{r}.\frac{rx}{\|x\|}\right)\right\| = \frac{\|x\|}{r}\left\|F\left(\frac{rx}{\|x\|}\right)\right\| \le \frac{\|x\|}{r}M = \alpha\|x\|, \quad \text{where } \alpha = \frac{M}{r}$$

It also holds for x=0. So (v) follows.

(v) → (iv): Let $\|F(x)\| \le \alpha\|x\| \quad \forall x \in X$ and some $\alpha > 0$. Then for $\varepsilon > 0$,

$$\|F(x) - F(y)\| = \|F(x - y)\| \le \alpha\|x - y\| < \varepsilon \text{ if } \|x - y\| < \frac{\varepsilon}{\alpha}$$

Hence F is uniformly continuous on X and (iv) follows.

(iv) → (iii). Suppose (iv) holds. Let $x_0 \in X$ be an arbitrary point of X. By (iv), for $\varepsilon > 0, \exists\delta > 0$ such that

$$\|F(x) - F(x_0)\| < \varepsilon \text{ when } \|x - x_0\| < \delta$$

So, F is continuous at x_0 and hence on X.

(iii)→ (ii) : If F is continuous on X then F is continuous at every point of X. Hence it is continuous at 0.

(ii) → (i) : Let F be continuous at the origin. For $\varepsilon > 0, \exists\delta > 0$ such that

$$\|F(x)\| < \varepsilon \text{ whenever } \|x\| \le \delta$$

Then $x \in \overline{B}_r(0)$ implies $\left\|\frac{\delta x}{r}\right\| < \delta$ and hence $\left\|F\left(\frac{\delta x}{r}\right)\right\| < \varepsilon$ and $\|F(x)\| < \frac{\varepsilon r}{\delta} = M$

Hence (i) is proved.

Lastly we prove that (iii) ↔ (vi). Let $F : X \to Y$ be continuous on X. Since $\{0\}$ is a closed set in Y and the inverse of a closed set under continuous map is closed, we see that the null space of F,

$N(F) = F^{-1}(\{0\})$ is a closed subspace of X and $X/N(F)$ is well defined. Also for $z \in N(F)$,

$$\|\overline{F}(x + N(F))\| = \|\overline{F}(x + z + N(F))\| = \|F(x + z)\| \le \alpha\|x + z\|$$

Since it holds for all $z \in N(F)$ we have

$$\|\bar{F}(x + N(F))\| \le \alpha \inf\|x + z\| = \alpha\|x + N(F)\|.$$

Hence $\bar{F}$ is continuous .

Conversely, assume that N(F) is closed and $\bar{F}$ is continuous . Then for $\alpha > 0$, we have

$$\|F(x)\| = \|\bar{F}(x + N(F))\| \le \alpha\|x + N(F)\| \le \alpha\|x\| \quad \forall x \in X.$$

By (v) F is continuous .

Theorem 6.3 A linear map F from a normed linear space X to a normed linear space Y is a homeomorphism from X onto R(F) if and only if there exist $\alpha, \beta > 0$ such that

$\beta\|x\| \le \|F(x)\| \le \alpha\|x\| \quad \forall x \in X$. In case, there is a linear homeomorphism from X to Y, X is complete if and only if Y is complete.

Proof: Let $F : X \to Y$ be a linear map and suppose

(*) $\quad \beta\|x\| \le \|F(x)\| \le \alpha\|x\| \quad \forall x \in X$

To show that $F: X \to R(F)(\subset Y)$ is a homeomorphism

F is one-to-one:

$F(x_1)= F(x_2)= F(x_1-x_2)=0.$ Hence from (*),

$$\beta\|x_1 - x_2\| \le 0 \le \alpha\|x_1 - x_2\| \Rightarrow \|x_1 - x_2\| = 0 \quad (\because \alpha,\ \beta > 0) \Rightarrow x_1 = x_2$$

Also, $F: X \to R(F)$ is obvious ly onto. Also from (*), F is continuous. $F^{-1} : R(F) \to X$ is also onto. Let $y_1, y_2 \in R(F)$. So, there exist $x_1, x_2 \in X$ such that

$y_1 = F(x_1)$ and $y_2 = F(x_2)$. For $\alpha, \beta \in K$, $\alpha y_1 + \beta y_2 = F(\alpha x_1 + \beta x_2)$

Hence, $F^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha F^{-1}(y_1) + \beta F^{-1}(y_2)$

Then from (*), $\beta\|F^{-1}(y)\| \le \|y\|$ if $y = F(x) \in R(F)$.

And this shows that F^{-1} is continuous by Theorem 6.2 (v). Thus $F: X \to R(F)$ is a linear homeomorphism .

Suppose $F : X \to Y$ is a linear homeomorphism and X is a complete normed linear space. Let $\{y_n\}$ be a cauchy sequence in Y and $y_n = F(x_n)$.

Then $\beta\|x_n - x_m\| \le \|F(x_n - x_m)\| = \|y_n - y_m\|$.

From this it follows that since $\{y_n\}$ is a cauchy sequence, $\{x_n\}$ is also a cauchy sequence .

Since X is complete, $x_n \to x \in X$. From (*) $\|F(x_n) - F(x)\| \le \alpha\|x_n - x\|$

It follows that $y_n = F(x_n) \to F(x) \in Y$ if $x_n \to x$

Hence Y is complete if X is complete. Similarly X is complete if Y is complete.

Corollary 6.4: Two norm $\|\,\|$ and $\|\,\|'$ are equivalent, that is, the same topology is generated by the two norms if and only if there are $\alpha,\beta > 0$ such that $\beta\|x\| \le \|x\|' \le \alpha\|x\|$, $\forall x \in X$.

In fact $\|\,\|$ and $\|\,\|'$ are equivalent if and only if the identity map

$I_x : (X,\|\,\|) \to (X,\|\,\|')$ is a homeomorphism that is, if and only if

$$\beta\|x\| \le \|I_x(x)\|' \le \alpha\|x\|$$
$$\beta\|x\| \le \|x\|' \le \alpha\|x\|, \quad \forall x \in X.$$

Theorem 6.5 : let X and Y be two normed linear space and X be finite dimensional. Then

(a) every bijective linear map from X to Y is a homeomorphism.

(b) All norms on X are equivalent

(c) If dim $X=n<\infty$, then there is a linear homeomorphism from K^n onto X and X is complete with respect to each norm on X.

Proof : (a) Let $F : X \to Y$ be a bijective linear map from finite dimensional normed space X onto Y By Theorem 6.1(a), F is continuous. Since X is finite dimensional and $F : X \to Y$ is a bijective linear map, Y is also finite dimensional normed space.

Again by Theorem 6.1(a), $F^{-1} : Y \to X$ is continuous.

Hence F is a linear homeomorphism.

(b) Let $\|\,\|$ and $\|\,\|'$ be two norms on X. The identity map $I_x : (X,\|\,\|) \to (X,\|\,\|')$ is a bijective linear map. By part (a), I_x is a linear homeomorphism. So the two topologies induced by $\|\,\|$ and $\|\,\|'$ must be identical. This means that norms $\|\,\|$ and $\|\,\|'$ are equivalent.

(c) Let dimX=n and $B=\{x_1, x_2,..., x_n\}$ be a basis for X. Then for $(\alpha_1, \alpha_2,..., \alpha_n) \in K^n$, $F(\alpha_1, \alpha_2,..., \alpha_n) = \alpha_1x_1 + \alpha_2x_2 + ... + \alpha_nx_n$ is a bijective linear map from K^n onto X. By part (a), F is a linear homeomorphism.

Again $(K^n, \|\,\|_p)$ $(1 \le p \le \infty)$ is a complete normed space(Theorem 5.3) Hence by Theorem 6.3, X is complete normed linear space with any norm in it.

Theorem 6.6 : Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range space R(F) of F is finite dimensional. Then F is continuous if and only if the null space N(F) of F is a closed subspace of X.

A linear functional f on X is continuous if and only if N(F) is closed in X.

Proof: Let $F : X \to Y$ be a linear map where $\dim R(F) < \infty$. Suppose F is a continuous map Then $N(F) = F^{-1}(\{0\})$ is a closed subspace of X since $\{0\}$ is a closed set and the inverse of a closed set under a continuous map is continuous.

Conversely, suppose N(F) is a closed subspace of X. If $R(F)=\{0\}$ then $F(x)=0$ for all $x \in X$. Then constant map F is continuous. Let $R(F) \neq 0$ and let $\{y_1, y_2, ..., y_m\}$ be a basis for R(F). Let $x_j \in X$ such that $F(x_j) = y_j (1 \le j \le m)$. It can be easily seen that

$$X/N(F) = \text{span}\{x_1 + N(F), x_2 + N(F), ..., x_m + N(F)\}$$

In fact, for $x \in X$,

$$F(x) = k_1 y_1 + k_2 y_2 + ... + k_m y_m = F(k_1 x_1 + k_2 x_2 + ... + k_m x_m)$$

T

So, $x - (k_1 x_1 + k_2 x_2 + ... + k_m x_m) \in N(F)$. Hence

$$x + N(F) = (k_1 x_1 + k_2 x_2 + ... + k_m x_m) + N(F) = k_1(x_1 + N(F)) + ... + k_m(x_m + N(F))$$

Hence $X/N(F)$ is a finite dimensional normed linear space since N(F) is a closed subspace of X. By Theorem 6.1, $\tilde{F} : X/N(F) \to Y$, $\tilde{F}(X + N(F)) = F(x)$ is continuous. So F is continuous by Theorem 6.2 (vi)

Let $f : X \to K$ be a linear functional on X. $\dim R(f) \le \dim K = 1$. Hence f is continuous if and only if the null space N(F) is closed.

Note 1. The result is not true if R(F) infinite dimensional. For example, let

$X = C^1[0,1]$, the space of scalar -valued function with continuous first derivative

$Y = C[0,1]$, the space of all continuous scalar -valued function on [0,1]

Both X and Y are normed spaces with supremum norm.

Consider $F : C^1[0,1] \to C[0,1]$ defined by

$F(x) = \dfrac{dx}{dt}$ $(x = x(t), (0 \le t \le 1)$. Clearly F is linear but not continuous. In fact, if $x_n(t) = t^n, t \in [0,1]$ then $\|x_n\| = \sup_{t \in [0,1]} |x_n(t)| = 1$

But $\|F(x_n)\| = \|nt^{n-1}\| = \sup_{t \in [0,1]} (nt^{n-1}) = n\|x_n\|$. So there is no M>0 satisfying $\|F(x)\| \le M\|x\| \ \forall x \in X$. Hence by Theorem 6.2 (v), F is not continuous.

We can observe that in this case N(F) is a closed subspace but R(F) is not finite dimensional. In fact,

$N(F) = \left\{ x \,\middle|\, \frac{dx}{dt} = 0 \right\}$ = the space of all constant functions on [0,1] . If $\{x_n | x_n(t) = \alpha_n, 0 \le t \le 1\}$ be a sequence of constant functions in N(F) and $x_n \to x$, then

$$x(t_1) - x(t_2) = \lim_{n\to\infty} x_n(t_1) - \lim_{n\to\infty} x_n(t_2) = \alpha_n - \alpha_n = 0$$

This shows that x is a constant function. Consequently, N(F) is closed. But R(F)=C[0,1] is infinite dimensional space. Thus the condition of finite dimensionality cannot be dropped and the condition that N(F) is closed alone is not sufficient for continuity.

Note 2: In case of linear functional, the condition of closedness of N(F) cannot be dropped. For example, consider $F : C^1[0,1] \to K$ defined by $F(x)=x'(1)$ $\left(x' = \frac{d}{dt}x(t)\right)$. In this case

$N(F) = \{x \in C^1[0,1] | x'(1) = 0\}$ which is not closed.

Infact, $\{x_n(t)\} = \left\{ t - \frac{t^n}{n} \right\} \subset N(F)$ so that $x_n \to x_1 = t$ but $t \notin N(F)$. So, N(F)is not closed. F is also not continuous as can be seen as in Note 1.

Riesz Lemma (1918) 6.7 Let X be a normed space , Y be a closed subspace of X and $Y \neq X$. Let r be a real number such that 0<r<1. Then there exists some $x_r \in X$ such that $\|x_r\| = 1$ and $r < \text{dist}(x_r, Y) \le 1$

Proof: Since $Y \neq X$, there is an $x \in X$ and $x \notin Y$

Also Y is a closed subspace, so we have $d(x, Y) = \inf_{y\in Y} \|x - y\| > 0$

Clearly $d(x, Y) < \frac{1}{r} d(x, Y)$. So there is $y_0 \in Y$ such that $\|x - y_0\| < \frac{1}{r} d(x, Y)$.

Put $x_r = \frac{x - y_0}{\|x - y_0\|}$ so that $\|x_r\| = 1$ and $\text{dist}(x_r, Y) = \inf_{y\in Y} \|x_r - y\| \le \|x_r\| = 1$

$$\textit{Further, } \text{dist}(x_r, Y) = \inf_{y\in Y} \|x_r - y\| = \inf \left\| \frac{x - y_0}{\|x - y_0\|} - y \right\| = \frac{1}{\|x - y_0\|} \inf_{u\in Y} \|x - u\| = \frac{d(x, Y)}{\|x - y_0\|} > r$$

This proves the lemma.

Proposition 6.8 : Let X be a normed space and Y be a subspace of X. Then

(a) for $x \in X$, $y \in Y$ and $k \in K$, $\|kx + y\| \ge |k| \text{dist}(x, Y)$

(b) if Y be finite dimensional then Y is complete and hence closed in X.

Proof: (a) For k=0, it is trivial

If $k \neq 0$, $\|kx+y\| = |k|\|x+k^{-1}y\| \geq |k|\inf_{u \in Y}\|x-u\|$ $(u = -k^{-1}y \in Y) = |k|$ dist(x,Y)

(b)The completeness of a finite dimensional subspace Y of normed space X can be proved with the help of principle of mathematical induction on the dimension m of Y.

If dimY=1,then $Y = \{ky_0 | k \in K\} = \text{span}\{y_0\}$ $(y_0 \in Y)$. Let $\{y_n\} = \{k_n y_0\}$ be a Cauchy sequence in Y.

Then from $\|y_n - y_m\| = |k_n - k_m|\|y_0\|$ it follows that $\{k_n\}$ is a Cauchy Sequence in K. So, $k_n \to k_0 \in K$ and hence $y_n \to k_0 y_0 \in Y$. This shows that Y is complete and hence it is a closed subspace of X.

Assume that every (m-1)dimensional subspace of X is complete. Let Y be an m-dimensional subspace of X and $B = \{y_1, y_2, ..., y_m\}$ be a basis for Y.

If $\{x_n\} = \{k_{n,1}y_1 + k_{n,2}y_2 + ... + k_{n,m}y_m\}$ be a Cauchy Sequence in Y.

$x_n = k_{n,1}y_1 + z_n$ where $z_n \in \text{span}\{y_2, ..., y_m\} = Z$.

Also, $\|x_n - x_p\| = \|(k_{n,1} - k_{p,1})y_1 + (z_n - z_p)\| \geq |k_{n,1} - k_{p,1}| dist(y_1, Z)$ $\quad$ (*part*(*a*))

and dist(y_1,Z)>0 since Z is closed and $y_1 \notin Z$. From this it follows that $\{k_{n,1}\}_{n\geq 1}$ is a cauchy sequence in K and hence $k_{n,1} \to k \in K$.

Consequently $\{z_n\} = \{x_n - k_{n,1}y_1\}$ is a Cauchy sequence in Z. Since dim Z=m-1, Z is complete and hence

$(x_n - k_{n,1}y_1) \to z \in Z$ $\quad$ and $x_n \to ky_1 + z \in Y$

Thus Y is complete and hence closed in X.

Exercise 6.9 In proposition 6.8, show that

(a) $x_n = k_{n,1}y_1 + ... + k_{n,m}y_m \to x = k_1y_1 + ... + k_my_m$ if and only if $k_{n,j} \to k_j$

(b) $\{x_n\}$ is bounded if and only if $\{k_{n,j}\}_{n\geq 1}$ is bounded for j=1,2,...,m.

Note 3: An infinite dimensional subspace of a normed space X may not be closed. Let

$X = l^\infty$ = the space of all bounded sequences of complex numbers

and Y= C_{00} = the space of sequences of finitely many non zero terms is a subspace of X.

dim Y= ∞ and Y is not closed. In fact,

$$x_n = \{1, 1/2, ..., 1/n, 0, 0, ...\} \in C_{00} (n \geq 1)$$

But $x_n \to \{1, 1/2, 1/3, ...\} \notin C_{00}$. So, the proposition 6.8 is not true for infinite dimensional subspace.

7. Compactness in finite dimensional normed linear space :

Definition 7.1 : A normed linear space is said to be **compact** if every sequence in X has a convergent subsequence. A subset M of a normed linear space X is compact if M is compact considered as a metric subspace X, that is , if every sequence in M has a convergent subsequence whose limit is an element of M.

Proposition 7.1: A compact subset M of a metric space (and hence with the metric induced by a norm) is closed and bounded.

Proof : Let $x \in \overline{M}$. There is a sequence $\{x_n\}$ in M such that $x_n \to x$. Since M is compact, a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to a point of M. But if a subsequence of a cauchy sequence converges to a point then the whole sequence converges to the same limit. From the uniqueness of limit of a convergent sequence , x must be in M. So, M is closed.

Suppose M is unbounded. Then for fixed $b \in M$ and $n \in N$ there exists $y_n \in M$ such that $d(y_n, b) > n$. The sequence $\{y_n\}$ cannot have a convergent subsequence. This contradicts the compactness of M. So, M must be bounded.

Remark: The converse of the above result is not true in general. Consider the space

$$l^2 = \left\{\{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\} \text{with norm defined by } \|\{a_n\}\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}}$$

If $e_n = \{0, 0, ..., 0, 1, 0, ...\}$, 1 being in the n^{th} place, then $M = \{e_1, e_2, e_3, ...\}$ is a closed and bounded subset of l^2 . M is closed as it has no accumulation point. In fact, if M has an accumulation point p then there exists a sequence in M which converges to p. Such sequence must be Cauchy. But this is not true since $\|e_n - e_m\| = \sqrt{2}$. Also M is bounded since $\|e_n\| = 1 \quad \forall n \geq 1$. But M is not compact since $\{e_n\}_{n \geq 1}$ has no convergent subsequence.

The following theorem shows that the converse is also partially true in case of finite dimensional normed linear space .

Theorem 7.2: In a normed linear space $(X, \|\|)$ the following conditions are equivalent.

(i) Every closed and bounded subset of X is compact.

(ii) The closed unit ball $\{x : \|x\| \leq 1\}$ of X is compact

(iii) X is finite dimensional

Proof :

(i)→ (ii) The closed unit ball is closed and bounded. So, by (i), it is compact.

(ii)→ (iii) Let the closed unit ball be compact. If X be infinite dimensional, there exists an infinite linearly independent set $\{z_1, z_2, ..., z_n, ...\}$ in X. Let $Z_n = \text{span}\{z_1, z_2, ..., z_n\}, n \geq 1$. Being finite dimensional Z_n is a closed subspace of Z_{n+1} by Theorem 6.8. Clearly $Z_n \neq Z_{n+1}$. Since $\{z_1, z_2, ..., z_n, z_{n+1}\}$ is linearly independent. By Riesz Lemma (6.7), there is some $x_n \in Z_{n+1}$ such that .

$$\|x_n\| = 1 \text{ and } \text{dist}(x_n, Z_n) \geq \frac{1}{2}$$

Now $\{x_n\}$ is a sequence in the closed unit ball having no convergent subsequence, because $\|x_n - x_m\| \geq \frac{1}{2}$ $\forall m \neq n$, the set $\{x : \|x\| \leq 1\}$ cannot be compact. Thus (ii) implies (iii)

(iii) → (i) Let E be a closed and bounded subset of finite dimensional space X. To prove that E is compact. Let $B = \{y_1, ..., y_m\}$ be a basis for X. Consider a sequence ,

$$\{x_n\} = \{k_{n,1}y_1 + k_{n,2}y_2 + ... + k_{n,m}y_m\}_{n\geq 1} \text{ in E.}$$

The boundedness of $\{x_n\}$ implies boundedness $\{k_{n,j}\}_{n\geq 1}(j = 1, ..., m)$ [exercise 6.9]. In fact,

$$\|x_n\| = \|k_{n,1}y_1 + k_{n,2}y_2 + ... + k_{n,m}y_m\| \geq |k_{n,j}|\, dist(y_j, Y_j) \text{ where } Y_j = \underset{i \neq j}{\text{span}}\{y_j\}_{i=1}^{m} (j = 1,2, ..., m)$$

By Bolzano-Weierstrass Theorem the bounded sequence $\{k_{n,j}\}_{n\geq 1}$ has a subsequence converging to k_1. Let $\{x_{1,n}\}_{n\geq 1}$ be the corresponding subsequence $\{x_n\}$. Again $\{k_{n,2}\}$ has a subsequence converging to k_2. Let $\{x_{2,n}\}$ be the corresponding subsequence of $\{x_{1,n}\}$. Proceeding in this way, after the m^{th} step ,we have the subsequence $\{x_{m,n}\}_{n\geq 1}$ of $\{x_n\}$. Here

$$x_{m,n} = k_{n_m,1}y_1 + k_{n_m,2}y_2 + ... + k_{n_m,m}y_m$$

$$\text{with } k_{n_m,j} \to k_j \text{ and } x_{m,n} \to k_1y_1 + k_2y_2 + ... + k_my_m = x$$

Thus the sequence $\{x_n\}$ has a convergent subsequence $\{x_{m,n}\}$ converging to $x \in X$. Since $\{x_{m,n}\}_{n\geq 1} \subseteq E$ and E is closed, we have $x \in E$. Hence E is compact.

8. Bounded Linear Maps:

Definition 8.1: A linear map $F:(X,\|\|) \to (Y,\|\|)$ is called bounded linear map if F maps a bounded subset of X into a bounded subset of Y.

Proposition 8.2 A linear map $F:(X,\|\|) \to (Y,\|\|)$ is bounded if and only if there is a positive number M>0 such that $\|F(x)\| \leq M\|x\|$ $\forall x \in X$..

Proof: Let F be bounded linear map . $\{x : \|x\| = 1\}$ is a bounded subset of X. So we have a positive number M such that

$$\|F(x/\|x\|)\| \le M \quad \forall x (\neq 0) \in X \qquad \text{So,} \qquad \|F(x)\| \le M\|x\| \quad \forall x \in X.$$

It is also noted that the inequality also holds for x=0. Conversely, let $\|F(x)\| \le M\|x\| \quad \forall x \in X$. and for some fixed M>0. Let $B = \{x : \|x\| \le r\}$ be a bounded subset of X. Then $\|F(x)\| \le Mr \quad \forall x \in B$. So, F is bounded in B and hence it is a bounded linear map.

Proposition 8.3 : A linear map $F : (X, \|\|) \to (Y, \|\|)$ is bounded if and only if F is continuous.

Proof: Follows from Theorem 6.2 (see equivalenceof (iii) and (v)) and proposition 8.1.

Examples of bounded linear maps 8.4 :

Example 1 If $(X, \|\|)$ is a normed linear space then the identity operator $I_X : X \to X$ on a normed space $X \neq \{0\}$ is bounded. $[I_x(x)=x \ \forall x \in X]$

Example 2 : The zero operator $O : X \to X$ on a normed space X is bounded.

$$[O(x)=0 \quad \forall x \in X]$$

Example 3: Let X be normed space of all polynomials on [0,1] with norm defined by $\|x\| = \sup_{t \in [0,1]} |x(t)|$ The differential operator $D : X \to X$,

$[D(x)](t) = \frac{d}{dt} x(t)$ is a linear operator but it is not bounded operator. If $x_n(t) = t^n, 0 \le t \le 1$, then $B = \{x_n(t) | n \ge 1\}$ is a bounded set, because

$$\|x_n\| = \sup_{t \in [0,1]} |x_n(t)| = \sup_{t \in [0,1]} t^n = 1. \text{ But, } \quad \|D(x_n)\| = \|nt^{n-1}\| = n = n\|x_n\|$$

So, there exists no M>0 satisfying $\|D(x_n)\| \le M\|x_n\| \quad \forall n \ge 1$

Example 4: If X=C[0,1], then the integral operator $T : C[0,1] \to C[0,1]$ defined by

$$[T(x)](t) = \int_0^1 K(t, \tau) x(\tau) d\tau$$

Here $K(t, \tau)$ is a continuous function on [0,1]x[0,1]

$$T(\alpha x_1 + \beta x_2)(t) = \int_0^1 K(t,\tau)(\alpha x_1 + \beta x_2)(\tau) d\tau = \alpha \int_0^1 K(t,\tau) x_1(\tau) d\tau + \beta \int_0^1 K(t,\tau) x_2(\tau) d\tau$$
$$= \alpha[T(x_1)](t) + \beta[T(x_2)](t) = [\alpha T(x_1) + \beta T(x_2)](t), \forall t \in [0,1]$$

So, T is a linear map. T is also bounded.

Since $K(t,\tau)$ is continuous on the compact subset $[0,1]X[0,1]$, there exists $k_0 \geq 0$ such that $K(t,\tau) \leq k_0 \quad \forall (t,\tau) \in [0,1]X[0,1]$. Also, $\quad |x(t)| \leq \sup_{0\leq t\leq 1} |x(t)| = \|x\|$

Hence $\|Tx\| = \sup_{0\leq t\leq 1} |(Tx)(t)| = \sup_{0\leq t\leq 1} \left| \int_0^1 K(t,\tau)x(\tau) d\tau \right| \leq \sup_{0\leq t\leq 1} \int_0^1 |K(t,\tau)||x(\tau)| d\tau \leq k_0 \|x\|$

Thus $\|Tx\| \leq k_0 \|x\| \quad \forall x \in C[0,1]$. Hence T is a bounded linear operator.

Theorem 8.5: If a normed linear space X is finite dimensional, then every linear operator on X is bounded.

Proof : It follows from Theorem 8.3 and Theorem 6.1 (a)

Linear Space of Bounded Linear operators 8.6 :

The set of all bounded linear maps from a normed linear space X into a normed linear space Y is denoted by B(X,Y). If Y=X, then we write B(X,Y)=B(X). If Y=K, then we write $B(X,K)=X^*$, called **dual space** or **conjugate space** of X. It will be shown that B(X,Y) is a linear space. For $T \in B(X,Y)$, we can define a norm for T, called **operator norm** , as follows : $\quad \|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$

Since $\|Tx\| \leq M\|x\| \leq M \quad \forall x$ with $\|x\| \leq 1$, $\|T\|$ is a finite unique number.

Further $\quad \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq \sup\{\|Tx\| : \|x\| \leq 1\} = \|T\| \qquad$ So, $\|T(x)\| \leq \|T\| \|x\|$.

Before proving B(X,Y) is a normed linear space , we see that the operator norm of $T \in B(X,Y)$ can be defined in four equivalent forms.

Proposition 8.7: For $T \in B(X,Y)$, if

$$\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}, \qquad \beta = \sup\{\|Tx\| : \|x\| = 1\}$$

$$\gamma = \sup\{\|Tx\| : \|x\| < 1\}, \qquad \alpha_0 = \inf\{\alpha > 0 : \|Tx\| \leq \alpha\|x\| \forall x \in X\}$$

then $\|T\| = \alpha_0 = \beta = \gamma$

Proof: Clearly, $\beta \leq \|T\|$ and $\gamma \leq \|T\|$

For a non-zero $x \in X$ and $0 < r \leq 1$, we have $\|Tx\| = \frac{\|x\|}{r} \left\| T\left(\frac{rx}{\|x\|}\right) \right\| \leq \frac{\|x\|}{r} \sup\{\|Tx\| : \|x\| = r\}$

Taking r=1, we get $\quad \|Tx\| \leq \|x\| \sup\{\|Tx\| : \|x\| = 1\} = \beta\|x\|$

so, $\alpha_0 \leq \beta$. Taking $0<r<1$, $\|Tx\| \leq \frac{\|x\|}{r} \sup\{\|Tx\| : \|x\| < 1\} = \frac{\gamma}{r}\|x\|$

Letting $r \to 1$, we see that $\|Tx\| \leq \gamma\|x\|$ for all $x \in X$ hence $\alpha_0 \leq \gamma$

Finally, we can show that $\|T\| \leq \alpha_0$. Consider $\alpha \geq 0$ such that $\|Tx\| \leq \alpha\|x\|$ for all $x \in X$. Taking supremum over all $x \in X$ with $\|x\| < 1$, we obtain $\|T\| \leq \alpha$. Since α_0 is the infimum of all such α', we obtain $\|T\| \leq \alpha_0$.

Thus $\|T\| \leq \alpha_0 \leq \min\{\beta, \gamma\}$ (since $\alpha_0 \leq \beta$ and $\alpha_0 \leq \gamma$)

$\leq \|T\|$ (since $\beta \leq \|T\|$ and $\gamma \leq \|T\|$) and the proof is complete.

Proposition 8.8 : If X and Y are normed linear spaces then

a) B(X,Y) is a Banach space if and only if Y is a Banach space with respect to operator norm;

b) The dual space X^* is a Banach space.

Proof : We know that the set L(X,Y) of all linear maps is a linear space with respect to the operations

$$(T_1 + T_2)x = T_1x + T_2x$$

$$(\alpha T_1)x = \alpha(T_1x)$$

B(X,Y) is a subspace of L(X,Y). In fact $B(X,Y) \neq \Phi$ and if $T_1, T_2 \in B(X,Y)$ and $\alpha, \beta \in K$ then

$$\begin{aligned}\|(\alpha T_1 + \beta T_2)x\| &\leq |\alpha|\|T_1x\| + |\beta|\|T_2x\| \\ &\leq |\alpha|M_1\|x\| + |\beta|M_2\|x\| \quad \text{for some } M_1 > 0 \text{ and } M_2 > 0 \\ &= (|\alpha|M_1 + |\beta|M_2)\|x\|\end{aligned}$$

This shows that $\alpha T_1 + \beta T_2 \in B(X,Y)$. Hence B(X,Y) is a linear subspace of L(X,Y).

B(X,Y) is a normed linear space . For $T \in B(X,Y)$ and $\alpha \in K$,

N1) $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\} \geq 0$

If $T=0$, $\|T\| = 0$ and conversely if $\|T\| = 0$, then $\|Tx\| = 0$ for each x with $\|x\| \leq 1$.

Hence for all $x \neq 0$, $\left\|T\left(\frac{x}{\|x\|}\right)\right\| = 0$ or $\|Tx\| = 0$

Also $\|T(0)\| = 0$. Thus $Tx=0$ for all $x \in X$ and hence $T=0$.

N₂) $\|\alpha T\| = \sup\{\|(\alpha T)x\| : \|x\| \le 1\} = \sup\{|\alpha|\ \|Tx\| : \|x\| \le 1\} = |\alpha| \sup\{\|Tx\| : \|x\| \le 1\} = |\alpha| \|T\|$

N₃) For $T_1, T_2 \in B(X, Y)$, $\|T_1 + T_2\| = \sup\{\|T_1x + T_2x\| : \|x\| \le 1\}$

For $\|x\| \le 1$, $\|T_1x + T_2x\| \le \|T_1x\| + \|T_2x\| < \|T_1\| + \|T_2\|$

Since it holds for all x with $\|x\| \le 1$, $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$.

Hence B(X,Y) is a normed linear space.

Let Y be a Banach Space. We have to show that B(X,Y) is a Banach Space. Let $\{F_n\}$ be a Cauchy sequence in B(X,Y). For $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \ge n_0$

$$\|F_m - F_n\| < \varepsilon$$

Then for all $x \in X$ and $m, n \ge n_0$

(*) $\|F_m(x) - F_n(x)\| \le \|F_m - F_n\| \|x\| < \varepsilon\|x\|$

this shows that for a fixed $x \in X$, $\{F_n(x)\}$ is a Cauchy sequence in Y. Since Y is a Banach Space, there exists $y \in Y$ such that $\lim_{n\to\infty} F_n(x) = y$.

Define $F : X \to Y$ by $F(x) = y = \lim_{n\to\infty} F_n(x)$.

It can be shown that (i) $\lim_{n\to\infty} F_n = F$ and (ii) $F \in B(X, Y)$.

F is linear. Indeed, for $x_1, x_2 \in X$ and $\alpha, \beta \in K$

$$F(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} F_n(\alpha x_1 + \beta x_2) = \alpha \lim_{n\to\infty} F_n(x_1) + \beta \lim_{n\to\infty} F_n(x_2) = \alpha F(x_1) + \beta F(x_2)$$

Letting $m \to \infty$, we have from (*)

(**) $\|F(x) - F_n(x)\| \le \varepsilon\|x\|$ $\forall x \in X$ and all $n \ge n_0$

Hence $(F - F_n) \in B(X, Y)$ $\forall n \ge n_0$ and since B(X,Y) is a linear space, we have

$$F = (F - F_{n_0}) + F_{n_0} \in B(X, Y)$$

Further from (**), $\|F_n - F\| = \sup\{\|F_n(x) - F(x)\| : \|x\| \le 1\} < \varepsilon$ $\forall n \ge n_0$.

Hence, $\lim_{n\to\infty} F_n = F$. Thus B(X,Y) is a Banach Space if Y is a Banach Space.

Conversely, let B(X,Y) be a Banach Space and $X \ne 0$. To prove that Y is a Banach Space. The proof of this result depends on the following fact which is not yet proved. The result will be proved at a later stage as a consequence of Hahn Banach Theorem. The result states that for $x_0(\neq 0) \in X$, there is some $f \in X^*$ such that $f(x_0) = \|x_0\| \neq 0$ and $\|f\| = 1$. Consider a cauchy sequence $\{y_n\}$ in Y and define $F_n(x) = f(x)y_n$ for $x \in X$. It can be easily checked that $F_n \in B(X, Y)$.

F_n is linear since

$$F_n(\alpha x_1 + \beta x_2) = f(\alpha x_1 + \beta x_2)y_n = \alpha f(x_1)y_n + \beta f(x_2)y_n = \alpha F(x_1) + \beta F(x_2)$$

for all $x_1, x_2 \in X$ and $\alpha, \beta \in K$

Also, $\|F_n(x)\| = \|f(x)y_n\| = |f(x)|\|y_n\| \leq \|f\|\|y_n\|\|x\| = M\|x\|$ where $M = \|f\|\|y_n\|$

Next we see that $\{F_n\}$ is a Cauchy sequence in B(X,Y). For $x \in X$,

$$\|F_n(x) - F_m(x)\| = \|f(x)(y_n - y_m)\| = |f(x)| \|y_n - y_m\| \leq \|f\| \|x\| \|y_n - y_m\|$$
$$= \|x\| \|y_n - y_m\| \quad (\because \|f\| = 1)$$

So, $\|F_n - F_m\| = \sup_{\|x\| \leq 1} \|F_n(x) - F_m(x)\| \leq \|y_n - y_m\| < \varepsilon \quad \forall \quad n, m \geq n_0,$

Because $\{y_n\}$ is a Cauchy sequence . Hence $\{F_n\}_{n \geq 1}$ is a cauchy sequence in Banach Space B(X,Y). So there exists $F \in B(X, Y)$ such that $F_n \to F$.

Consequently, $\quad F_n(x_0/\|x_0\|) \to F(x_0/\|x_0\|)$

Hence $y_n \to y_0 = F(x_0)/\|x_0\| \in Y$ and this shows that Y is a Banach Space

(c) Since K is a Banach Space , X^*=B(X,K) is a Banach Space .

□□□□

Unit 2

Set II:

The Hahn Banach Theorem, the natural embedding of N in N^{**}, weak and weak* topologies, the open mapping Theorem, the closed Graph Theorem, The Banach Steinhaus Theorem and the Conjugate of an operator.

9. The Hahn Banach Theorem

We shall need Zorn's Lemma in the proof of the Hahn Banach Theorem. The setting for the lemma requires the knowledge of a partially ordered set.

Definition 9.1: A partially ordered set (p.o.set) in a non-empty set P together with a (binary) relation denoted by '$\leq$', which satisfies the conditions:

(i) $x \leq x$ for all $x \in P$(reflexivity)

(ii) If $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry)

(iii) If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity)

'Partially' emphasizes that there may exist two elements x and y in P such that neither $x \leq y$ nor $y \leq x$. Two elements x and y are said to be **comparable** if either $x \leq y$ or $y \leq x$. So in a p.o. set two elements may not be comparable. A p.o. set is called a **totally ordered set** or a **chain** if any two elements of the set are comparable. If M is a subset of a p.o. set $(P, \leq)$, an element $u \in P$ is called **upper bound** of M if $x \leq u$ for all $x \in M$. An element $m \in P$ is called a **maximal element** of the p.o. set $(P, \leq)$ if $m \leq x$ implies $m = x$. A poset may or may not have a maximal element. The poset $(3, \leq)$ has no maximal element. The power set $(P(X), \subseteq)$ of X is a p.o. set for which X is the only maximal element.

Zorn's lemma 9.2: Let $(P, \leq)$ be a p.o. set. Suppose every chain in P has an upper bound in P. Then P has a maximal element.

The Hahn Banach Theorem is an extension theorem. It asserts that every continuous linear functional on a subspace of a normed linear space can be extended to the whole space with preservation of the norm. This theorem also shows that there is a plenty of bounded linear functionals on a normed linear space and facilitates the study of adequate theory of dual spaces. This theorem is due to H. Hahn (1927) for real normed linear space. S.Banach (1929) rediscovered a more general form of it. H.F. Bohnenblust and A.Sobeczyk (1938) generalised to complex normed linear space.

Theorem 9.3 (The Hahn Bananch Theorem): Let $(X, \|.\|)$ be a normed linear space and M a subspace of X. Let f be a continuous linear functional on M. Then there exists a continuous linear functional g on X such that f = g on M and $\|f\| = \|g\|$.

Proof: We complete the proof into three parts. Firstly we extend f from M to N =span $M \cup \{z\}$, $z \in X \setminus M$ Then we extend to the whole space X in case X is a real normed linear space. Lastly, we extend to complex normed linear space X.

Step I : Let X be a real normed linear space and $z \in X \setminus M$. Let N be the subspace generated by M and z. We can show that f can be extended from M to N by $\bar{f}$ with $\|\bar{f}\| = \|f\|$. For y_1, $y_2 \in M$, We have

$$f(y_1) - f(y_2) = f(y_1 - y_2) \le \|f\| \cdot \|y_1 - y_2\|$$

$$= \|f\| \|y_1 + z - (y_2 + z)\| \le \|f\| \; \|y_1 + z\| + \|f\| \; \|y_2 + z\|;$$

hence $$-\|f\| \|y_2 + z\| - f(y_2) \le \|f\| \|y_1 + z\| - f(y_1).$$

So $$\alpha = \sup_{y \in M}\left(-\|f\| \|y + z\| - f(y)\right) \le \inf_{y \in M}\left(\|f\| \|y + z\| - f(y)\right) = \beta$$

Let μ be any real number such that $\alpha \le \mu \le \beta$. An element of N can be uniquely expressed as $y + \lambda z, y \in M$ and $\lambda \in 3$. Define $\bar{f} : N \to 3$ by $\bar{f}(y + \lambda z) = f(y) + \lambda \mu$

Clearly $\bar{f}$ is linear on N. If $\lambda > 0$,

$$\bar{f}(y + \lambda z) = f(y) + \lambda\mu = \lambda\{f(\lambda^{-1}y) + \mu\}$$
$$\le \lambda\{f(\lambda^{-1}y) + \beta\} \qquad (\because \alpha \le \mu \le \beta)$$

$$\le |\lambda| \{f(\lambda^{-1}y) \|f\| \|\lambda^{-1}y + z\| - f(\lambda^{-1}y)\} \quad (\text{Since, } \beta \le \|f\| \|y + z\| - f(y))$$

$$\le |\lambda| \|f\| \|\lambda^{-1}y + z\| = \|f\| \|y + \lambda z\| \quad \text{......(1)}$$

and $$\bar{f}(y + \lambda z) = \lambda\{f(\lambda^{-1}y) + \mu\}$$

$$\ge \lambda\{f(\lambda^{-1}y) + \alpha\} \qquad (\because \alpha \le \mu \le \beta)$$

$$= -\lambda\{-f(\lambda^{-1}y) - \alpha\}$$

$$\therefore -\bar{f}(y + \lambda z) \le \lambda\{-f(\lambda^{-1}y) - \alpha\} \le \lambda \|f\| \|\lambda^{-1}y + z\| \quad (\because -\|f\| \|y + z\| - f(y) \le \alpha)$$

$$= \|f\| \|y + \lambda z\| \qquad \text{.......(2)}$$

So, from (1) and (2) $\left|f^{-1}(y + \lambda z)\right| \le \|f\| \|y + \lambda z\|$ if $\lambda > 0$

If $\lambda < 0$, $\bar{f}(y + \lambda z) = f(y) + \lambda\mu = \lambda\{f(\lambda^{-1}y) + \mu\} \ge \lambda\{f(\lambda^{-1}y) + \beta\}$ $(\because \mu \le \beta \text{ and } \lambda < 0)$

$$\ge \lambda \|f\| \|\lambda^{-1}y + z\| \qquad (\because \beta \le \|f\| \|y + z\| - f(y) \text{ and } \lambda < 0)$$

So, $$-\bar{f}(y + \lambda z) \le -\lambda \|f\| \|\lambda^{-1}y + z\| = |\lambda| \|f\| \|\lambda^{-1}y + z\| = \|f\| \|y + \lambda z\|$$

and $\bar{f}(y+\lambda z)=\lambda\{f(\lambda^{-1}y)+\mu\}$

$$\leq\lambda\{f(\lambda^{-1}y)+\alpha\} \qquad (\because \alpha\leq\mu \quad \text{and} \quad \lambda<0)$$

$$\leq\lambda\{-\|f\|\|\lambda^{-1}y+z\|\} \qquad (\because -\|f\|\|y+z\|-f(y)\leq\alpha \quad \text{and} \quad \lambda<0)$$

$$=|\lambda|\,\|f\|\|\lambda^{-1}y+z\|=\|f\|\,\|y+\lambda z\|$$

So, if $\lambda<0, |\bar{f}(y+\lambda z)|\leq\|f\|\,\|y+\lambda z\|$

Hence $\bar{f}$ is continuous and linear. Further,

$$|f^{-1}(y+\lambda z)|\leq\|f\| \quad \text{if} \quad \|y+\lambda z\|\leq 1$$

So, $\sup\limits_{\|y+\lambda z\|\leq 1}|\bar{f}(y+\lambda z)|\leq\|f\|$. That is, $\|\bar{f}\|\leq\|f\|$

Also $\|f\|=\sup\limits_{\substack{y\in M\\ \|y\|\leq 1}}|f(y)|\leq\sup\limits_{\substack{y\in M, y\in R\\ \|y+\lambda z\|\leq 1}}|\bar{f}(y+\lambda z)|=\|\bar{f}\|$. So, $\|\bar{f}\|=\|f\|$

This shows that f can be extended to $\bar{f}$ from M to N= span $(M\cup\{z\})$. Continuing the process and applying Zorn's Lemma f can be extended to the whole space X.

Step II: If N= span $(M\cup\{z\})\neq X$, there exists $z_1\in X\setminus N$. As in step I, $\bar{f}$ can be extended to $\bar{f}'$ from N to N' =span $(N\cup\{z_1\})$. Continuing the process we get a set

P=$\{(N,\bar{f})|\ \bar{f}$ is an extension of f from M to N such that $M\subset N$, $\bar{f}=f$ on M and $\|\bar{f}\|=\|f\|\}$.

Define an order '$\leq$' on P by $(N,\bar{f})\leq(N',\bar{f}')$ if and only if $N\subset N'$ and $\bar{f}=\bar{f}'$ on N. It is easy to check that $(P,\leq)$is a partially ordered set. We can show that every chain in P has an upper bound .

Consider an arbitrary chain $\ell=\{(N_\alpha,\bar{f}_\alpha)\}$ in $(P,\leq)$.

Set $N_0=U_\alpha N_\alpha$ and $\bar{g}:N_0\to K$ be defined as follows:

For $x\in N_0$, $x\in N_\alpha$ for some α define, $\bar{g}(x)=\bar{f}_\alpha(x)$

$\bar{g}$ is well defined and linear since ℓ is a chain .

Further $\bar{g}$= f on M and $|\bar{g}(x)|=|\bar{f}_\alpha(x)|\leq\|\bar{f}_\alpha\|\|x\|=\|f\|\|x\|$.

Taking supremum over all x with $\|x\| \le 1$, we have $\|\bar{g}\| \le \|f\|$. Also from definition $\|f\| \le \|\bar{g}\|$, since the domain of f is contained in the domain of $\bar{g}$, hence, $\|\bar{g}\| = \|f\|$. Consequently,

(N_0, $\bar{g}$) is an upper bound of $\{(N_\alpha, \bar{f}_\alpha)\}$. Zorn's lemma guaranties that the partially ordered set $(P, \le)$ has a maximal element (N', g) in P. We claim that N′ =X. If not, there exists $z_0 \in X \setminus N'$ and as in step I, there exists a continuous linear extension g_0 on $N'' = \text{span}(N' \cup \{z_0\})$ with $f=g_0$ on M and $\|f\| = \|g_0\|$. Thus $(N'', g_0) \in P$ and $(N', g) \le (N''_0, g_0)$.

This contradicts maximality of (N', g). So, $N'=X$ and g is the required extension of f from M to X. This proves the Hahn Banach theorem for real normed linear space. In the next step we shall prove it for complex normed linear space.

Step III: We note that if X is a vector space over ∀ then by restricting to real scalars only X is also a vector space over 3. Let X_0 be the underlying real normed linear space of the complex normed linear space (X, $\|.\|$). We consider the set M as a subspace M_0 of X_0.

$$\text{Let } f(x) = f_1(x) + i\, f_2(x) = \text{Re } f(x) + i \text{ Im } f(x).$$

Then $x \to f_1(x) = \text{Re} f(x)$ is clearly real linear and $|f_1(x)| \le |f(x)| \le \|f\| \|x\|$. So, $\|f_1\| \le \|f\|$. Thus f_1 is a real continuous linear functional on M_0. Further, $if(x) = f(ix) = f_1(ix) + i\, f_2(ix)$. Substituting for f(x), $if_1(x) - f_2(x) = f_1(ix) + if_2(ix)$. Comparing the real parts

We have, $f_2(x) = -f_1(ix)$. Thus, $f(x) = f_1(x) - if_1(ix)$

By step II, there exists an extension g_1 of f_1 on X_0 with $\|g_1\| = \|f_1\|$. Define, $g(x) = g_1(x) - i\, g_1(ix)$ for $x \in X$.

We can show that g is the required extension. For this, we need to show (a) f=g on M (b) $g \in X^*$ and (c) $\|f\| = \|g\|$

(a) for $x \in M$, $g(x) = g_1(x) - ig_1(ix) = f_1(x) - if_1(ix)$ ($\because g_1$ is extension of f_1)

$$= f(x)$$

(b) $g(x_1 + x_2) = g_1(x_1 + x_2) - ig_1(ix_1 + ix_2)$

$$= g_1(x_1) + g(x_2) - ig_1(ix_1) - ig_1(ix_2) = g(x_1) + g(x_2)$$

For $r \in R$, $g(rx) = g_1(rx) - ig_1(irx) = rg_1(x) - rig_1(ix) = r\, g(x)$

$$g(ix) = g_1(ix) - ig_1(-x) = g_1(ix) + ig_1(x) = i\,\{g_1(x) - ig_1(ix)\} = ig(x)$$

For $x \in X$, suppose, $g(x) = re^{i\theta}$ where r is positive real.

Then, $|g(x)| = r = e^{-i\theta}g(x) = g(e^{-i\theta}x)$

$$= g_1(e^{-i\theta}x) \qquad (\because g(e^{-i\theta}x) \text{ is positive real})$$

$$= |g_1(e^{-i\theta}x)| \le \|g_1\| \|x\| \qquad (\because g_1 \in X_0^*)$$

$$= \|f_1\| \|x\| \le \|f\| \|x\|$$

This shows that $g \in X^*$ and $\|g\| \le \|f\|$. Also from definition $\|f\| \le \|g\|$. So, $\|g\| = \|f\|$ and the Hahn Banach theorem is proved completely.

Consequences of Hahn Banach Theorem :

Theorem 9.4 If N is a normed linear space and x_0 is a non zero vector in N then there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Proof: Consider the subspace $M = \langle x_0 \rangle$, the linear space spanned by x_0 and define f from M to K by $f(\alpha x_0) = \alpha \|x_0\|$. Since, for a fixed x in M , α is fixed, f is well-defined. Then $f[a(\alpha x_0) + b(\beta x_0)] = (a\alpha + b\beta)\|x_0\| = af(\alpha x_0) + bf(\beta x_0)$ which shows that f is a linear functional on M. Further,

$f(x_0) = \|x_0\|$ and $|f(\alpha x_0)| = |\alpha| \|x_0\| = \|\alpha x_0\|$. This shows that $f \in M^*$ and $f(x_0) = \|x_0\|$. By Hahn Banach Theorem , f can be extended to a continuous linear functional f_0 in N^* with $f = f_0$ on M and $\|f\| = \|f_0\|$

So, $f_0(x_0) = f(x_0) = \|x_0\|$ and $|f(\alpha x_0)| = \|\alpha x_0\|$ which shows that $\|f\| = 1$. Thus $\|f_0\| = \|f\| = 1$

Corollary : N^* separates points of N. That is, for $x \neq y$ there exists $f \in N^*$ such that $f(x) \neq f(y)$.

Proof: Since $x - y \neq 0$ and $x - y \in N$, there exists by Theorem 9·4 , $f \in N^*$ such that

$f(x-y) = \|x - y\| \neq 0$. Consequently $f(x) \neq f(y)$.

Note (1) $f(x) = 0$ for all $f \in N^*$ implies $x = 0$.

Theorem 9·5 If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$.

Proof: Consider the natural mapping T from N onto N/M defined by $T(x) = x + M$ for $x \in N$. Recall that N/M is a vector space with M as its zero element and norm on N/M is defined by $\|x + M\| = \inf\{\|x + m\| : m \in M\}$.

Since $m \in M$, $T(m)=m + M = M$ we write $T(M)=0$

$$T(x_0) = x_0 + M \neq 0 (\text{Since } x_0 \notin M)$$

So by consequence 1, there exists a continuous linear functional f in $(N/M)^*$ such that

$$F(x_0+M)=\| x_0 + M \| \neq 0.$$

Define, $f_0 : N \to K$ by $f_0 = f oT$ clearly, $f_0 \in N^*$ and $f_0(M) = T(M) = 0$

$$N \xrightarrow{T} N/M \xrightarrow{f} K$$

$$f_0 = f oT$$

Further $f_0(x_0) = f(T(x_0)) = f(x_0 + M) = \| x_0 + M \| \neq 0,$

Thus $f_0 \in N^*$ such that $f_0(M)=0$ and $f_0(x_0) \neq 0$. This proves the result completely.

Theorem 9.6 Let N be a normed linear space and M a subspace of N. $x_0 \in N$ such that $d(x_0,M)>0$. Then there is a bounded linear functional f on N such that $f(x) = 0$ for all $x \in M, f(x_0) = d$ and $\|f\| = 1$.

Proof: Let $M_0 = \text{span } (M \cup \{x_0\})$. Clearly $x_0 \notin M$. If $x_0 \in M$ then $d(x_0,M)=\inf\{d(x,m) | m \in M\} = 0$, a contradiction . Hence $x_0 \notin M$. Every element of M_0 can be uniquely expressed as $y + \alpha x_0$ where $\alpha \in K$. Define, $f_0(y + \alpha x_0) = \alpha \, d$ for all $y \in M$ and $\alpha \in K$.

Then $f_0 \in M^*$. Clearly f_0 is linear. Further ,for $\alpha = 0$, $f_0(y)=0 \leq \| y \|$. Also for $\alpha \neq 0$,

$$\| y + \alpha x_0 \| = |\alpha| \left\| \frac{1}{\alpha} y + x_0 \right\| \geq |\alpha| d \text{ and so}$$

$$|f_0(y + \alpha x_0)| = |\alpha| d \leq \| y + \alpha x_0 \|.$$

Hence $|f_0(u)| \leq \| u \|$ for all $u \in M_0$. This shows that $f_0 \in M_0^*$ and $\|f_0\| \leq 1$.

Since, $f_0(y + \alpha x_0) = \alpha d$ for all $y \in M$, we have $f_0(-y+x_0) = d$. Hence ,

$$d = |f_0(-y + x_0)| \leq \| f_0 \| \, \| x_0 - y \| \text{ for all } y \in M.$$

then $\dfrac{d}{\| f_0 \|} \leq \inf_{y \in M} \| x_0 - y \| = d$. So, $1 \leq \| f_0 \|$

Thus we have $f_0 \in M_0^*$ with $\|f_0\| = 1$. By Hahn Banach theorem, there exists $f \in N^*$ such that $f_0(y) = f(y) = 0\ \forall y \in M$ and $\|f_0\| = \|f\| = 1$.

And $f_0(x_0) = f_0(0 + 1.x_0) = 1.d = d$. This proves the theorem

Theorem 9·7 If X be a normed linear space over K and let $x \in X$. Then

$$\|x\| = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \le 1\}.$$

proof: For x=0, it is trivial that $\|x\| = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \le 1\}$

Suppose $x \neq 0$.

$|f(x)| \le \|f\|\,\|x\| \le \|x\|$ for all $f \in X^*$ with $\|f\| \le 1$

Hence $\sup\{|f(x)| : \|f\| \le 1, f \in X^*\} \le \|x\|$. If possible,

Let $\sup\{|f(x)| : \|f\| \le 1, f \in X^*\} < \|x\|$.

So, $|f(x)| < \|x\|$ for $f \in X^*$ with $\|f\| \le 1$

This leads to a contradiction to Theorem 9·4 . Hence

$$\|x\| = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \le 1\}.$$

Theorem 9·8 If N is a normed linear space and suppose that N^* is separable. Then N is separable. But the converse is not true in general.

Proof: Let $\{f_n : n \in N\}$ be a countable dense subset of N^* .

Since $\|f_n\| = \sup\{|f_n(x)| : \|x\| = 1\}$, there exists $x_n \in N$ with $\|x_n\| = 1$ and $\frac{\|f_n\|}{2} < |f_n(x_n)|$. Let F be the closed linear hull of $\{x_n\}$. We claim F = N. if possible let $F \neq N$. By Theorem 9.5, there is a bounded linear functional $f_0 \in N^*$ such that

$f_0(x) = 0$ for all $x \in F$ but $f_0 \neq 0$. Since $\{f_n : n \in N\}$ is dense in N^* there is a positive integer n_0 with $\|f_0 - f_{n_0}\| < \frac{1}{4}\|f_0\|$, then

$$\frac{1}{2}\|f_{n_0}\| \le |f_{n_0}(x_{n_0})| \text{ with } \|x_{n_0}\| = 1$$

$$= |f_{n_0}(x_{n_0}) - f_0(x_{n_0})| \quad (\because x_{n_0} \in F)$$

$$\leq \| f_{n_0} - f_0 \| \| x_{n_0} \| < \frac{1}{4} \| f_0 \|$$

and hence

$$\| f_0 \| \leq \| f_0 - f_{n_0} \| + \| f_{n_0} \| < \frac{1}{4} \| f_0 \| + \frac{1}{2} \| f_0 \| = \frac{3}{4} \| f_0 \|$$

Which is not possible. Hence, $F = \overline{F} = N$ and N is a separable space.

SECOND PART: If N be a separable normed linear space then N^* may not be separable.

Consider $N = \ell = \left\{ \{x_n\} : \sum_{n \geq 1} |x_n| < \infty \right\}$. In the next section, it would be shown that $\ell^* = m$ or ℓ_∞ in the sense that ℓ^* is isometrically isomorphic to m. We can show that ℓ is separable but m is not.

Let $\overline{x} = \{x_n\} \in \ell$ and $\varepsilon > 0$. There is an integer n_0 with $\sum_{m=n+1}^{\infty} |x_n| < \varepsilon/2$ for $n \geq n_0$.

For each $x_i \in R$ there exists $y_i \in R$ such that $\sum_{i=1}^{n_0} | x_i - y_i | < \varepsilon/2$.

Let $\overline{y} = \{y_1, y_2, \ldots y_n, 0, 0, 0 \ldots\ldots\ldots\}$

Then $d(\overline{x}, \overline{y}) = \sum_{1 \leq i \leq n_0} | x_i - y_i | + \sum_{n_0+1}^{\infty} | x_i | < \varepsilon$. This shows that every ε-nbhd of $\overline{x}$ contains a $\overline{y} \in M = \{\{y_1, y_2, \ldots y_n, 0, 0, 0 \ldots\ldots\ldots\} \mid n \in N\}$. Hence $\overline{M} = \ell$ and M is countable. This proves that ℓ is a separable space.

But $\ell^* = m$ is not separable. Let $y = \{\alpha_1, \alpha_2, \ldots\}$ be a sequence of zeros and ones. Then $y \in m$. With y,a real number $\hat{y}$ can be associated whose binary representation is $\hat{y} = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \ldots \frac{\alpha_n}{2^n} + \ldots + \infty$.

We now use the fact that the set of points in the interval [0,1] is uncountable, each $\hat{y} \in [0,1]$ has binary representation. Hence there are uncountably many sequences of zeros and ones. The metric on m, $d(\overline{x}, \overline{y}) = \sup | x_i - y_i |$ shows that any two of them which are not equal must be of distance 1 apart. If we let each of these sequences be the center of a small ball, say, of radius $\frac{1}{3}$, these balls do not intersect and we have uncountably many of them. If M be any dense subset in m, each of these non intersecting balls must contain an element of M. Hence M can not have subsets which are countable. Consequently, m is not separable.

10. Natural imbedding of N in N**:

If N be a normed linear space then the set N* of all bounded linear functionals is a Banach space. N* is called **dual space or conjugate space** of N. Similarly, dual space of N^* is denoted by N** and it is called the **second dual of N**. Each $x \in N$ induces a continuous linear functional F_x on N* defined by

$$F_x(f) = f(x) \text{ for all } f \in N^*.$$

Theorem 10·1: N is a normed linear space, N* and N** are first and second dual of N.

Then the mapping $x \to F_x$ is an isometric isomorphism.

Proof : To show $J : N \to N^{**}$ is well defined, it is enough to show that for each $x \in N$, $J(x) = F_x$ is in N**.

$F_x : N^* \to K$ defined by $F_x(f) = f(x)$ is linear. For $f, g \in N^*$ and $\alpha, \beta \in K$

$F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha F_x(f) + \beta F_x(g)$ and
$|F_x(f)| = |f(x)| \le \|f\| \|x\|$. Hence F_x is a bounded linear map on N*.

Clearly $J:N \to N^{**}$ is an isometry, that is $\|x\| = \|F_x\|$. In fact,

$$\|F_x\| = \sup\{|F_x(f)| : \|f\| = 1\} = \sup\{|f(x)| : \|f\| = 1\} = \|x\| \text{ (Theorem 9·7)}$$

J is an isomorphism into N.**

$$[J(\alpha x_1 + \beta x_2)](f) = F_{\alpha x_1 + \beta x_2}(f) = f(\alpha x_1 + \beta x_2)$$

$$= \alpha f(x_1) + \beta f(x_2) = \alpha F_{x_1}(f) + \beta F_{x_2}(f) = [\alpha F_{x_1} + \beta F_{x_2}](f) = [\alpha J_{x_1} + \beta J_{x_2}](f)$$

So, J is linear in N**.

J is one-one: If $Jx_1 = Jx_2$, then $F_{x_1} = F_{x_2}$ and so $F_{x_1}(f) = F_{x_2}(f)$ for all $f \in N^*$. Hence $f(x_1) = f(x_2)\ \forall\ f \in N^*$.

or $\quad f(x_1 - x_2) = 0$ for all $f \in N^*$

Then by corollary to Theorem 9·5, $x_1 - x_2 = 0$ or $x_1 = x_2$. Hence J is an isometric isomorphism of N into N**. In general J is not onto.

11. Reflexive Spaces :

A normed linear space N is said to be **reflexive** if $J:N \to N^{**}$ defined by $[J(x)](f) = f(x)$, $f \in N^*$ is an **onto** isometric isomorphism.

Proposition 11·1. A reflexive space is a Banach space.

Proof : Dual X^{**} of X^* is a Banach space. The natural imbedding $J:X \to X^{**}$ is an isometric isomorphism. Hence completeness of X^{**} implies completeness of X.

Proposition 11·2 : every finite dimensional normed linear space X is reflexive.

Proof : In case of finite dimensional normed space, every linear functional is continuous. So, $B(X,K) = X^* = L(X,K) = X'$

Also $\dim X = \dim X^* = \dim X^{**}$. The one- one linear map J from X to X^{**} must be onto. Hence X is reflexive.

Some facts on reflexivity

1. A reflexive space is complete but a complete space may not be reflexive. For example the Banach space C_0 is not reflexive.
2. ℓ^p $(1<p<\infty)$ is reflexive but ℓ^1 is not reflexive.

12. Some Dual Spaces :

The dual space X^* of a normed linear space X contains the zero linear functional, θ, given by $\theta(x)=0$ for all $x \in X$. The Hahn Banach theorem (9.4) has established that for non-zero space X, X^* contains non-zero bounded linear functional. We shall obtain representations for the bounded linear functional on certain normed linear space and identify their dual spaces.

Theorem 12.1: Let X be a finite dimensional normed linear space over K. Then every linear functional on X is bounded, the dual space X^* is finite dimensional and X and X^* have the same dimension.

Proof : Let $(X, \|.\|)$ be an n-dimensional normed linear space with base $\{e_1, e_2, \ldots, e_n\}$. For $x \in X$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ in K such that

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n \text{. Define } \|x\|_1 = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$$

which is a norm on X. Since any two norms on a finite dimensional space are equivalent, there exists M>0 such that

$$\|x\|_1 \le M\|x\| \quad \text{for all } x \in X.$$

Now, we construct a basis $\{f_r\}_{r=1}^n$ consisting of n elements for X^*. This would prove that dim X= dim X^*.

Define $\qquad f_r(x) = \alpha_r$ if $x = \sum_{k=1}^{n} \alpha_k e_k$

It is easy to check that f_r is a linear functional on X. Also, $|f_r(x)| = |\alpha_r| \le \|x\|_1 \le M\|x\|$ for all $x \in X$.

Thus $f_r \in X^*$. Next we show that $\{f_r\}_{r=1}^n$ is linearly independent and $X^* = \text{span}\,[\{f_r\}_{r=1}^n]$.

If $\sum_{k=1}^n a_k f_k = 0$, then $0 = O(a_r) = (\sum_{k=1}^n a_k f_k)(a_r) = a_r (r = 1,2,\ldots,n)$. Hence $\{f_r\}_{r=1}^n$ is linearly independent.

Lastly, $X^* = \text{span}\,(\{f_r: 1 \le r \le n\})$. For $x \in X$ if $x = \sum_{r=1}^n \alpha_r e_r$, then for any linear functional f, we have

$$f(x) = f(\sum_{r=1}^n \alpha_r e_r) = \sum_{r=1}^n \alpha_r f(e_r) = \sum_{r=1}^n f_r(x)\beta_r \quad \text{where} \quad \beta_r = f(e_r)$$

$$= \sum_{r=1}^n (\beta_r f_r)(x)$$

So, $f = \sum_{r=1}^n \beta_r f_r$. This shows that $f \in X^*$ and $\{f_1, f_2, \ldots\ldots, f_n\}$ in a basis for X^*. Thus dim X = dim X^* and every linear functional on X is bounded.

Theorem 12.2: The dual space of R^n is R^n.

Proof: Let $(R^n)'$ and $(R^n)^*$ be algebraic and topological dual of R^n.

By Theorem 12.1, $R^{n'} = R^{n^*}$. $B = \{e_1 = (1,0,\ldots,0), e_2 = (0,1,0,\ldots,0), \ldots, e_n = (0,0,\ldots,0,1)\}$ is a basis for R^n. For $x \in R^n$, $x = \sum_{k=1}^n \alpha_k e_k$ and $f(x) = \sum_{k=1}^n \alpha_k f(e_k)$

$$|f(x)| = \left|\sum_{k=1}^n \alpha_k f(e_k)\right| \le \left(\sum_{k=1}^n \alpha_k^2\right)^{1/2} \left(\sum_{k=1}^n (f(e_k))^2\right)^{1/2} = \|x\| \left(\sum_{k=1}^n (f(e_k))^2\right)^{1/2}$$

$$\text{Hence} \quad \|f\| \le \left(\sum_{k=1}^n (f(e_k))^2\right)^{1/2}$$

However, since for $x = (f(e_1), \ldots, f(e_n))$ equality is achieved in the Cauchy Schawrz inequality, we must in fact have $\|f\| = \left(\sum_{k=1}^n |f(e_k)|^2\right)^{1/2}$. Hence the mapping of $(R^n)'$ onto R^n defined by $f \to (f(e_1), \ldots, f(e_n))$ is norm preserving and since it is linear and bijective, it is an isomorphism. This proves that $(R^n)^* = R^n$

Theorem 12.3: $\ell^* = m$. The equality is not in the set theoretic sense but in the sense that ℓ^* is isometric isomorphic to m.

Proof : Let $e_n = \{0,0,\ldots\ldots,0,1,0,\ \ldots\}$,1 being in the nth place. We recall $\ell = \{\{x_n\} : \sum_{n\geq 1}|x_n| < \infty\}$ and m the space of all bounded scalar sequences. Clearly $e_n \in \ell$ and

$\|e_n\| = 1$. If $\bar{x} = \{x_n\} \in \ell$ then $\bar{x} = \sum_{k=1}^{\infty} x_k e_k$. Put $S_n = \sum_{k=1}^{n} x_k e_k$.

Then $\|\bar{x} - S_n\| = \sum_{k=n+1}^{\infty}|x_k|$. Since $\sum_{k=1}^{\infty}|x_k|$ converges, given $\varepsilon > 0$ there exists an integer n_o

such that $\|\bar{x} - S_n\| < \varepsilon$ for all $n \geq n_o$. Hence $\lim_{n\to\infty} S_n = \bar{x}$ or $\bar{x} = \sum_{k=1}^{\infty} x_k e_k$ .Define T: $\ell^* \to m$,

by $$T(f) = \{f(e_k)\}_{k=1}^{\infty}$$

We can show that T is the required isometric isomorphism.

T is well defined. $|f(e_k)| \leq \|f\|\,\|e_k\| = \|f\|$ for $k \geq 1$.

This shows that $\{f(e_k)\} \in m$. Further,

$$\|T(f)\| = \sup_{k\geq 1}|f(e_k)| \leq \|f\| \ldots\ldots\ldots\ldots(*)$$

T is linear : For $f, g \in \ell^*$ and $\alpha, \beta \in K$

$$T(\alpha f + \beta g) = \{(\alpha f + \beta g)(e_k)\}_{k\geq 1} = \alpha\{f(e_k)\}_{k=1}^{\infty} + \beta\{g(e_k)\}_{k=1}^{\infty} = \alpha\, T(f) + \beta T(g)$$

T is one-one: Since $\bar{x} = \sum_{k=1}^{\infty} x_k e_k$, for $f \in \ell^*$, $f(\bar{x}) = \sum_{k=1}^{\infty} x_k f(e_k)$.

If $T(f) = \{f(e_k)\}_{k\geq 1} = 0$, the zero sequence, then $f(\bar{x}) = 0$ for all $\bar{x} \in \ell$. Hence $f=0$ and T is an one-one mapping.

T is into:

Let $z = \{\alpha_k\} \in m$. Then $\sum_{k\geq 1} x_k\alpha_k$ is absolutely convergent for each $\{x_k\}_{k\geq 1} \in \ell$. We define g: $\ell \to K$ by $g(\{x_k\}) = \sum_{k\geq 1} x_k \alpha_k$. We claim $g \in \ell^*$ and $T(g) = z$. Clearly g is linear in ℓ and

$$|g(\bar{x})| = \left|\sum_{K\geq 1} x_k\alpha_k\right| \leq \sum_{k\geq 1}|x_k|\,|\alpha_k| \leq \|z\| \sum_{k\geq 1}|x_k| = \|z\|\,\|\bar{x}\|$$

This proves that $g \in \ell^*$ and $T(g) = \{g(e_k)\}_{k\geq 1} = \{\alpha_k\}_{k\geq 1} = z$

So, T maps ℓ^* onto m. Further $\|g\| \leq \|z\|$

T is norm preserving or isometry:

Let $f \in \ell^*$ and let $z = T(f)$. As seen in the proof of T is onto, $f = g$ where $g(\bar{x}) = \sum_{k\geq 1}^{n} x_k (Tf)_k$

So, $\| f \| = \| g \| \leq \| z \| = \| T(f) \| \leq \| f \|$ (from (*)). This shows that $\| T(f) \| = \| f \|$. That is, T is norm preserving or isometry. This proves that ℓ^* is isometrically isomorphic to m.

Theorem 12.4: $C_o^* = \ell$ in the sense that C_o^* is isometrically isomorphic to ℓ.

Proof: As in Theorem 12.3, $e_n \in C_o$ and $\| e_n \| = 1$. Let $\bar{x} = \{x_k\} \in C_o$ and let $S_n = \sum_{K=1}^{n} x_k e_k$.

Since $\lim_{n\to\alpha} x_n = 0$ there exists an integer n_o such that $| x_k | < \varepsilon$ for $k \geq n_o$.

$\| \bar{x} - S_n \| = \sup_{k\geq n+1} | x_k | < \varepsilon$ for $n \geq n_o$. Thus $\bar{x} = \lim_{n\to\alpha} S_n = \lim_{n\to\alpha} \sum_{K=1}^{n} x_k e_k = \sum_{k\geq 1} x_k e_k$

For $f \in C_o^*$, we have, $F(\bar{x}) = \sum_{k\geq 1} x_k f(e_k)$ ....(*)

We claim that $\{f(e_k)\}_{k\geq 1} \in \ell$. Let n be a positive, and

$$\alpha_r = \begin{cases} o \text{ if } f(e_r) = 0 \\ \frac{|f(e_r)|}{f(e_r)} \text{ if } f(e_r) \neq 0 \end{cases}$$

and let $y_n = \sum_{r=1}^{n} \alpha_r e_r$. Obviously, $\| y_n \| \leq 1$.

From (*) $f(y_n) = \sum_{k=1}^{n} | f(e_k) |$

Therefore, $\sum_{k=1}^{n} | f(e_k) | = | f(y_n) | \leq \| f \| \| y_n \| \leq \| f \|$

This shows that $\sum_{k=1}^{\infty} | f(e_k) |$ converges and $\sum_{k=1}^{\infty} | f(e_k) | \leq \| f \|$ ....(**)

Define $T: C_o^* \to \ell$, by $T(f) = \{f(e_n)\}_{n\geq 1}$. We have seen that T is well defined and

$$\| T(f) \| = \| \{f(e_n)\} \| = \sum_{n=1}^{\alpha} | f(e_n) | \leq \| f \| \text{ ....(***)}$$

Clearly T is linear. We can show that T is also one-one onto and isometric.

T is one-one: From (*), $f(\bar{x})=\sum_{k\geq 1} x_k f(e_k)$.

$$T f=\{f(e_k)\}=\bar{0}=\{0,0,0,\ldots\} \Rightarrow f(e_k)=0 \text{ for } k\geq 1$$

Hence $f(\bar{x})=0$ for all $\bar{x}\in C_o$. So, $f=0$ and T is one-one.

T is onto:

Let $\{y_k\}\in \ell$. For each $\bar{x}=\{x_k\}\in C_o$, the series $\sum_{k=1}^{\infty} x_k y_k$ converges absolutely.

Define, $g:C_o\to K$ by

$$g(\bar{x})=\sum_{k=1}^{\infty} x_k y_k \text{ for each } \{x_k\}\in C_o$$

We claim $g\in C_o^*$ and $T(g)=\bar{y}=\{y_k\}_{k=1}^{\infty}$. It is clear that g is linear in C_o

$$|g(\bar{x})|=\left|\sum_{k=1}^{\infty} x_k y_k\right|\leq \sum_{k=1}^{\infty}|x_k||y_k|\leq\|\bar{x}\|\sum_{k=1}^{\infty}|y_k|$$

So, g is bounded linear on C_o and $T(g)=\{g(e_k)\}=\{y_k\}$. Hence T is onto. Further, $\|g\|\leq\sum_{k=1}^{\infty}|y_k|$. Lastly, **T is norm preserving or isometric.**

Let $f\in C_0^*$ then $T(f)=\{f(e_k)\}$ and $\|f\|\leq\sum_{k=1}^{\infty}|f(e_k)|=\|T(f)\|\leq\|f\|$.

Thus $\|T(f)\|=\|f\|$. So, T is isometric isomosphism.

Theorem 12.5 $(\ell^p)^*=\ell^q$ where, $\frac{1}{p}+\frac{1}{q}=1, 1<p<\infty$

Proof: As in the preceding theorems, every $\bar{x}=\{x_k\}\in\ell^p$ has unique representation

$$\bar{x}=\sum_{k=1}^{\infty} x_k e_k$$

For $f\in\ell^{p^*}$, $f(\bar{x})=\sum_{k=1}^{\infty} x_k f(e_k)$ ....(*)

Let q be the conjugate of p and consider $x_n=\{\alpha_k^{(n)}\}$ with

$$\alpha_k^{(n)}=\begin{cases}\frac{|f(e_k)|^q}{f(e_k)} \text{ if } k\leq n \text{ and } f(e_k)\neq 0\\ 0 \text{ if } k>n \text{ or } f(e_k)=0\end{cases} \quad\text{..........(**)}$$

By substituting this in (*), we obtain

$$f(x_n)=\sum_{k=1}^{\infty}\alpha_k^{(n)}f(e_k)=\sum_{k=1}^{n}|f(e_k)|^q$$

we also have, using (**) and $(q-1)p = q$,

$$f(x_n)\le \|f\|\,\|x_n\|=\|f\|\left(\sum_{k=1}^{n}|\alpha_k^{(n)}|^p\right)^{1/p}=\|f\|\left(\sum_{k=1}^{n}|f(e_k)|^{(q-1)p}\right)^{1/p}=\|f\|\left(\sum_{k=1}^{n}|f(e_k)|^q\right)^{1/p}$$

$$\text{So, } f(x_n)=\sum_{k=1}^{n}|f(e_k)|^q\le\|f\|\left(\sum_{k=1}^{n}|f(e_k)|^q\right)^{1/p}$$

Dividing by the last factor and using $1-\frac{1}{p}=\frac{1}{q}$,

$$\text{We get, } \left(\sum_{k=1}^{n}|f(e_k)|^q\right)^{1-\frac{1}{p}}=\left(\sum_{k=1}^{n}|f(e_k)|^q\right)^{1/q}\le\|f\|$$

As n is arbitrary letting $n\to\infty$, we obtain

$$\left(\sum_{k=1}^{\infty}|f(e_k)|^q\right)^{1/q}\le\|f\|\text{............(***)}$$

Hence, $\{f(e_k)\}\in \ell^q$

Conversely, for any $b=\{\beta_k\}\in\ell^q$ we can get a corresponding bounded linear functional g on ℓ^p. In fact, we may define g on ℓ^p by setting $g(\overline{x})=\sum_{k=1}^{\infty}x_k\beta_k$ where $\overline{x}=\{x_k\}\in\ell^p$. Then g is linear and bounded follows from the Holder inequality . Hence $g\in\ell^{p*}$.

Next we show that the norm of $f\in C_0$ is the norm of $\{f(e_k)\}_{k=1}^{\infty}$ in ℓ^q

$$|f(x)|=\left|\sum_{k=1}^{\infty}x_kf(e_k)\right|\le\left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}\left(\sum_{k=1}^{\infty}|f(e_k)|^q\right)^{1/q}=\|x\|\left(\sum_{k=1}^{\infty}|f(e_k)|^q\right)^{1/q}$$

$$\text{Hence, } \|f\|\le\left(\sum_{k=1}^{\infty}|f(e_k)|^q\right)^{1/q}\text{ from (***)}$$

We see that equality holds, i.e., $\|f\|\le\left(\sum_{k=1}^{\infty}|f(e_k)|^q\right)^{1/q}$

this can be written as $\|f\| = \|T(f)\|_q$ where T: $\ell^{p*} \to \ell^q$ defined by

$$T(f)=\{f(e_k)\}_{k=1}^{\infty}.$$

The map T is linear one-one, onto and norm preserving. So, ℓ^{p*} is

isometrically isomorphic to ℓ^q with $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$.

Examples of reflexive and non-reflexive spaces:

Example 12.6: ℓ^p ($p>1$)is a reflexive space.

Proof: Let $p>1$ and $q = p(p-1)$. As seen in Theorem 12.5, there is linear isometry T_p of ℓ^q onto ℓ^{p*} such that $(T_p\bar{y})\bar{x} = \sum_{i=1}^{\infty} x_i y_i$ for all $\bar{x} \in \ell^q$ and for all $\bar{y} \in \ell^p$. Similarly there is a linear isometry T_q of ℓ^p onto ℓ^{q*} such that $(T_q\bar{x})\bar{y} = \sum_{i=1}^{\infty} x_i y_i$ for all $\bar{x} \in \ell^p$ and for all $\bar{y} \in \ell^q$, we have to show that J: $\ell^p \to \ell^{p**}$ is onto.

Let $F \in \ell^{p**}$. To show that there exists $\bar{x} \in \ell^p$ such that $F=F_{\bar{x}}$. Let $g=F_o T_p : \ell^q \to K$. Clearly $g \in \ell^{q*}$, So there exists $\bar{x} \in \ell^p$ such that $g=T_q\bar{x}$

Let $f \in (\ell^p)^*$. Then $f = T_p\bar{y}$ for some $\bar{y} \in \ell^q$

and we have $F(f)=f(T_p(\bar{y}))=g(\bar{y}) =(T_q\bar{x})\bar{y} = \sum_{i=1}^{\infty} x_i y_i =(T_p\bar{y})(\bar{x}) =f(\bar{x})$

This shows that $F=F_{\bar{x}}$ or $J(\bar{x})$ and J is onto. Hence ℓ^p is a reflexive space.

Example 12.7 The Banach space c_o is non reflexive :

Proof : We have seen in Theorem 12.4 that $C_o^* = \ell$ and that there is a isometric isomorphism T of C_0^* onto ℓ. For any $f \in C_o^*$,$T(f)=\{f(e_i)\} \in \ell$. Since T is one-one and onto,

$$f(\bar{x})= \sum_{i=1}^{\infty} x_i f(e_i) \text{ for all } \bar{x}=\{x_i\} \in C_o$$

For each $f \in C_o^*$, $F_{\bar{x}}(f)= f(\bar{x}) = \sum_{i=1}^{\infty} x_i f(e_i)$ ....(*)

Let $\bar{y}=\{y_i\} \in m \setminus C_o$ and let $G(f)= \sum_{i=1}^{\infty} y_i f(e_i)$ for all $f \in C_o^*$ ....(**)

We claim $G \in C_o^{**}$. G is clearly linear and

$$|G(f)| \le \sum_{i=1}^{\infty} |y_i| |f(e_i)| = \|y\| \sum_{i=1}^{\infty} |f(e_i)| = \|y\| \, \|T(f)\| = \|y\| \, \|f\|.$$

We can show that G has no pre image under the natural imbedding $J: C_0 \to C_0^{**}$. If possible, let, $G = F_{\overline{x}}$ for some $\overline{x} \in C_0$

From (*)and (**), $\sum_{i=1}^{\infty} x_i f(e_i) = \sum_{i=1}^{\infty} y_i f(e_i)$ for $f \in C_0^*$.

Since T maps C_0^* and ℓ, $\sum_{i=1}^{\infty} x_i e_i = \sum_{i=1}^{\infty} y_i e_i$ (i=1,2,3,......)

Hence $\overline{x} = \overline{y}$ which is impossible as $\overline{y} \notin C_0$ and $\overline{x} \in C_0$.

Exercises

1. Is the dual space of every finite dimensional space, the space itself?
2. Obtain the dual space of each of the following normed linear spaces:
   - (i) C[a,b]
   - (ii) ℓ_1
   - (iii) ℓ_∞
   - (iv) $L_p (1 \le p < \infty)$
3. Show that extension in the Hahn Banach Theorem is not unique, state conditions under which the extension is unique?

●●●

13 Weak and Weak* topology for a normed linear space

A normed linear space $(N, \|.\|)$ is a topological space, the topology being induced by the metric $d(x,y) = \|x - y\|$. All members of N^* are continuous with respect to this topology which is referred as **strong topology**. There may exist another topology on N with respect to which all the members of N^* are continuous. Our interest is to investigate the weakest topology on N with respect to which all the members of N^* are continuous. This topology is known as **weak topology** of the normed linear space $(N, \|.\|)$ which has very important role in functional analysis. For instance, a Banach space is reflexive if and only if the closed unit ball $\{x: \|x\| \leq 1\}$ is weakly compact which means that the ball is compact with respect to the weak topology.

Definition 13.1

Weak Topology : $(N, \|.\|)$ is a normed linear space, $x \in N$ and $f \in N^*$.

Consider $S(x,f,\varepsilon) = \{y \in N: |f(x)-f(y)| < \varepsilon\}, \varepsilon>0$. The unions of finite intersections of all such $S(x,f,\varepsilon)$ on varying x, f and ε generate a topology with the family of $S(x,f,\varepsilon)$ as a subbasis of it. This topology is called weak topology on N.

Proposition 13.2: The weak topology is weaker than the norm (strong) topology.

Proof: $S(x,f,\varepsilon)$ is a subbasic neighbourhood of x. By the continuity of f, there exists $\delta>0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $\|x - y\| < \delta$.

Hence the open ball $B_\delta(x) \subset S(x,f,\varepsilon)$ and therefore $S(x,f,\varepsilon)$ is a neighbourhood of x with respect to norm topology. This shows that the weak topology is weaker than the norm topology.

Proposition 13.3 Weak topology is the weakest topology on a normed linear space $(N, \|.\|)$ with respect to which all the members of N^* are continuous.

Proof: Let T be a topology on N such that all $f \in N^*$ are continuous with respect to T. We show that the weak topology is weaker than the topology T. Consider weak subbasic neighbourhood $S(x,f,\varepsilon)$ of $x \in N$. Since $f:(N,T) \rightarrow K$ is continuous, $B = f^{-1}\{y \in K: |y-f(x)| < \varepsilon\}$ is a T-neighbourhood of x. Then $B \subset S(x,f,\varepsilon)$ since $u \in B$ implies $|f(u)-f(x)| < \varepsilon$. Hence $S(x,f,\varepsilon)$ is a T-nbhd of x and consequently weak topology is weaker than the topology T and all $f \in X^*$ are also continuous with respect to weak topology. This proves the result.

Throughout weakly convergent, weakly cauchy etc. mean convergent and Cauchy with respect to weak topology.

Proposition 13.4 : In a normed linear space $(N, \|.\|)$,

(a) a sequence $\{x_n\}$ weakly converges to x, in symbol, $x_n \xrightarrow{w} x$ if and only if $f(x_n) \to f(x)$ for all $f \in N^*$,

(b) weak limit of a weakly convergent sequence is unique,

(c) every subsequence of a weakly convergent sequence is bounded,

(d) if $x_n \xrightarrow{w} x$ then $\{\|x_n\|\}$ is bounded.

Proof: $x_n \xrightarrow{w} x$ if and only if everysubbasic weak nbhd $S(x,f,\varepsilon)$ of x contains all but a finite number of terms of $\{x_n\}$. Equivalently, there exists $n_0 \in \angle$ such that $x_n \in S(x,f,\varepsilon)$ for all $n \geq n_0$.

This is equivalent to $|f(x_n) - f(x)| < \varepsilon$ for $n \geq n_0$, that is, $f(x_n) \to f(x)$.

Thus $x_n \xrightarrow{w} x$ iff $f(x_n) \to f(x)$ for all $f \in N^*$.

(b) Suppose $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$

$\Rightarrow f(x_n) \to f(x)$ and $f(x_n) \to f(y)$ for all $f \in X^*$

$\Rightarrow f(x) = f(y)$ for all $f \in X^*$

$\Rightarrow f(x-y) = 0$ for all $f \in X^*$

$\Rightarrow x - y = 0$ (consequence of Hahn Banach Theorem)

$\Rightarrow x = y$. So weak limit is unique.

(c) $x_n \xrightarrow{w} x \Rightarrow f(x_n) \to f(x)$ for all $f \in X^*$

$\Rightarrow f(x_{n_k}) \to f(x)$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

$\Rightarrow x_k \xrightarrow{w} x$

(d) The proof of this part follows from the principle of uniform boundedness which states that

"If $\{T_i\}$ is a non empty set of continuous linear transformations of Banach space B into normed linear space N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers"

$x_n \xrightarrow{w} x \Rightarrow f(x_n) \to f(x)$ for all $f \in N^*$

$\Rightarrow |f(x_n)| \leq M_f$ for each $f \in N^*$, for all $n \geq 1$

$\Rightarrow |F_{x_n}(f)| \leq M_f$ for each $f \in N^*$

$\Rightarrow \{\|F_{x_n}\|\}$ is bounded (principle of uniform boundedness)

$\Rightarrow \{\|x_n\|\}$ is bounded (since the natural embedding $x \to F_x$ is an isometry)

Hence proved.

Theorem 13.4: Let $\{x_n\}$ be a sequence in a normed linear space N. Then

(a) Strong convergence implies weak convergence with the same limit

(b) The converse of (a) is not true

(c) If dim $x < \infty$, then the weak convergence implies strong convergence.

Proof:

(a) If $\{x_n\}$ strongly converges to x, then for $\varepsilon > 0$, there exists $n_0 \in \angle$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq n_0$.

For $f \in N^*$,

$$|f(x_n)-f(x)| = |f(x_n-x)| \leq \|f\| \|x_n - x\| \to 0 \text{ as } n \to \infty$$

So $f(x_n) \to f(x)$ for each $f \in N^*$ and $x_n \xrightarrow{w} x$

(b) Let $X = \ell^p$, $1<p<\infty$ and $e_n = \{0,0,..,0,1,0,...\}$, 1 being in the n^{th} place. $\|e_n\|_p = 1$ for each n, so that $e_n \not\to 0$ in X with respect to norm topology.

In the proof of $\ell^{p^*} = \ell^q$, we have seen that for $f \in \ell^{p^*}$ there exists $y = \{y_j\} \in \ell^q$ such that

$$f(x) = \sum_{j=1}^{\infty} x_j y_j \text{ for each } x = \{x_j\} \in \ell^p.$$

So, $f(e_n) = y_n \to 0$ since $\sum_{j=1}^{\infty} |y_j|^q < \infty$.

Hence $e_n \xrightarrow{w} 0$ in ℓ^p but $e_n \not\to 0$ in $(\ell^p, \|.\|_p)$.

(c) Suppose that $x_n \xrightarrow{w} x$ and $\dim X = k < \infty$.

If $\{e_1, e_2, \ldots e_n\}$ be a basis for X and

$$x_n = \alpha_1^{(n)} e_1 + \alpha_2^{(n)} e_2 + \cdots + \alpha_k^{(n)} e_k \text{ and } x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k$$

By assumption $f(x_n) \to f(x)$ for each $f \in X^*$.

Consider $f_1, f_2, \ldots, f_k$ in X^* defined by

$$f_j(e_j)=1,\ f_j(e_i)=0 \text{ if } i \neq j$$

$$\text{So, } f_j(x_n)=\alpha_j^{(n)},\ f_j(x)=\alpha_j$$

Hence, $f_j(x_n) \to f_j(x) \Rightarrow \alpha_j^{(n)} \to \alpha_j$. We readily obtain

$$\| x_n - x \| = \left\| \sum_{j=1}^{k} (\alpha_j^{(n)} - \alpha_j) e_j \right\| \leq \sum_{j=1}^{k} \left| \alpha_j^{(n)} - \alpha_j \right| \quad \| e_j \| \to 0 \text{ as } n \to \infty$$

So, weak convergence implies norm convergence

13.5 Weak* Topology :

For each $x \in N$, if $F_x(f)=f(x)$, $f \in N^*$ then $F_x \in N^{**}$. Weak topology is defined for $(N, \| . \|)$ as the weakest topology on N with respect to which all $f \in N^*$ are continuous. Similarly, weakest topology on N^* can be defined so that all $F_x \in N^{**}$ are continuous.

Definition 13.6: For fixed $f \in N^*$, $x \in N$ and $\varepsilon > 0$, let $S^*(f,x,\varepsilon)=\{g \in N^* : |F_x(g)-F_x(f)| = |g(x)-f(x)| < \varepsilon\}$. Varying f, x, and ε, the family $B^*=\{ S(f,x,\varepsilon) \}$ generates a topology on N^*. An open subset of this topology is union of finite intersections of members of B^* and B^* is a subbasis for this topology. This is the weakest topology on N^* with respect to which all $F_x \in N^{**}$ are continuous. This topology is called the weak* topology on N^*.

Proposition 13.7 Weak* topology is Hausdorff.

Proof: Let $f, g \in N^*$ and $f \neq g$. Then there exists $x \in N$ such that $f(x) \neq g(x)$. Suppose $|f(x)-g(x)|=3\varepsilon$. Consider, $S_1^*(f,x,\varepsilon)$ and $S_2^*(f,x,\varepsilon)$. Then S_1^* and S_2^* are disjoint neighbourhoods of f and g in the weak* topology of N^*. Suppose $h \in S_1^* \cap S_2^*$. Then $|h(x)-f(x)| < \varepsilon$ and $|h(x)-g(x)| < \varepsilon$.

So, $3\varepsilon = |f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)| < 2\varepsilon$, which is not possible. Hence $S_1^* \cap S_2^* = \phi$ and N^* is weak* Hausdorff.

14. B(N) as a Banach Algebra:

14.1 Definition : A linear space X is called an **algebra** if a multiplication of the elements of X be defined such that for x, y, z in X,

(1) $(xy)z = x(yz)$

(2) $(x + y) z = xz + yz$ and $x (y + z) = xy + xz$

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every scalar α.

The algebra is real or complex according as scalars are real or complex.

An algebra X is called a **commutative algebra**

If (4) $xy = yx$ for all x and y in X.

An algebra X is called an **algebra with identity** if there exists $1 \in X$ such that

$1.x = x = x.1$ for all $x \in X$.

A Banach Algebra is a Complex algebra with identity which is a Banach space with respect to a norm $\|.\|$ such that (1) $\|xy\| \le \|x\|\,\|y\|$ and (2) $\|1\| = 1$.

Proposition 14.2: B(N) is a Banach Algebra where N is a normed linear space.

Proof: We have already seen that B(N) the vector space of all bounded linear operators forms a Banach space with supremum norm defined by

$\|T\| = \text{Sup}\{\|Tx\| : \|x\| \le 1\} = \text{Sup}\{\|Tx\| : \|x\| = 1\}$ It can be easily seen that B(N) is an algebra if multiplication is defined by

$(T_1T_2)x = (T_1(T_2(x)))$ for $T_1, T_2 \in B(N)$

(1) $[(T_1T_2)T_3](x) = (T_1T_2)(T_3x) = T_1(T_2(T_3x))$

and $[(T_1(T_2T_3)](x) = T_1((T_2T_3)x) = T_1(T_2(T_3x))$

So, $(T_1T_2)T_3 = T_1(T_2T_3)$

Similarly

(2) $T_1(T_2+T_3) = T_1T_2+T_1T_3$ and $(T_1+T_2)T_3 = T_1T_3+T_2T_3$

(3) $\alpha(T_1T_2) = (\alpha T_1)T_2$

(4) The identity operator $I \in B(N)$ and $TI = T = IT$

(5) $\|T_1T_2\| = \sup\limits_{\|x\| \le 1} \|(T_1T_2)x\|$

$= \sup\limits_{\|x\| \le 1} \|T_1(T_2x)\| \le \sup\limits_{\|x\| \le 1} \|T_1\|\,\|T_2x\| \le \sup\limits_{\|x\| \le 1} \|T_1\|\,\|T_2\|\,\|x\| \le \|T_1\|\,\|T_2\|$

(6) $\|1\| = \sup\limits_{\|x\| = 1} \|I(x)\| = \sup\limits_{\|x\| = 1} \|x\| = 1$

So, B(N) is a Banach Algebra.

Note: (1) In a Banach Algebra multiplication is continuous. That is $T_n \to T$ and $S_n \to S \Rightarrow T_nS_n \to TS$.

$\|T_nS_n - TS\| = \|T_n(S_n - S) + (T_n - T)S\| \le \|T_n\|\,\|S_n - S\| + \|T_n - T\|\,\|S\|$

$$\leq M\|S_n - S\| + \|T_n - T\|\,\|S\| \quad (\text{since } \sup_n \|T_n\| \leq M < \infty)$$

$\to 0$ as $n \to \infty$. So, $T_nS_n \to TS$.

15. The open mapping Theorem :

This theorem provides a set of sufficient conditions under which a bounded linear map becomes an open map. The Baire's Theorem plays the main role in this theorem which states that

"If a complete metric space is the union of a sequence of its subsets then the closure of at least one set in the sequence must have non empty interior".

We also recall that under a homeomorphism $f(\overline{A}) = \overline{f(A)}$.

Theorem 15.1 If B and B′ are Banach spaces, and T is a continuous linear transformation of B onto B′, then T is an open mapping.

The most difficult part of the theorem will be covered in the following lemma.

Lemma: If B and B′ are Banach spaces, and if T is a continuous linear transformation of B onto B′, then the image of each open sphere centred on the origin of B contains an open sphere centered on the origin of B′.

Proof of the Lemma:

Let $S_r = \{x \in B \mid \|x\| < r\}$ and $S'_r = \{x \in B' \mid \|x\| < r\}$. Clearly $T(S_r) = T(rS_1) = rT(S_1)$. So, it is enough to show that $\overline{T(S_1)}$ contains S_r' then we shall show that $T(S_1)$ contains S_r'. Since T is onto, we see that $B' = \bigcup_{n=1}^{\infty} T(S_r)$. Since B′ is complete, by Baire's Theorem, some $\overline{T(S_{n_0})}$ has an interior point y_0, which may be assumed to lie in $T(S_{n_0})$. [If y_0 does not lie in $T(S_{n_0})$, it must be a limit point of $T(S_{n_0})$. So, the open sphere centred at y_0 and contained in $\overline{T(S_{n_0})}$ contains a point of $T(S_{n_0})$ which in turn becomes limit point of $\overline{T(S_{n_0})}$]. Clearly, $0 \in T(S_{n_0}) - y_0$. Since translation is a homeomorphism, 0 is an interior point $\overline{T(S_{n_0})} - y_0$.

Also $T(S_{n_0}) - y_0 \in T(S_{n_0})\, T(S_{n_0}) = T(S_{n_0} + S_{n_0}) = T(2S_{n_0})$, since $-S_{n_0} = S_{n_0}$. Then

$\overline{T(S_{n_0})} - y_0 = \overline{T(S_{n_0}) - y_0}$ (since $y \to y - y_0$ is a homeomorphism and under homeomorphism f, $f(\overline{A}) = \overline{f(A)}$) $\subseteq \overline{T(S_{2n_0})}$. So, the origin is an interior point of $\overline{T(S_{2n_0})} = \overline{2n_0 T(S_1)} = 2n_0 \overline{T(S_1)}$,

Since $y \to (2n_0)y$ is homeomorphism. It follows from this that the origin is also an interior point of $\overline{T(S_1)}$, So $S'_{r_0} \subset \overline{T(S_1)}$ for some $r_0 > 0$.

We conclude the proof by showing that $S'_{\eta_0} \subset T(S_3)$, equivalently, $S'_{\eta_0/3} \subset T(S_1)$. Let y be a vector in $S'_{\eta_0} \subset \overline{T(S_1)}$. So, there exists a vector $x_1 \in B, \|x_1\| < 1$ and $\|y - y_1\| < \eta_0/2$ where $y_1 = T(x_1)$. Next we observe that $S'_{\eta_0/2} \subset \overline{T(S_{1/2})}$. Then $y - y_1 \in S'_{\eta_0/2}$ implies there exists $x_2 \in X$, $\|x_2\| < 1/2$ and $\|(y - y_1) - y_2\| < \eta_0/4$ where $y_2 = Tx_2$. Proceeding in this we get a sequence $\{x_n\} \subset B$ such that $\|x_n\| < 1/2^{n-1}$ and $\|y - (y_1 + y_2 + \cdots + y_n)\| < \eta_0/2^n$, where $y_n = Tx_n$.

If we put $S_n = x_1 + x_2 + \ldots + x_n$, then for $n > m$

$$\|S_n - S_m\| = \|x_{m+1} + \ldots + x_n\| \le \|x_{m+1}\| + \cdots + \|x_n\| < \frac{1}{2^m} + \cdots + \frac{1}{2^{n-1}} \to 0 \text{ as } n,m \to \infty.$$

So, $\{S_n\}$ is a Cauchy sequence in B. Since B is complete there exists a vector x in B such that $x_n \to x$ and

$$\|x_n\| = \left\|\lim_n S_n\right\| = \lim_n \|S_n\| = \lim_n \|x_1 + x_2 + \ldots + x_n\| \le \lim_n (\|x_1\| + \|x_2\| + \ldots + \|x_n\|)$$

$$< \lim_n \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right) \le 2 < 3$$

This shows that x is in S_3. Finally

$T(x) = T\left(\lim_n S_n\right) = \lim(TS_n) = \lim(y_1 + y_2 + \ldots + y_n) = y$ from which we see that $y = Tx \in T(S_3)$. Thus $S'_{\eta_0} \subset T(S_3)$ or $S'_{\eta_0/3} \subset T(S_1)$. This proves the lemma and with the help of it the open mapping Theorem can be proved easily.

Let G be an open susbet of B. We have to show that T(G) is open in B′. If y is a point in T(G), y = T(x) and x is in G. Since x is an interior point of the set G, there exists an open sphere S_r in B such that $x + S_r \subset G$. By the lemma, there exists $r_1 > 0$ such that $S'_{r_1} \subset T(S_r)$. Finally, $y + S'_{r_1} \subset T(x) + T(S_r) = T(x + S_r) \subset T(G)$. This shows that every $y \in T(G)$ is an interior point of T(G). Hence T(G) is open and the Theorem is proved.

16. The Closed Graph Theorem :

If B and B′ are Banach spaces, and if T is a linear transformation of B and B′, then T is continuous if and only if its graph is closed.

Proof: Suppose $T: B \to B'$ is continuous linear. To prove that graph of T = G= $\{(x, T(x) \mid x \in B\}$ is a closed subset of the product space of BxB′, the product topology of which is equivalent to the topology included by the norm defined on BxB′ by

$$\|(x,y)\|_{B \times B'} = \|x\|_B + \|y\|_{B'}$$

Let $\{(x_n,Tx_n)\}$ be sequence in G and $(x_n,Tx_n) \to (x,y)$. It is sufficient to show that $(x,y)\in G$, equivalently, $y=Tx$.

Since, $\|(x_n,Tx_n)-(x,y)\| = \|(x_n - x, Tx_n - y)\| = \|x_n - x\| + \|Tx_n - y\|$,

$(x_n,Tx_n) \to (x,y) \Rightarrow x_n \to x$ and $Tx_n \to y$. Since T is continuous, $x_n \to x \Rightarrow Tx_n \to Tx$. From the uniqueness of limit of a convergent sequence in a normed linear space, it follows that $y = Tx$. Hence $(x,y)=(x,Tx)\in G$ and G is a closed subset of $B\times B'$.

Conversely, suppose the graph of T is closed. To prove that the linear map $T:B\to B'$ is continuous. Define a new norm $\|.\|$ in B by $\|x\| = \|x\|_B + \|Tx\|_{B'}$ where $\|.\|_B$ and $\|.\|_{B'}$ are original norms of B and B′ respectively. With this new norm $\|.\|_B$, the continuity of T can be proved immediately. It follows from

$\|Tx\|_{B'} = \|x\|_B + \|Tx\|_{B'} = \|x\|$ that $T(B, \|.\|) \to T(B', \|.\|_{B'})$ is continuous . ...(*)

The proof can be completed by showing that $\|.\|$ -topology and $\|\|_B$ -topology are the same. Equivalently, the identity mapping $I_B:(B, \|.\|) \to (B, \|.\|_B)$ is a homeomorphism. This will follow from the open mapping theorem if we can show that (i) I_B is continuous linear and onto map and (2) $(B, \|.\|)$ is a Banach space. The continuity of $I_B:(B, \|.\|) \to (B, \|.\|_B)$ follows from the fact that $\|I_B(x)\| = \|x\|_B \le \|x\|_B + \|Tx\|_{B'} = \|x\|$. Also I_B is linear and onto. For completeness of $(B, \|.\|)$, let $\{x_n\}$ be a Cauchy sequence in $(B, \|.\|)$. Since,

$$\|x_n - x_m\| = \|x_n - x_m\| + \|Tx_n - Tx_m\|_{B'}$$

$\{x_n\}$ be a Cauchy sequence in $(B, \|.\|)$ and $\{Tx_n\}$ be a Cauchy sequence in $(B', \|.\|_{B'})$. Since both the spaces are complete, there exists $x\in B$ and $y\in B'$ such that $\|x_n - x\|_B \to 0$ and $\|Tx_n - y\|_{B'} \to 0$ as $n\to\infty$. Then,

$$\|(x_n,Tx_n)-(x,y)\|_{B\times B'} = \|x_n - x\|_B + \|Tx_n - y\|_{B'} \to 0 \text{ as } n\to\infty.$$

Thus the sequence $\{(x_n,Tx_n)\}$ in G converges to (x,y). Since G is closed, $(x,y)\in G$ and $y = Tx$. We complete the proof, showing that $\|x_n - x\| \to 0$ as $n\to\infty$.

$$\|x_n - x\| = \|x_n - x\|_B + \|T(x_n - x)\|_{B'} = \|x_n - x\|_B + \|Tx_n - Tx\|_{B'}$$

$$= \|x_n - x\|_B + \|Tx_n - y\|_{B'} \to 0 \text{ as } n\to\infty.$$

So, $x_n \to x$ in $(B, \|.\|)$. Hence B is a Banach space. By the open mapping theorem, $I_B:(B, \|.\|) \to (B, \|.\|_B)$ is a homeomorphism. So, $\|.\|$ and $\|.\|_B$ are equivalent.

So, $T:(B, \|\cdot\|_B) \to (B', \|\cdot\|_{B'})$ is continuous as seen in (*). This proves the theorem.

17. Banach Steinhaus Theorem (The uniform Boundedness Theorem):

Theorem 17.1: Let B be s Banach space and N be a normed linear space. If $\{T_i\}$is a non-empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$is a bounded subset of N for each x in B, then $\{\|T_i\|\}$is a bounded set of numbers; that is, $\{T_i\}$is bounded as a subset of B(B,N).

Proof : For each positive integer n, the set $F_n=\{x:x\in B \text{ and } \|T_i(x)\| \le n \text{ for all } i\}$

is clearly a closed subset of B. Indeed, for any $x\in \overline{F}_n$,there is a sequence $\{x_j\}$in F_n such that $x_j \to x$ as $j\to\infty$. For each fixed i, $\|T_i(x_j)\| \le n$. Taking limit as $j\to\infty$, we obtain $\|T_i(x)\| \le n$ for all i. Hence $x\in F_n$ and F_n is closed.

For fixed $x\in B$, there exists $n\in N$ such that $\|T_i(x)\| \le n$ for all i.

So, $x\in F_n$ and hence $B=\bigcup_{n=1}^{\infty} F_n$. Since B is complete, by Baire's Theorem some F_k contains an open ball, $B_0=B_r(x_0)\subset F_k$

Let $x\in B$ be arbitrary, not zero.

We set $z=x_0+\gamma x$ where $\gamma = \frac{r}{2\|x\|}$. Then $\|z-x_0\| = \|\gamma\|\,\|x\| = \frac{r}{2} < r$ so that

$z\in B_0 \subset A_k$. We thus have $\|T_i z\| \le k$ for all i. Also $\|T_i x_0\| \le k$ since $x_0\in B_0$.

For all i,

$$\|T_i(x)\| = \left\| T_i\left(\frac{z-x_0}{\gamma}\right)\right\| \qquad (\text{since, } z = x_0+\gamma x)$$

$$= \frac{1}{\gamma}\ \|T_i(z-x_0)\|$$

$$\le \frac{1}{\gamma}(\ \|T_i z\| + \|T_i(x_0)\|) < \frac{2\|x\|}{r} \times 2k$$

Hence, for all i $\qquad \|T_i\| = \sup_{\|x\|\le 1} \|T_i(x)\| < \frac{4k}{r}$

Thus we have that $\{\|T_i\|\}$is a bounded sequence of numbers.

Applying Banach Steinhaus Theorem, we have the following useful result.

17.2 Theorem : A non-empty set A of a normed linear space N is bounded if and only if f(A) is a bounded set of numbers for each $f\in N^*$.

Proof: Suppose A is bounded. Then $\|x\|\le k$ for all $x\in A$ and a fixed $k\in 3_+$. Then

$$|f(x)|\le\|f\|.\|x\|\le\|f\|k \text{ for all } x\in A$$

Hence $f(A)=\{f(x):x\in A\}$ is a bounded subset of numbers.

Conversely, suppose $\{f(x):x\in A\}$ be a bounded set of numbers for each $f\in N^*$. We recall, $F_x\in N^{**}$ where $F_x(f)=f(x)$ for all $f\in N^*$. So, $\{F_x(f)\}$ is bounded at each $f\in N^*$. Applying Banach Steinhaus Theorem $\{\|F_x\|:x\in A\}=\{\|x\|:x\in A\}$ is bounded. Hence A is bounded and the result is proved.

18. Conjugate Operator T^*:

Let N' and N^* be the algebraic dual (vector space of all linear functionals on N) and topological dual (vector space of bounded linear functionals on N) respectively. Clearly $N^*\subset N'$. Let $T:N\to N$ be a linear operator. The operator T can be associated with an operator $T':N'\to N'$ defined by $[T'(f)](x)=f(T(x))$.

T′ is well defined: To show $T'(f)\in N'$. For $x_1,x_2\in N$ and scalars α and β,

$$[T'(f)](\alpha x_1+\beta x_2)=f[T(\alpha x_1+\beta x_2)]=f[\alpha T(x_1)+\beta T(x_2)]$$

$$=\alpha f[T(x_1)]+\beta f[T(x_2)]=\alpha[T'(f)](x_1)]+\beta[T'(f)](x_2)$$

T′ is a linear map: For $f,g\in N'$ and scalars α,β if $x\in N$, then

$$[T'(\alpha f+\beta g)](x)=(\alpha f+\beta g)(T(x))=\alpha f(T(x))+\beta g(T(x))$$

$$=\alpha[T'(f)](x)+\beta[T'(g)](x)=[\alpha T'(f)+\beta T'(g)](x)$$

This shows that the map T′ is also linear.

Proposition 18.1: If $T\in B(N)$ then $T^*=T'_{|N^*}:N^*\to N^*$ is

(a) well defined, that is, $T^*(N^*)\subset N^*$ (b) T^* is bounded linear (c) $\|T^*\|=\|T\|$

($T'_{|N^*}$ is the restriction of T′ from N′ to N^*)

Proof: (a) Let $f\in N^*$. To show that $T^*(f)\in N^*$. As seen in the preceding discussion, $T^*(f)$ is linear.

Let S be the closed unit ball in N.

T is continuous linear$\Rightarrow$T(S) is bounded in N$\Rightarrow$f(T(S)) is bounded in K

$$\Rightarrow (T'f)(S) \text{ is bounded} \Rightarrow T'f \text{ is continuous.}$$

Hence $T^*(N^*)\subset N^*$ and T^* is well defined.

Remark: Retracing back, one can easily see that $T^*:N^*\to N^*$ is continuous.

(b) T^* is linear as we have seen that T' is linear. The continuity of T^* follows from the fact that

$$\|T^*(f)\| = \sup_{\|x\|\le 1}|[T^*(f)](x)| = \sup_{\|x\|\le 1}|f(T(x))| \le \sup_{\|x\|\le 1}\|f\|\|Tx\|$$

$$= \sup_{\|x\|\le 1}\|f\|\|T\|\|x\| = \|f\|\|T\|$$

So, $\|T^*(f)\|\le\|f\|\|T\|$ and T^* is continuous linear. Also $\sup_{\|f\|\le 1}\|T^*(f)\|\le\|T\|$. Hence $\|T^*\|\le\|T\|$.

We recall that, $J:N\to N^{**}, x\to F_x$ is an isometric isomorphism.

Then $\|Tx\| = \|F_{Tx}\| = \sup_{\|f\|\le 1}|F_{Tx}(f)| = \sup_{\|f\|\le 1}|f(Tx)| = \sup_{\|f\|\le 1}|(T^*f)(x)|$

$$= \sup_{\|f\|\le 1}\|f\|\|T^*\|\|x\| = \|x\|\|T^*\|$$

So, $\sup_{\|x\|\le 1}\|Tx\|\le\|T^*\|$ and $\|T\|\le\|T^*\|$. Thus, $\|T^*\| = \|T\|$.

Proposition 18.2: If T_1, T_2 are bounded linear transformations and I_N the identity operator on a normed linear space N, then

(a) $(\alpha T_1 + \beta T_2)^* = \alpha T_1^* + \beta T_2^*$ (b) $(T_1T_2)^* = T_2^*T_1^*$ (c) $I_N^* = I_{N^*}$.

Proof:(a)

$$[(\alpha T_1 + \beta T_2)^*(f)](x) = f[(\alpha T_1 + \beta T_2)(x)] = f[\alpha T_1(x) + \beta T_2(x)] = \alpha(T_1^*(f)(x) + \beta(T_2^*(f))(x)$$

$$= [\alpha(T_1^*(f) + \beta(T_2^*(f))](x) = [\alpha T_1^* + \beta T_2^*)(f)](x) \qquad \text{for all } f\in N^* \text{ and all } x\in N$$

Hence (a) follows.

(b) $[(T_1T_2)^*(f)](x) = f[(T_1T_2)x] = f[T_1(T_2x)] = (T_1^*f)(T_2x) = [T_2^*(T_1^*f)](x) = [(T_2^*T_1^*)(f)](x)$

for all $f\in N^*$ and for all $x\in N$. Hence $(T_1T_2)^* = T_2^*T_1^*$

(b) $[I_N^*(f)](x) = f(I_N(x)) = f(x) = [I_{N^*}(f)](x)$

So, $I_N^* = I_{N^*}$.

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Exercises (problems)

(1) Produce counter examples to show that open mapping theorem and closed graph theorem may not be true in ordinary normed linear spaces (i.e. in non-Banach spaces).

(2) Explain with examples the differences between algebraic isomorphism and topological isomorphism. Describe the connection of topological isomorphism with the continuity of a linear operator.

(3) Construct an unbounded linear operator. Can we construct an unbounded linear operator on a finite dimensional Banach space?

(4) State two Theorems guaranteeing that a Banach space will be a Hilbert space.

(5) State five results which are true in a Banach space but not true in a Hilbert space.

❑❑❑

HILBERT SPACES and FINITE DIMENSIONAL

SPECTRAL THEORY

1. Definition and Examples :

We have already studied normed linear spaces in which, we feel the absence of analogues of the familiar dot product

$$\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3,$$
$$|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

and the condition of orthogonality , namely, $\bar{a}.\bar{b} = 0$. In this part we shall study the generalization of such notions in the form of **inner product space and Hilbert space.**

Finally we shall study the generalization of the notion of eigenvector for Hilbert space operators which we have studied in the matrix theory in under graduate course . Historically this study is older than that of the general normed linear spaces. Hilbert spaces is named in the honour of David Hilbert who initiated the whole theory in 1912 on integral equation . However the present form of notations and terminology is due to E.Schmidt (1908).

Definition 1.1 . An inner product on a conplex vector space is a mapping of $X \times X$ into the scaler field C ; that is , with every pair of vectors x and y there is associated a scalar written as $< x , y >$ and is called the inner product of x and y, such that for all vectors x , y , z and scalars α we have ,

1. $< x + y, z > = < x , z > + < y , z >$ **[Linearity in first variable]**

2. $< \alpha x , y > = \alpha < x , y >$

3. $< x , y > = \overline{< y , x >}$ **[Conjugate - Symmetry]**

4. $< x , x > \geq 0$ and $< x , x > = 0$ if and only if $x = 0$

[Positive - Definiteness]

Note : Another notation for inner product used by many authors is (x ,y).To avoid the confusion of inner product with ordered pair (x, y), we shall strictly use the symbol $< x , y >$ for inner product .

Deduction : In an inner product space

1. $< \alpha x + \beta y, z > = \alpha < x , z > + \beta < y , z >$

2. $< x , \beta y > = \bar{\beta} < x , y >$

3. $< x , \beta y + \gamma z > = \bar{\beta} < x , y > + \bar{\gamma} < x, z >$

Proof : 1. $< \alpha x + \beta y, z > = < \alpha x, z > + < \beta y, z >$

$$= \alpha < x, z > + \beta < y, z >$$

2. $< x, \beta y > = \overline{<\beta y, x>} = \overline{\beta <y, x>}$

$= \bar{\beta} \overline{<y, x>} = \bar{\beta} <x, y>$

3. $< x, \beta y + \gamma z > = <x, \beta y> + <x, \gamma z>$

$= \bar{\beta} <x, y> + \bar{\gamma} <x, z>$

Proposition : In an inner product space (X , <>) if $\| x \| = \sqrt{<x, x>}$ then

a. $\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2$ **[Parallelogram Law]**

b. $4 <x, y> = \| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i \| x - iy \|^2$

[Polarization Identity]

c. $| <x, y> | \le \| x \| \| y \|$ **[Schwarz Inequality]**

d. $<x, y> = 0$ for all $y \in X$ if and only if $x = 0$

e. $(x, \| . \|)$ is a normal linear space .

f. inner product is continuous

Proof :

a. $\| x + y \|^2 + \| x - y \|^2$

$= <x + y, x + y> + <x - y, x - y>$

$= <x,x> + <x,y> + <y,x> + <y,y> + <x,x> + <x,-y> + <-y,x> + <-y,-y>$

$= 2 \| x \|^2 + <x, y> + <cy, x> - <x, y> - <y, x> + \| y \|^2 + <y, y>$

$= 2 \| x \|^2 + 2 \| y \|^2$

b $\| x + y \|^2 = <x + y, x + y> = \| x \|^2 + <x, y> + <y, x> + \| y \|^2$

$i\| x + iy \|^2 = i \| x \|^2 + i<x, iy> + i<iy, x> + i\| iy \|^2$

$= i \| x \|^2 + <x, iy> - <y, x> + i\| iy \|^2$

$- \| x - y \|^2 = - \| x \|^2 + <x, -y> - <-y, x> - \| -y \|^2$

$= - \| x \|^2 + <x, y> + <y, x> - \| y \|^2$

$i\| x - iy \|^2 = - i \| x \|^2 + i<x, iy> + i<iy, x> - i\| iy \|^2$

$= -i \| x \|^2 + <x, y> - <y, x> - i\| y \|^2$

Adding we have

$\| x + y \|^2 - \| x - y \|^2 + \| x + iy \|^2 - \| x - iy \|^2 = 4 <x, y>$

c. For any scalar $\lambda \in C$

$$0 \le \| \lambda x + y \|^2 = < \lambda x + y , \lambda x + y >$$

$$= \lambda \bar{\lambda} < x, x> + \lambda < x, y> + \bar{\lambda} < y, x> + < y, y>$$

$$= | \lambda |^2 \| x \|^2 + \lambda < x, y> + \bar{\lambda} < y, x> + \| y \|^2$$

Take $\lambda = \dfrac{\overline{-< x, y >}}{\| x \|^2}$ where $x \ne 0$. Then we have,

$$0 \le \frac{| < x, y > |^2}{\| x \|^2} - \frac{| < x, y > |^2}{\| x \|^2} - \frac{| < x, y > |^2}{\| x \|^2} + \| y \|^2$$

$$\text{or } \frac{|< x, y >|}{\| x \|^2} \le \| y \|^2 \qquad \text{or } |<x,y>|^2 \le \|x\|^2 \, \|y\|^2$$

Hence, $|<x,y>| \le \|x\| \, \|y\|$. The inequality becomes equality if $x = 0$.

Second part : If { x , y } is linear dependent , then $x = \alpha y$ and

$$|<x, y>| = |<\alpha y, y>| = |\alpha| \, |<y,y>|$$

$$\|x\|^2 \, \|y\|^2 = <\alpha y, \alpha y> <y,y> = |\alpha|^2 \, \overline{<y,y>} \, <y,y> = |\alpha|^2 <y,y>^2$$

So $|<x,y>| = \|x\| \, \|y\| = |\alpha| \, |<y,y>|$

Let $|<x,y>|^2 = < x , x > < y , y >$ .Then if $z = < y , y > x - < x , y > y$,

We have $< z , z > = <<y,y>x - <x,y>y, <y,y>x - <x,y>y >$

$$= |<y,y>|^2 < x , x > - < y , y > < y , x > < x , y >$$

$$- < x , y > < y , y > < y ,x > + |<x,y>|^2 < y , y >$$

$$= 0 .$$

Hence $z = 0$ and $< y , y > x = < x , y > y$ so that { x , y } is a linear dependent set.

d . Suppose $< x , y > = 0$ for every $y \in X$. Then in particular $< x , x > = 0$ so that $x = 0$.

Conversely , suppose $x = 0$ .Then $< x , y > = 0$ for every $y \in X$.

e .N_1) $\quad \| x \| = \sqrt{< x, y >} \ge 0$

$x = 0 \Rightarrow < x , x > = 0$ and $\quad \|x\| = 0 \Rightarrow \sqrt{< x, x >} = 0 \Rightarrow <x,x> = 0 \Rightarrow x = 0$

N₂) $\|\alpha x\|^2 = \langle\alpha x, \alpha x\rangle = \alpha\,\bar{\alpha}\langle x,x\rangle = |\alpha|^2\,\|x\|^2$ So that $\|\alpha x\| = |\alpha|\,\|x\|$

N₃) $\|x+y\|^2 = \langle x+y, x+y\rangle$

$$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 = \|x\|^2 + 2\,\text{Re}\langle x, y\rangle + \|y\|^2$$

$$\leq \|x\|^2 + 2\,|\langle x,y\rangle| + \|y\|^2 \leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 \quad \text{[Schwarz Inequality]}$$

$$= (\|x\| + \|y\|)^2$$

Hence , $\|x + y\| \leq \|x\| + \|y\|$. Thus $(X , \|.\|)$ is a normed linear space .

f. To prove the continuity of inner product , it is enough to show that

$$x_n \to x,\ y_n \to y \Rightarrow \langle x_n, y_n\rangle \to \langle x, y\rangle$$

$$|\langle x_n, y_n\rangle - \langle x, y\rangle| = |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$$

$$\leq |\langle x_n, y_n - y\rangle| + |x_n - x, y| \leq \|x_n\|\ \|y_n - y\| + \|x_n - x\|\ \|y\|$$

$$\to 0 \text{ since } x_n \to x,\ y_n \to y \text{ and } \{\|x_n\|\} \text{ is bounded .}$$

Definition : An inner product space (X , < >) is called a Hilbert space if $(X,\|.\|)$ is a Banach space with the norm induced by $\|x\| = \sqrt{\langle x,y\rangle}$.

Example 1. : The unitary space C^n is a Hilbert space with inner product defined by

$$\langle x , y\rangle = x_1\,\bar{y}_1 + x_2\,\bar{y}_2 + \ldots\ldots\ldots + x_n\,\bar{y}_n$$

where $x = (x_1, x_2, x_3, \ldots, x_n)$ and $y = (y_1, y_2, y_3, \ldots, y_n)$

1. $\langle x , x\rangle = x_1^2 + x_2^2 + \cdots + x_n^n \geq 0$ and $\langle x , x\rangle = 0$ if and only if $x = 0$.

2. $\langle\alpha x, y\rangle = \alpha x_1\,\bar{y}_1 + \alpha x_2\,\bar{y}_2 + \ldots\ldots + \alpha x_n\,\bar{y}_n = \alpha\langle x , y\rangle$

3. $\langle x+y, z\rangle = (x_1+y_1)\,\bar{z}_1 + (x_2+y_2)\,\bar{z}_2 + \ldots\ldots + (x_n+y_n)\,\bar{z}_n$

$$= (x_1\,\bar{z}_1 + \ldots\ldots + x_n\,\bar{z}_n) + (y_1\,\bar{z}_1 + \ldots\ldots y_n\,\bar{z}_n)$$

$$= \langle x , z\rangle + \langle y , z\rangle$$

Hence $(C^n,\langle\bullet\rangle)$ is an I P S and $\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{x_1^1 + x_2^2 + \cdots x_n^n}$

so that $(C^n, \|.\|)$ is a Banach space .Thus C^n is a Hilbert space .

Example 2. : $L^2[a,b]$, the vector space of all continuous complex valued functions on [a , b]

forms a normed space with normed defined by $\|x\| = \left(\int_a^b |x(t)|^2\,dt\right)^{\frac{1}{2}}$

This space is not complete. But this space is an inner product space if we define the inner product as follows :

$$<x,y> = \int_a^b x(t)\overline{y(t)}\, dt$$

Hence $L^2[a, b]$ is an I P S but not a Hilbert space .

Example 3. : ℓ^2 is a Hilbert space with the inner product define by

$$<x, y> = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

ℓ^2 is already seen to be a Banach space with norm $\|x\| = \left[\sum_{i=1}^{\infty}|x_i|^2\right]^{\frac{1}{2}}$ induced by inner product .

We have seen that in I P S , the parallelogram law

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

holds. Hence , if in a vector space parallelogram law does not hold then it can not be an inner product space. This idea is used to find examples of vector which are not inner product spaces.

Example 4 . The space ℓ^p with $p\neq 2$ is not an inner product space and hence not a Hilbert space.

Proof : It is enough to show that the parallelogram law is not satisfied for $p\neq 2$.

Take $x = \{1, 1, 0, 0, \ldots\} \in \ell^p$ and $y = \{1, -1, 0, 0, \ldots\} \in \ell^p$.

$$\|x\| = 2^{\frac{1}{p}}, \quad \|y\| = 2^{\frac{1}{p}}$$

$$\|x + y\| = \|\{2, 0, 0, \ldots\}\| = 2, \quad \|x - y\| = \|\{0, 2, 0, \ldots\}\| = 2$$

Therefore , $\|x + y\|^2 + \|x - y\|^2 = 8$

But $2\|x\|^2 + 2\|y\|^2 = 2.2^{\frac{2}{p}} + 2.2^{\frac{2}{p}} = 4.2^{\frac{2}{p}} \neq 8$, if $p \neq 2$.

Thus $\|x + y\|^2 + \|x - y\|^2 \neq 2\|x\|^2 + 2\|y\|^2$. So , $\ell^p (p\neq 2)$ is not an I P S and hence not a Hilbert space.

Example 5 . : C [a , b] is not an I P S and hence not a Hilbert space .

For $x \in C[a, b]$, $\|x\| = \sup_{a\leq t\leq b}|x(t)|$.

If $x(t) = t\ \forall\ t \in [a,b]$ then $\|x\| = 1$ and if $y(t) = (t-a)/(t-b)$,

we have, $\|y\| = \sup\limits_{a \le t \le b} |y(t)| = 1$

$$\|x+y\| = \sup\left|t + \frac{t-a}{t-b}\right| = 2$$

$$\|x-y\| = \sup\limits_{a \le t \le b}\left|t - \frac{t-a}{t-b}\right| = 1$$

and $\|x+y\|^2 + \|x-y\|^2 = 5 \neq 2\|x\|^2 + 2\|y\|^2$. So, C[a , b] is not an I P S.

Theorem : If the norm of a Banach space X satisfies the parallelogram law then the Banach space is a Hilbert space with the inner product defined by

$$4 < x, y > = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Proof : (1) $\quad 4 < x, x > = \|2x\|^2 + i\|x+ix\|^2 - i\|x-ix\|^2$

$$= 4\|x\|^2 + i\,2\|x\|^2 - i\,2\|x\|^2 = 4\|x\|^2$$

So, $4\|x\|^2 = < x, x >$. Hence $< x, x > \geq 0$ and $< x, x > = 0$ if and only if $x = 0$.

(2) $\quad 4\overline{< x, y >} = \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2$

$$= \|y+x\|^2 - \|y-x\|^2 - i\|y-ix\|^2 + i\|y+ix\|^2$$

$$= 4 < y, x >$$

So, $\overline{< x, y >} = < x, y >$

(3) $\quad 4 < x+z, y > = \|x+z+y\|^2 - \|x+z-y\|^2 + i\|x+z+iy\|^2 - i\|x+z-iy\|^2$

$$= \left\|\left(x+\frac{y}{2}\right)+\left(z+\frac{y}{2}\right)\right\|^2 - \left\|\left(x-\frac{y}{2}\right)+\left(z-\frac{y}{2}\right)\right\|^2$$

$$+ i\left\|\left(x+\frac{iy}{2}\right)+\left(z+\frac{iy}{2}\right)\right\|^2 - \left\|\left(x-\frac{iy}{2}\right)+\left(z-\frac{iy}{2}\right)\right\|^2$$

$$= 2\left\|x+\frac{y}{2}\right\|^2 + 2\left\|z+\frac{y}{2}\right\|^2 - \|x-z\|^2 - 2\left\|x-\frac{y}{2}\right\|^2 - 2\left\|z-\frac{y}{2}\right\|^2 + \|x-z\|^2 +$$

$$i2\left\|x+\frac{iy}{2}\right\|^2 + i2\left\|z+\frac{iy}{2}\right\|^2 - i\|x-z\|^2 - i2\left\|x-\frac{iy}{2}\right\|^2 - i2\left\|z-\frac{iy}{2}\right\|^2 + i\|x-z\|^2$$

$$= 8 < x, \frac{y}{2} > + 8 < z, \frac{y}{2} >$$

So, $<x+z, y> = 2<x, \frac{y}{2}> + 2<z, \frac{y}{2}>$

Putting $z = 0, <x, y> = 2<x, \frac{y}{2}>$ [Since from the polarization formula $<0,u> = 0\ \forall u \in X$.]

Hence, $<x+z, y> = <x, y> + <z, y>$ for all x, y, z in H

4. For positive integer n, $<nx, y> = n<x, y>$

It is true for n = 2, since $<2x, y> = <x+x, y> = <x, y> + <x, y> = 2<x, y>$

Suppose $<nx, y> = n<x, y>$ then

$$<(n+1)x, y> = <nx+x, y> = <nx, y> + <x, y>$$
$$= n<x, y> + <x, y>$$
$$= (n+1)<x, y>$$

So, $<nx, y> = n<x, y>$ for all positive integers n. It is also true for n = 0.

From polarization formula,

$$4<-x, y> = \|-x+y\|^2 - \|-x-y\|^2 + i\|-x+iy\|^2 - i\|-x-iy\|^2$$
$$= -\{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\} = -4<x, y>$$

So, $<-x, y> = -<x, y>$

If $n = -m$ is a negative integer then

$$<nx, y> = <-mx, y> = -<mx, y> = -m<x, y> = n<x, y>$$

If $n = \frac{p}{q}$ is a rational number, then

$$p<x, y> = <px, y> = <q.\frac{p}{q}x, y> = q<\frac{p}{q}x, y>$$

so that $<nx, y> = <\frac{p}{q}x, y> = \frac{p}{q}<x, y> = n<x, y>$.

If α be any real then $<\alpha x, y> = <\lim_{n\to\infty} r_n x, y>$ where r_n is rational.

$$= \lim_{n\to\infty} <r_n x, y> = \lim_{n\to\infty} r_n <x, y> = \alpha<x, y>$$

Again from the polarization formula,

$$4<ix, y> = \|ix+y\|^2 - \|ix-y\|^2 + i\|ix+iy\|^2 - i\|ix-iy\|^2$$

$= i\{\|x+y\|^2 - \|x-y\|^2 + i\|x-iy\|^2 - i\|x-iy\|^2\}$

$= i4<x,y>$, So that $<ix,y> = i<x,y>$

Lastly if $n = \alpha + i\beta$ be a complex number, then

$$<nx,y> = <(\alpha+i\beta)x, y> = <\alpha x + i\beta x, y>$$

$$= <\alpha x, y> + <i\beta x, y> = \alpha< x, y> + i<\beta x, y>$$

$$= \alpha< x, y> + i\beta< x, y> = (\alpha+i\beta)< x, y>.$$

Thus the Banach space X satisfying the parallelogram law is an inner product space and norm is induced by the inner product by the relation.

$<x,y> = \|x\|^2$. Hence X is a Hilbert space.

Orthogonal Complements : An element x of a Hilbert space H is said to be **orthogonal** to a point y in H if $<x,y> = 0$. Since $\overline{<x,y>} = <y,x>$, x is orthogonal to y if and only if y is orthogonal to x. Without ambiguity, we can say x and y are othogonal. In symbol, we write $x \perp y$.

Given a non empty subset A of Hilbert space H, we shall write

$A^{\perp} = \{x \in H : <x,y> = 0 \text{ for all } y \in A\}$. The set $A^{\perp}$ is called the orthogonal complement of A in H. Since $<0,x> = 0$, $0 \in A^{\perp}$ for any subset A of H. If $x \in A^{\perp}$, we can also write that $x \perp A$.

Proposition : In a Hilbert Space H, if S, S_1, S_2 are subsets of H, then

1. $\{0\}^{\perp} = H$ 2. $H^{\perp} = \{0\}$ 3. $S \cap S^{\perp} \subseteq \{0\}$

4. $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

5. $S^{\perp}$ is a closed subspace of H. 6. $S \subseteq (S^{\perp})^{\perp} = S^{\perp\perp}$

Proof : 1. By definition $\{0\}^{\perp} \subseteq H$. Also $x \in H$ implies $<x,0> = 0$ so that, $x \in \{0\}^{\perp}$ and hence $H \subseteq \{0\}^{\perp}$. Hence, $\{0\}^{\perp} = H$

2. $x \in H^{\perp} \Rightarrow <x,y> = 0$ for all $y \in H$.

$\Rightarrow <x,x> = 0$ when $y = x$

$\Rightarrow x = 0$

So, $H^{\perp} \subseteq \{0\}$. Also $0 \in H^{\perp}$. So $H^{\perp} = \{0\}$.

3. $x \in S \cap S^{\perp} \Rightarrow x \in S$ and $x \in S^{\perp}$

$\Rightarrow <x,x> = 0$

$\Rightarrow x = 0,$ So, $S \cap S^{\perp} \subseteq \{0\}^{\perp}$.

4. Let $S_1 \subseteq S_2$. $x \in S_2^{\perp} \Rightarrow \langle x, y \rangle = 0$ for all $y \in S_2$

$\Rightarrow \langle x, y \rangle = 0$ for all $y \in S_1$

$\Rightarrow x \in S_1^{\perp}$ Hence, $S_2^{\perp} \subseteq S_1^{\perp}$.

5. $S^{\perp}$ is a closed subspace of H. For $\alpha, \beta \in K$, $x_1, x_2 \in S^{\perp}$ and $y \in S$, we have

$$\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$$

$$= \alpha.0 + \beta.0 = 0,$$

So, $\alpha x_1 + \beta x_2 \in S^{\perp}$. Hence $S^{\perp}$ is a subspace. To show that $S^{\perp}$ is closed, let $\{y_n\} \subset S^{\perp}$ and $y_n \rightarrow y$.

Then for $x \in H$, $\langle y, x \rangle = \langle \lim_{n\rightarrow\infty} y_n, x \rangle = \lim_{n\rightarrow\infty} \langle y_n, x \rangle = 0$

This shows that $y \in S^{\perp}$. Hence $S^{\perp}$ is a closed subspace of H.

6. Let $x \in S$. Then $\langle y, x \rangle = 0$ for $y \in S^{\perp}$.

Then $x \in (S^{\perp})^{\perp} = S^{\perp\perp}$ so that $S \subseteq S^{\perp\perp}$.

Proposition : In an inner product space, the pythagoras theorem holds. If $\{x_1, x_2, \cdots, x_n\}$ be an orthogonal set, that is, $\langle x_i, x_j \rangle = 0$ for $i \neq j$ then

$$\| x_1 + x_2 + \ldots + x_n \|^2 = \| x_1 \|^2 + \| x_2 \|^2 + \ldots + \| x_n \|^2$$

Proof : $\| x_1 + x_2 + \ldots + x_n \|^2 = \langle x_1 + x_2 + \ldots + x_n, x_1 + x_2 + \ldots + x_n \rangle$

$$= \sum_{i,j=1}^{n} \langle x_i, x_j \rangle = \sum_{i=1}^{n} \langle x_i, x_i \rangle \text{ (Since } \langle x_i, x_j \rangle = 0 \text{ for } i \neq j)$$

$$= \| x_1 \|^2 + \| x_2 \|^2 + \ldots + \| x_n \|^2$$

We recall the definition of convex subset.

Definition : A subset C of a vector space X (K) is called a **convex subset** of X if $x, y \in C$ and $0 \leq t \leq 1$ implies $tx + (1 - t)y \in C$.

Proposition : A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Proof : Let $d = \inf\{\|x\| : x \in C\}$. For $n \in N$, $d + \frac{1}{n}$ is not a lower bound of $\{\|x\| : x \in C\}$.

So, there exists $x_n \in C$ such that $\|x_n\| > d + \frac{1}{n}$. Taking $n = 1, 2, 3, \ldots$ there exists a sequence $\{x_n\}$ in C s.t. $\lim_{n\to\alpha}\|x_n\| = d$ By the convexity of C, $\frac{x_m + x_n}{2} \in C$.

So $\left\|\frac{x_m + x_n}{2}\right\| \geq d$ and $\|x_n + x_m\| \geq 2d$. Using the parallelogram law ,

$$\|x_n - x_m\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_n + x_m\|^2$$

$$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0 \qquad \text{as } m, n \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in C. Since C is is complete there exists, $x \in C$ such that $x_n \to x$.

It is clear that $\|x\| = \left\|\lim_{n\to\infty} x_n\right\| = \lim_{n\to\infty}\|x_n\| = d$. This shows that x is the vector in C with the smallest norm.

Uniqueness : To show that x is unique , suppose that x^1 is a vector in C other than x which also has norm d. Then $\frac{(x + x^1)}{2}$ is in C. Applying parallelogram law , we obtain ,

$$\left\|\frac{x + x^1}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{x^1}{2}\right\|^2 - 2\left\|\frac{x - x^1}{2}\right\|^2 < \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^1\|^2 = d^2$$

So , $d < \left\|\frac{x + x^1}{2}\right\|^2$ which is contrary to the definition of d . So x is unique and this completes the proof of the result .

Proposition : Let M be a closed linnear subspace of a Hilbert space H , let x be a vector not in M , and let d be the distance from x to M .Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

Proof : Clearly $C = x + M$ is a closed and convex set and

$d = d(x, M) = \inf\{\|x - m\| : m \in M\} = \inf\{\|x + m\| : m \in M\}$ $(\because M = -M) = d(0, x + M)$

So, there exists a unique vector z_0 in C such that $\|z_0\| = d$.

Then $y_0 = x - z_0 \in M$ and $\|x - y_0\| = \|z_0\| = d$. Uniqueness of y_0 follows from the uniqueness of z_0. If y_1 is a vector in M such that $z_1 = x - y_1$, is a vector in C such that $z_1 \neq z_0$ and $\|z_1\| = d$,which contradicts uniqueness of z_0.

Proposition : If M is a proper closed linear subspace of a Hilbert space H ,then there exists a non – zero vector z_0 is H such that $z_0 \perp M$.

Proof : Let x be a vector not M , let d be the distance from x to M .So there exists a vector y_0 in M such that $\|x - y_0\| = d$. Put $z_0 = x - y_0$. z_0 is a non–zero vector since $d > 0$. We complete the proof by showing that if y is an arbitrary vector in M , then $z_0 \perp M$. For any scalar α, we have

$$\|z_0 - \alpha y\| = \|x - (y_0 + \alpha y)\| \geq d = \|z_0\|. \quad \text{So, } \|z_0 - \alpha y\|^2 - \|z_0\| \geq 0.$$

and $-\bar{\alpha} < z_0, y > -\alpha \overline{< z_0, y >} + |\alpha|^2 \| y \|^2 \geq 0.$

Putting $\alpha = \beta < z_0, y>$ for an arbitrary real number β , then it becomes

$$2\beta|<z_0, y>|^2 + \beta^2|<z_0, y>|^2 \ \|y\|^2 \geq 0.$$

If we put $a = |<z_0, y>|^2$ and $b = \|y\|^2$, we obtain $-2\beta a + \beta^2 ab \geq 0$ or $\beta a (\beta b - 2) \geq 0$ for all real β .

However if $a > 0$, then $\beta a (\beta b - 2) \geq 0$ is false for all sufficiently small positive β . This shows that $a = 0$ which means that $z_0 \perp y$.

Definition : Two non empty subsets S_1 and S_2 of a Hilbert space H are said to be orthogonal and written as $S_1 \perp S_2$ if $x \perp y$ for x in S_1 and y in S_2 .

Proposition : If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace M + N is also closed .

Proof : Let u be a limit point of M + N . There exists a sequence $\{u_n\}$ in M+N such that $\lim_{n\to\infty} u_n = u$. By the assumption that $M \perp N$, $M \cap N = \{0\}$.

So, u_n can be uniquely expressed as $x_n + y_n$, $x_n \in M$ and $y_n \in N$. By pythagorean theorem

$$\|u_m - u_n\|^2 = \|x_m + y_m - x_n - y_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2.$$

So, $\{x_n\}$and $\{y_n\}$ are cauchy sequences in M and N . M and N are being closed subspaces are complete . So , there exists vectors x and y in M and N such that $x_n \to x$ and $y_n \to y$.

Finally ,$u = \lim_{n\to\infty} u_n = \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = x + y \in M + N$.

This shows that every limit point of M + N is in M + N and hence M + N is closed.

Proposition : If M is a closed subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$.

Proof : M and $M^{\perp}$ are closed subspaces of H and $M \perp M^{\perp}$. Hence $M + M^{\perp}$ is a closd subspace of H . Since $M \cap M^{\perp} = \{0\}$, it is enough to show that $H=M+M^{\perp}$. If possible $M + M^{\perp}$ be a proper closed subspace of H . Then there exists a non zero z_0 in H such that $z_0 \perp (M + M^{\perp})$. Clearly $z_0 \perp M$ and $z_0 \perp M^{\perp}$. So , $z_0 \in M^{\perp} \cap M^{\perp\perp}$ and this is impossible . Hence $H = M + M^{\perp}$ and $M \cap M^{\perp} = \{0\}$. This proves that $H = M \oplus M^{\perp}$.

Proposition : If S is a non empty subset of Hilbert space then $S^\perp = S^{\perp\perp\perp}$

Proof : We have $S \subseteq S^{\perp\perp}$. Replacing S by $S^\perp$, we have $S^\perp \subseteq S^{\perp\perp\perp}$.

Also, we have, $S_1 \subseteq S_2 \Rightarrow S_2^\perp \subseteq S_1^\perp$. So, $S \subseteq S^{\perp\perp} \Rightarrow S^{\perp\perp\perp} \subseteq S^\perp$. It follows that

$S^\perp = S^{\perp\perp\perp}$.

Proposition : M is a linear subspace of a Hilbert space H .Then M is closed if and only if

$M = M^{\perp\perp}$.

Proof : We know that $S^\perp$ is a closed linear subspace . So, $M = (M^\perp)^\perp$ is a closed linear subspace . Conversely , let M be a closed linear subspace . We have to show that $M = M^{\perp\perp}$.

For any set S in H , we have $S \subseteq S^{\perp\perp}$. So , $M \subseteq M^{\perp\perp}$ and $M^{\perp\perp}$ is a Hilbert space . If possible let M be a proper closed linear subspace of the Hilbert space $M^{\perp\perp}$. So, there exists a non zero vector z_0 in $M^{\perp\perp}$ s. t $z_0 \perp M$. Then $z_0 \in M^\perp \cap M^{\perp\perp} = \{0\}$. This shows that $z_0 = 0$, a contradiction . So, $M = M^{\perp\perp}$.

Orthonormal Set :

Definition : A non empty subset A of a Hilbert space H is said to be orthonormal if and only if

a. $<x , y> = 0$ for all $x , y \in A$ with $x \neq y$, and

b. $<x , x> = 1$ for all $x \in A$.

Proposition : Each orthonormal subset of Hilbert space H is linearly independent.

Proof : Let A be an orthonormal subset of H and $\{x_1, x_2, \ldots, x_n\}$ be a finite subset of a A . If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are element of K with

$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = 0$ then for m = 1 , 2 ,..., n , we have

$$0 = <\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n , x_m> = \lambda_1 < x_1, x_m > + \lambda_2 < x_2, x_m > + \ldots\ldots + \lambda_n < x_n, x_m > = \lambda_m$$

This shows that A is a linear independent.

Theorem : (Gram – Schmidt Orthonormalization)

Let $\{x_1, x_2, \ldots\}$ be a linear independent subset of an inner product space X . Define $y_1 = x_1$.

$u_1 = \dfrac{y_1}{\| y_1 \|}$ and for n = 2 , 3 ....

$$y_n = x_n - <x_n, u_1> u_1 - \ldots\ldots - <x_n, u_{n-1}> u_{n-1},$$

$$u_n = \frac{y_n}{\| y_n \|}.$$

Then $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and for n = 1 , 2 ,.......

$$\text{Span }\{u_1, u_2, \ldots u_n\} = \text{Span }\{x_1, x_2, x_3, \ldots x_n\}.$$

Proof : We prove it by the method of induction .

As $\{x_1\}$ is a linearly independent set , $y_1 = x_1 \neq 0$,

$$\|x_1\| = \frac{\|y_1\|}{\|y_1\|} = 1 \text{ and span } \{u_1\} = \text{span } \{x_1\}.$$

For $n \geq 1$, assume that we have defined y_n and u_n as given such that $\{u_1, \ldots u_n\}$ is an orthonormal set satisfying

$$\text{span }\{u_1, \ldots, u_n\} = \text{span }\{x_1, \ldots\ldots, x_n\}.$$

Define $y_{n+1} = x_{n+1} - \langle x_{n+1}, u_1\rangle u_1 - \ldots - \langle x_{n+1}, u_n\rangle u_n$.

As $\{x_1, \ldots, x_{n+1}\}$ is a linearly independent set, x_{n+1} does not belong to span $\{u_1, \ldots, u_n\}$= span $\{x_1, \ldots, x_n\}$. Hence $y_{n+1} \neq 0$. Let $u_{n+1} = \frac{y_{n+1}}{\|y_{n+1}\|}$. Then $\|u_{n+1}\| = 1$ and for all $j \leq n$, we have

$$\langle y_{n+1}, u_j\rangle = \langle x_{n+1}, u_j\rangle - \sum_{k=1}^{n} \langle x_{n+1}, u_k\rangle\langle u_k, u_j\rangle = \langle x_{n+1}, u_j\rangle - \langle x_{n+1}, u_j\rangle = 0$$

Since $\langle u_k, u_j\rangle = 0$ for $k \neq j$, k = 1 , 2 , ..., n . Thus $\langle u_{n+1}, u_j\rangle = \frac{\langle y_{n+1}, u_j\rangle}{\|y_{n+1}\|} = 0$ for j = 1 , 2 , ..., n as well. Hence $\{u_1, \ldots, u_{n+1}\}$ is a orthonormal set . Also

$$\text{span }\{u_1, \ldots, u_{n+1}\} = \text{span }\{x_1, \ldots, x_n, u_{n+1}\} = \text{span }\{x_1, x_2, \ldots, x_n, x_{n+1}\}.$$

By mathemetical induction , the proof is complete .

The following basic inequality generalizes the schwarz inequality $|\langle x,y\rangle| \leq \|x\|\,\|y\|$

Bessel's Inequality : Let $\{u_1, u_2, \ldots\}$ be a countable othonormal set in an inner product space X and $x \in X$. Then $\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2$ where equality holds if and only if $x = \sum_{n=1}^{\infty} |\langle x, un\rangle|\, un$.

Proof : For m = 1 , 2 , 3 , ... , let $\sum_{n=1}^{m} \langle x, un\rangle un$. Since $\{u1, \ldots, um\}$ is an orthonormal set , we have $\langle x, xm\rangle = \langle x, \sum_{n=1}^{m} \langle x, un\rangle un\rangle = \sum_{n=1}^{m} \overline{\langle x, u_n\rangle} \langle x, u_n\rangle = \sum_{n=1}^{m} |\langle x, u_n\rangle|^2$

Similarly, $< x , x_m > = < x_m, x > = < x_m, x_m > = \sum_{n=1}^{m} |< x,u_n >|^2$

Hence, $0 \le \|x - x_m\|^2 = < x - x_m, x - x_m > = < x , x > - < x , x_m > - < x_m, x > + < x_m, x_m >$

$$= \|x\|^2 - \sum_{n=1}^{m} |< x,u_n >|^2 - \sum_{n=1}^{m} |< x,u_n >|^2 + \sum_{n=1}^{m} |< x,u_n >|^2$$

$$= \|x\|^2 - \sum_{n=1}^{m} |< x,u_n >|^2 \text{ . So, } \sum_{n=1}^{m} |< x,u_n >|^2 \le \|x\|^2$$

Taking limit as $m \to \infty$ $\quad \sum_{n=1}^{\infty} |< x,u_n >|^2 \le \|x\|^2$

If equality holds, $\lim_{m\to\infty} \sum_{n=1}^{m} |< x,u_n >|^2 = \|x\|^2$ and from

$$\|x - x_m\|^2 = \|x\|^2 - \sum_{n=1}^{m} |< x,u_n >|^2 \text{ , we have } = 0 \text{ that is,}$$

$$x = \lim_{m\to\infty} x_m = \lim_{m\to\infty} \sum_{n=1}^{m} < x,u_n > u_n = \sum_{n=1}^{\infty} < x,u_n > u_n .$$

Conversely, suppose, $\quad x = \sum_{n=1}^{\infty} < x,u_n > u_n$. Then

$$\|x\|^2 = < x , x > = < \sum_{n=1}^{\infty} < x,u_n > u_n , \sum_{n=1}^{\infty} < x,u_n > u_n > = \sum_{n=1}^{\infty} |< x,u_n >|^2$$

by the continuity, the linearity in first varible and conjugate linearity is second variable.

Proposition :Let X be a inner product space, $\{u_1, u_2, u_3, \ldots\}$ be a countable orthnormal set in X and $k_1, k_2, \ldots$ belong to K.

a. If $\sum_{n=1}^{\infty} k_n u_n$ converges to x in X then $\{k_n\} \in \ell^2$

b. If X is a Hilbert space and $\{k_n\} \in \ell^2$ then $\sum_{n=1}^{\infty} k_n u_n$ converges in X. **[Riesz-Firsher Theorem]**

Proof : a. Suppose $\sum_{n=1}^{\infty} k_n u_n = x$. Then

$$< x, u_n > = < \sum_{n=1}^{\infty} k_n u_n, u_n > = \lim_{m\to\infty} < \sum_{i=1}^{m} k_i u_i, u_n > = \lim_{m\to\infty} k_i = k_i$$

and $x = \sum_{n=1}^{\infty} < x, u_n > u_n$. So, by the preceding proposition,

$$\sum_{i=1}^{\infty} |k_i|^2 = \sum_{i=1}^{\infty} |< x, u_i >|^2 = \|x\|^2 \text{ so that } \{k_n\} \in \ell^2 .$$

b. Put $x_m = \sum_{n=1}^{m} k_n u_n$. Then $x_m - x_\ell = \sum_{n=\ell+1}^{m} k_n u_n$ and

$$\|x_m - x_\ell\|^2 = < \sum_{n=\ell+1}^{m} k_n u_n, \sum_{n=\ell+1}^{m} k_n u_n > = \sum_{n=\ell+1}^{m} |k_n|^2$$

Since $\{k_n\} \in \ell^2$, $\sum_{n=\ell+1}^{m} |k_n|^2 \to 0$ as $m, \ell \to \infty$. This shows that $\{x_n\}$ is a Cauchy sequence in the Hilbert space H .Since H is complete

$\{x_m\} = \left\{\sum_{n=1}^{m} k_n u_n\right\}$ converges in H . This proves that $\sum_{n=1}^{\infty} k_n u_n$ is a convergent series.

Proposition : Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Then $E_x = \{u_\alpha : <x, u_\alpha> \neq 0\}$ is a countable set . If E_x is denumerable say $\{u_n\}$ then $< x , u_n > \to 0$ as $n \to \infty$.

If H is a Hilbert space , then $\sum_n < x, u_n > u_n$ converges in X to some y such that $x - y \perp u_\alpha$ for every α.

Proof : If $x = 0$, $E_x = \phi$ which is countable . Suppose $x \neq 0$.

For $j = 1 ; 2 , \ldots\ldots.$, let $E_j = \{u_\alpha : \|x\| \leq j\ |<x, u_\alpha>|\}$.

For fixed j ,suppose Ej contains distinct elements $u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_m}$. Then

$$0 < m\|x\|^2 \leq j^2 \sum_{n=1}^{m} |< x, u_{\alpha_n} >|^2 \leq j^2 \|x\|^2 \text{, by Bessel's inequality.}$$

So, $m \leq j^2$. This shows that E_j contains atmost j^2 elements. Since $E_x = \bigcup_{j=1}^{\infty} E_j$, we see that E_x is a cuontable set .

Second part : Suppose $E_x = \{u_1, u_2, \ldots\}$ is denumerable .Then $\sum_{n=1}^{\infty} |< x, u_n >|^2 \le \|x\|^2 < \infty$.

From the convergence of the series , we obtain that $\lim_{n\to\infty} < x, u_n > = 0$.

Third part : Since $\{<x, u_n>\} \in \ell^2$ and the space is a Hilbert space , by Reisz – Fisher Theorem $\sum_{n=1}^{\infty} < x, u_n > u_n$ converges to some y in X and

$$<y, u_\alpha> = <\sum_{n=1}^{\infty} < x, u_n > u_n , u_\alpha> = <x, u_\alpha> \text{ so that}$$

$<x-y, u_\alpha> = 0$ for every α.This proves that $x - y \perp u_\alpha$ for every α.

Definition : An orthonormal set $\{u_\alpha\}$ in a Hilbert space H is said to be complete orthonormal set or orthonormal basis if it is maximal in the sense that if $\{u_\alpha\}$ is contained in some orthonormal subset E of H , then $E = \{u_\alpha\}$.

Proposition : Every non–zero Hilbert space H contains a complete orthonormal set.

Proof : If $H \neq \{ 0 \}$ then H has a non – zero element x and $\left\{\frac{x}{\|x\|}\right\}$ is an orthonormal set in H .

Let P be the set of all orthonormal sets in H containing the orthonormal set $\left\{\frac{x}{\|x\|}\right\}$. Then ($P ,\subseteq$) is a partially ordered set and if C is any chain in P then the union of its members is a upper bound of the chain .By Zorn's lemma P has a maximal element which is the complete orthonormal set.

Proposition : Let $\{u_\alpha\}$ be an orthonormal set is a hilbert space H .Then the following conditions are equivalent:

1. $\{u_\alpha\}$ is an orthonormal basis for H
2. For every $x \in H$, $x = \sum_n < x, u_n > u_n$ [Fourier expansion]
3. For every $x \in H$, $\|x\|^2 = \sum_n |< x, u_n >|^2$ [Parseval Formula]
4. Span $\{u_\alpha\}$ is dense in H
5. If $x \in H$ and $<x,u_\alpha> = 0$ for all α , then $x = 0$.

Here , $\{u_n\} = \{u_\alpha : <x, u_\alpha> \neq 0\}$.

Proof : (1) $\Rightarrow$ (2): Let $\{u_\alpha\}$ be a maximal orthonormal set in H .If $x \in H$, then

$$\sum_n < x,u_n > u_n = y \text{ for some } y \text{ in } H \text{ with } x - y \perp u_\alpha .$$

If $y \neq x$, $u = \frac{y-x}{\| y-x \|}$ is unit vector since $\|u\| = 1$ and also, $u \perp \{u_\alpha\}$, so that $\{u_\alpha\} \cup \{u\}$ is an orthonormal set in H ,contradicting the maximality of $\{u_\alpha\}$. Hence $y = x$ and $\sum_n < x,u_n > u_n = x$.

(2) $\Leftrightarrow$ (3) : $\sum_n < x,u_n > u_n \Leftrightarrow$ the equality of the Bessel's inequality, that is, $\|x\|^2 = \sum_n |< x,u_n >|^2$

(2) $\Rightarrow$ (4) : Suppose $x = \sum_n < x,u_n > u_n = \lim_{m\to\infty} \sum_{n=1}^{m} < x,u_n > u_n$

$$= \lim_{m\to\infty} v_m \quad \text{where } v_m = \sum_{n=1}^{m} < x,u_n > u_n \in \text{span}\,\{u_1, u_2, u_3, \ldots\}.$$

This proves (4).

(4) $\Rightarrow$ (5) : Suppose $\overline{\text{span}\{u_\alpha\}} = H$ and $< x,u_\alpha >= 0$ for all α

$$x \in H \Rightarrow x = \lim_{m\to\infty} x_m \text{ where } x_m \in \text{span}\{u_\alpha\}$$

$$< x , u_\alpha > = 0 \Rightarrow < x , x_m > = 0$$

$$\Rightarrow \lim_{m\to\infty} < x,x_m >= 0 \Rightarrow < x , x > = 0 \Rightarrow x = 0$$

(5) $\Rightarrow$ (1) : Suppose $< x , u_\alpha > = 0$ for all $\alpha \Rightarrow x = 0$.

If $\{u_\alpha\}$ be contained in an orthonormal set E then there exists $u \in E$,

$u \neq u_\alpha$ and $< x , u_\alpha > = 0$ for all α. Then by (5) , $u = 0$ and hence $\|u\| = 0$. This is not possible since u is an unit vector .So, $\{u_\alpha\}$ is not contained in any orthonormal set .This proves that $\{u_\alpha\}$c is a maximal orthonormal set.

Proposition : Let H be a Hilbert space . Then

a. If H is a separable , then H has a countable complete orthonormal set.

b. If H has a countable complete orthonormal set then H is separable .

c. If H has a countable orthonormal basis then H is linearly isometic to k^n for some n or to ℓ^2.

Proof : a. Let H be a separable space and $H \neq \{0\}$, then H has an orthonormal set M . To show M is countable let B be any dence subset of H .If possible let M be uncountable .Any two distinct elements x and y of M have distance $\sqrt{2}$ since $\|x - y\|^2 = <x - y, x - y> = <x, x> + <y, y> = 2$.

The spherical balls N_x of x and N_y of y of radius $\frac{\sqrt{2}}{3}$ are disjoint .Since B is dence in H , there is a $b \in B$ in N_x and a $\bar{b} \in B$ in N_y and $b \neq \bar{b}$ since $N_x \cap N_y = \phi$.

Hence if M were uncountable , we would have uncountably many such pairwise disjoint spherical neighbourhoods (for each $x \in M$ one of them), so that B would be uncountable .Thus B is any dense set which is not countable, contradicting separability. So, we conclude that M must be countable. This proves that H has a conntable orthonormal set and hence it has a complete orthonormal set.

b. Let $\{e_k\}$ be a complete orthonormal sequence in H and A the set of all linear combinations $c_{r_1}e_1 + c_{r_2}e_2 + ... + c_{r_n}e_n$ $(n = 1, 2, ...)$

Where $c_{r_j} = a_{r_j} + ib_{r_j}$ with $a_{r_j} \in Q$ and $b_{r_j} \in Q$. Clearly A is countable. We prove that A is dense in H by showing that for $x \in H$ and $\in > 0$ there is a $v \in A$ such that $\|x - v\| < \varepsilon$.We have $\overline{\text{span}\{e_n\}} = H$. So for $\in > 0$ there is an y in span $\{e_n\}$such that $\|x - y\| < \varepsilon/2$.

$$\text{Let} \quad y = \sum_{i=1}^{n} a_i e_i \ \text{ then } <y,e_j> = \left\langle \sum_{i=1}^{n} a_i e_i, e_j \right\rangle = a_i.$$

$$\text{So,} \quad y = \sum_{j=1}^{n} < y,e_j > e_j$$

It can be chosen that $y = \sum_{j=1}^{n} < x,e_j > e_j$ for which

$$z = x - y \perp y. \text{ Hence , we have} \left\| x - \sum_{k=1}^{n} < x,e_k > e_k \right\| < \frac{\varepsilon}{2}.$$

Since the rationals are dense on R , for each $<x,e_k>$ there is a c_{k_n} (with rational real and imaginary parts) such that $\left\| \sum_{k=1}^{n} [< x,e_k > - c_{k_n}] e_k \right\| < \frac{\varepsilon}{2}$

Hence $v \in A$ defined by $v = \sum_{k=1}^{n} c_{k_n} e_k$ satisfies

$$\| x - v \| = \left\| x - \sum_{k=1}^{n} c_{k_n} e_k \right\| \leq \left\| x - \sum_{k=1}^{n} < x,e_k > e_k \right\| + \left\| \sum_{k=1}^{n} < x,e_k > e_k - \sum_{k=1}^{n} < c_{k_n} e_k > \right\|.$$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

This proves that A is dense in H, and since A is countable, H is separable.

c. Let $\{e_1, e_2, \ldots\}$ be countable complete orthonormal set.

For $x \in H$, let

$F(x) = \{<x, e_i>\}$, $\{e_i\}$ is either a finite set or a denumerable set. Accordingly F is maps H into K^n or ℓ^2 since by parseval's formula

$$\| F(x) \|_2^2 = \sum_{n=1}^{\infty} |<x, e_n>|^2 = \| x \|^2 < \infty$$

F is clearly linear and isometric.

$$F(x) = F(y) \Rightarrow <x, e_i> = <y, e_i>$$

$$\Rightarrow <x - y, e_i> = 0 \text{ for all } e_i. \Rightarrow x - y = 0 \Rightarrow x = y$$

and $\{\alpha_i\} \in \ell^2 \Rightarrow \sum_{i=1}^{\infty} \alpha_i e_i$ converges to some $x \in H$ and $\alpha_i = <x, \alpha_i>$

So, $F(x) = \{<x, e_i>\}$. Hence F is onto. Thus H is linearly isometric to K^n or to ℓ^2.

Representation of Functionals on Hilbert Spaces :

Proposition : If z is any fixed element of an inner product space X, then $f(x) = <x, z>$ defines a bounded linear functional f on X, of norm $\|z\|$.

Proof : The inner product space X is a normal space with $\|x\|^2 = <x, x>$.

$f(\alpha x_1 + \beta x_2) = \alpha f(x_1, z) + \beta f(x_2, z)$ for all $\alpha, \beta \in K$ and x_1, x_2 in X.

This shows that f is linear. And $|f(x)| = |<x, z>| \leq \|z\| \|x\|$ (Scharz Inequality)

shows that f is bounded. Hence f is a bounded linear mapping and

$$\sup_{x \neq 0} \frac{|f(x)|}{\| x \|} \leq \| z \| \text{ shows that } \| f \| \leq \| z \|.$$

If $z \neq 0$, $\| f \| = \sup \{|f(x)| : \| x \| = 1\}$

$$\geq \left| f\left(\frac{z}{\| z \|}\right) \right| = \left| \left\langle \frac{z}{\| z \|}, z \right\rangle \right| = \frac{1}{\| z \|} <z, z> = \| z \|$$

So, $\| f \| = \| z \|$.

We now establish the converse of it , known as Reisz's representation Theorem for functionals on Hilbert space .

Theorem : Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product , namely ,

$$f(x) = \langle x, z \rangle$$

where z depends on f , is uniquely determined by f and has norm $\| z \| = \| f \|$

Proof :a. f has a representation $f(x) = \langle x, z \rangle$:

If f = 0 then $f(x) = \langle x, 0 \rangle$ for all $x \in H$ and $\| z \| = \| 0 \| = 0$. Suppose $f \neq 0$. The null space N (f) of f is a proper closed linear subspace of H. So, there exists a non – zero vector y_0 which is orthogonal to N (f). We can show that $z = \alpha y_0$ is the required vector for some suitable α.

For any $x \in N(f)$, $f(x) = 0 = \langle x, \alpha y_0 \rangle$ for any α since $y_0 \perp N(f)$.

Next for $x = y_0$, $f(y_0) = \langle y_0, \alpha y_0 \rangle = \bar{\alpha} \| y_0 \|^2$ if $\alpha = \dfrac{\overline{f(y_0)}}{\| y_0 \|^2}$.

Finally , we show that $z = \alpha y_0$ with $\alpha = \dfrac{\overline{f(y_0)}}{\| y_0 \|^2}$ satisfies

$$f(x) = \langle x, z \rangle \text{ for all } x \in H.$$

Each x in H can be written in the form $x = m + \beta y_0$ where $m \in N(f)$. Then $0 = f(m) = f(x - \beta y_0) = f(x) - \beta f(y_0)$ so that $\beta = \dfrac{f(x)}{f(y_0)}$. Then

$$f(x) = f\left(m + \frac{f(x)}{f(y_0)} y_0 \right) = f(m) + \frac{f(x)}{f(y_0)} f(y_0)$$

$$= \langle m, \alpha y_0 \rangle + \frac{f(x)}{f(y_0)} \langle y_0, \alpha y_0 \rangle = \langle m + \frac{f(x)}{f(y_0)} y_0, \alpha y_0 \rangle$$

$$= \langle x, \alpha y_0 \rangle = \langle x, z \rangle \quad \text{where } z = \alpha y_0 \text{ and } \alpha = \frac{\overline{f(y_0)}}{\| y_0 \|^2}$$

b. Representation is unique :

Let z_1 and z_2 be two vector such that

$f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle$. Then $\langle x, z_1 - z_2 \rangle = 0$ for $x \in H$. So , $z_1 - z_2 = 0$ and $z_1 = z_2$. This shows that the representation is unique .

c. $\| f \| = \| z \|$ follows from the preceding results .

Sesquilinear and Bilinear form :

Let X and Y be vector spaces over the same field K . A mapping $h : X \times Y \to K$ is called a sesquilinear form if h is linear in the first variable but conjugete linear in the second variable. Explicitely, for all $x_1, x_2, x \in X$; $y_1, y_2, y \in Y$ and for all scalars α, β

1. $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$
2. $h(\alpha x, y) = \alpha h(x, y)$
3. $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$
4. $h(x, \beta y) = \bar{\beta} h(x, y)$

A Sesquilinear form h is called a bilinear form if

4a. $h(x, \beta y) = \beta h(x, y)$.

If X and Y are normed linear spaces , h is called a bounded functional if there exists $c > 0$ such that $|h(x, y)| \le c\|x\| \, \|y\|$ for all (x, y) in $X \times Y$.

For example , the inner product is sesquilinear and bounded . The number

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x,y)|$$ is called the norm of h .

Theorem (Reisz representation) :

Let H_1, H_2 be Hilbert spaces and $h : H_1 \times H_2 \to K$ a bounded sesquilinear form .Then h has representation

$$h(x, y) = \langle Sx, y \rangle$$

Where $S : H_1 \to H_2$ is a bounded linear operator . S is uniquely determined by h and norm $\|S\| = \|h\|$

Proof : $f(y) = \overline{h(x,y)}$ is a bounded linear form in Y . So , there exists unique z in H_1, such that $f(y) = \overline{h(x,y)} = \langle y, z \rangle$.

Hence , $h(x, y) = \langle z, y \rangle$. $z \in H_1$ is unique but depends on $x \in H_1$. Define $S : H_1 \to H_2$ by $S(x) = z$.

So , $h(x, y) = \langle z, y \rangle = \langle Sx, y \rangle$

S is linear : $\langle S(\alpha x_1 + \beta x_2), y \rangle = h(\alpha x_1 + \beta x_2, y) = \alpha h(x_1, y) + \beta h(x_2, y)$

$= \alpha \langle Sx_1, y \rangle + \beta \langle Sx_2, y \rangle = \langle \alpha Sx_1 + \beta Sx_2, y \rangle$ for all $y \in H_2$.

So , $S(\alpha x_1 + \beta x_2) = \alpha Sx_2 + \beta Sx_2$.

S is bounded :

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|<Sx,y>|}{\|x\|\|y\|} \geq \sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{|<Sx,Sx>|}{\|x\|\|Sx\|} = \sup_{x\neq 0}\frac{\|Sx\|}{\|x\|} = \|S\|$$

This proves the boundedness of h and $\|h\| \geq \|S\|$.

Also , $\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|<Sx,y>|}{\|x\|\|y\|} \leq \sup_{x\neq 0}\frac{|<Sx,y>|}{\|x\|\|y\|} = \|S\|$. Hence , $\|S\| = \|h\|$.

S is unique : Suppose $T : H_1 \to H_2$, such that for all $x \in H_1$, and $y \in H_2$, we have

$$h(x,y) = <Sx,y> = <Tx,y>.$$

This shows that $Sx = Tx$ for all $x \in H_1$ and hence $S = T$.

HILBERT – ADJOINT OPERATOR :

Definition : Let $T : H_1 \to H_2$ be a bounded linear operator , where H_1 and H_2 are Hilbert spaces .Then the Hilbert–adjoint operator T^* of T is the operation

$T^* : H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$, $<Tx, y> = <x, T^*y>$

The following theorm shows that for a given T such a T^* does exist .

Theorem : The Hilbert adjoint T^* of T exists , is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$.

Proof : $h : H_1 \times H_2 \to K$, defined by $h(y,x) = <y, Tx>$ is a sesquilinear form on $H_1 \times H_2$ and it is bounded .

$$h(y x_1 + \beta x_2) = <y, T(\alpha x_1 + \beta x_2)> = <y, \alpha Tx_1 + \beta Tx_2>$$

$$= \bar{\alpha}<y, Tx_1> + \bar{\beta}<y, Tx_2> = \bar{\alpha}h(y, x_1) + \bar{\beta}(y, x_2)$$

So, h is conjugate linear in second variable and it is clearly linear in first variable.

Also by schwarz inequality

$$|h(y,x)| = |<y, Tx>| \leq \|y\| \ \|Tx\| \leq \|y\| \ \|T\| \ \|x\|$$

This shows that $\|h\| \leq \|T\|$. Also $\|h\| \geq \|T\|$ follows from

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|<y,Tx>|}{\|y\|\|x\|} \geq \sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{|<Tx,Tx>|}{\|Tx\|\|x\|} = \|T\|$$ Together , $\|h\| = \|T\|$.

The representation theorem gives that there exists a bounded linear operator

$S : H_2 \rightarrow H_1$, uniquely determined such that

$$h(y,x) = < Sy, x >. \text{ Writing } T^* \text{ for } S,$$

we have $h(y', x) = < T^* y, x >$, $T^* : H_2 \rightarrow H_1$ is uniquely determined bounded linear operator such that

$$\|T^*\| = \|S\| = \|h\| = \|T\|$$

Also $< y, Tx > = h(y,x) = < T^* y, x>$

So, $< Tx, y > = < x, T^* y >$ taking conjugate. Thus T^* exists with $\|T^*\| = \|T\|$.

Properties of Hilbert – adjoint operators :

Proposition:

Let H_1, H_2 be Hilbert spaces, $S : H_1 \rightarrow H_2$ and $T : H_1 \rightarrow H_2$ bounded linear operators and α any scalar. Then we have

1. $< T^* y, x > = < y, Tx >$
2. $(S+T)^* = S^* + T^*$
3. $(\alpha T)^* = \bar{\alpha} T^*$
4. $(T^*)^* = T$
5. $\|T^* T\| = \|TT^*\| = \|T\|^2$
6. $T^* T = 0 \Leftrightarrow T = 0$
7. $(ST)^* = T^* S^*$ (assuming $H_2 = H_1$)

Proof : 1. From definition of T^*,

$$< Tx, y > = < x, T^* y >$$

$$\Rightarrow \overline{< Tx, y >} = \overline{< x, T^* y >}$$

$$\Rightarrow < y, Tx > = < T^* y, x >.$$

2. $< x, (S+T)^* y > = < (S+T)x, y >$ [Definition of T^*]

$$= < Sx, y > + < Tx, y >$$

$$= < x, S^* y > + < x, T^* y > = < x, (S^* y + T^* y) >$$

$$= < x, (S^* + T^*) y >$$

for all x and y. Hence $(S+T)^* y = (S^* + T^*) y$ for all y.

3. $<(\alpha T)^* y, x> = <y, (\alpha T)x> = <y, \alpha(Tx)> = \bar{\alpha}<y, Tx>$

$$= \bar{\alpha}<T^*y, x> = <(\bar{\alpha}T^*)y, x>$$

for all x and y. So, $(\alpha T)^* = \bar{\alpha} T^*$.

4. $<(T^*)^*x, y> = <x, T^* y> = <Tx, y>$

So, $<(T^{**} - T)x, y> = 0$ for all x and y.

Hence, $T^{**} - T = 0$ or $T^{**} = T$.

5. We see that $T^*T : H_1 \to H_1$ and $TT^* : H_2 \to H_2$. By the Schwarz inequality,

$$\|Tx\|^2 = <Tx, Tx> = <T^*Tx, x> \leq \|T^*Tx\| \ \|x\| \leq \|T^*T\| \ \|x\|^2$$

So, $\sup_{\|x\|=1} \|Tx\|^2 \leq \|T^*T\|$. This implies $\|T^*\| \leq \|T^*T\|$.

Further, $\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \ \|T\| = \|T\|^2$ (Since $\|T^*\| = \|T\|$)

This shows that $\|T^* T\| = \|T\|^2$. Replacing T by T^*, we have

$$\|T^{**} T^*\| = \|T^*\|^2 \qquad \text{or} \qquad \|T T^*\| = \|T\|^2$$

6. From 5, $T^*T = 0 \Leftrightarrow T = 0$.

7. $<x, (ST)^* y> = <(ST)x, y> = <Tx, S^* y>$

$$= <x, T^* S^* y> \text{ for all x and y.}$$

This implies that $(ST)^* = T^* S^*$.

Exercise :

1. Show that $0^* = 0$ and $I^* = I$.

2. Let $T: H \to H$ be a bijective, bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$.

3. If $\{T_n\}$ is a sequence of bounded linear operators on a complex Hilbert space H and $T_n \to T$. Show that $T_n^* \to T^*$.

4. Let T_1 and T_2 be bounded linear operators on a complete Hilbert space H into itself. If $<T_1 x, x> = <T_2 x, x>$ for all $x \in H$. Show that $T_1 = T_2$.

Self – Adjoint, Unitary and Normal Operators :

Definition : A bounded linear operator $T: H \to H$ on a Hilbert space H is said to be

1. Self adjoint if $T^* = T$, that is, $<Tx, y> = <x, Ty>$.

2. Unitary if T is bijective and $T^* = T^{-1}$.

3. Normal if $T T^* = T^* T$.

Note : If an operator is self adjoint or unitary then the operator is normal .But the converse is not true . For example 2 i I is normal , but it is neither self adjoint nor unitary .

Theorem : Let $T:H \to H$ be a bounded linear operator on a Hilbert space H Then

a. If T is self adjoint , $< Tx , x >$ is a real for all $x \in H$.

b. If H is complete and $< Tx , x >$ is a real for all $x \in H$, the

operator T is self – adjoint .

Proof :a. If T is self adjoint , then for all x ,

$$\overline{< Tx, x >} = < x , Tx > = < Tx , x >.$$

Hence $< Tx , x >$ is equal to its complex conjugate , So that it is real .

b. If $< Tx , x >$ is a real for all x , then

$$< Tx , x > = \overline{< Tx, x >} = \overline{< x, T^* x >} = < T^* x , x >.$$

Hence , $< (T - T^*) x , x > = 0$ or $T = T^*$ since H is complex.

Theorem : The product of two bounded self adjoint linear operators S and T on a Hilbert space H is self adjoint if and only if the operators commute,

$$S T = T S .$$

Proof : For self adjoint operators S and T ,

$$(S T)^* = T^* S^* = T S .$$

Hence , S T is self adjoint or $(S T)^* = T S$. iff $T S = S T$.

Theorem : Let $\{ T_n \}$ be a sequence of bounded self adjoint linear operator $T_n : H \to H$ **on a Hilbert space H** .Suppose that $\{ T_n \}$ converges , say ,

$T_n \to T$ .Then T is bounded self adjoint linear operator on H .

Proof : $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$. By using triangle ineqality in B(H,H),

$$\|T - T^*\| \le \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\| = 2\|T - T_n\| \to 0 \text{ as } n \to \infty$$

Hence , $\|T - T^*\| = 0$ and $T = T^*$. This shows that T is self adjoint .

Theorem : Let the operators $U : H \to H$ and $V : H \to H$ be unitary . Then

a. U is isometric

b. $\|U\| = 1$, provided $H \neq \{0\}$

c. U^{-1} ($= U^*$) is unitary

d. U V is unitary

e. U is normal

Proof : a. U is isometry follows from

$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, U^* Ux \rangle = \langle x, I x \rangle = \|x\|^2$.

b. From (a), $\sup_{x\neq 1} \frac{\|Ux\|^2}{\|x\|^2} = 1$. So, $\|U\| = 1$.

c. U is bijective $\Rightarrow U^{-1}$ is bijective

Also, $(U^{-1})^* = (U^*)^*$ $\quad$ ($\because$ U is unitary $U^{-1} = U^*$)

$= U = (U^{-1})^{-1}$ $\quad$ So, U^{-1} is unitary.

d. U and V are bijective $\Rightarrow$ U V is bijective and

$(UV)^* = V^* U^* = V^{-1} U^{-1} = (UV)^{-1}$

e. U is unitary $\Rightarrow U^* = U^{-1}$

So, $\quad UU^* = UU^{-1} = I \quad$ and $\quad V^*V = V^{-1}V = I$.

Theorem : A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and surjective.

Proof : Suppose T is isometric and surjective .T is isometric $\Rightarrow$T is one – one . So T is bijective . We have to show that $T^* = T^{-1}$ .By the isomety

$\langle T^* T x, x \rangle = \langle Tx, Tx \rangle = \langle x, x \rangle = \langle Ix, x \rangle$

$\Rightarrow \langle (T^* T - I)x, x \rangle = 0$ for x .

So, $T^* T = I$. Als0 $TT^* = TT^*(TT^{-1}) = T(T^*T)T^{-1} = T I T^{-1} = T T^{-1} = I$

Together, $T^* T = TT^* = I$. Hence $T^* = T^{-1}$, so that T is unitary .The converse is clear by (a) and definition .

Exercise :

1. If S and T are bounded self adjoint linear operators on a Hilbert space H and α and β are real , show that $\alpha S + \beta T$ is self adjoint .

2. Show that if $T : H \to H$ is bounded self adjoint linear operator , so is T^2 where n is a positive integer.

3. Show that for any bounded linear operator T on H , the operators $T_1 = \frac{1}{2}(T + T^*)$ and $T_2 = \frac{1}{2i}(T - T^*)$ are self adjoint. Show that $T = T_1 + iT_2$ and $T^* = T_1 - iT_2$.

4. If $T_n : H \to H$ ($n = 1, 2, 3, \ldots\ldots\ldots$) are normal linear operators and

   $T_n \to T$, show that T is a normal linear operator .

5. If S and T are normal linear operators satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that S + T are product S T are normal .

6. Show that a bounded linear operator $T : H \to H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this show that for a normal operator $\|T^2\| = \|T\|^2$.

Projection Operator :

Definition : A linear operator P on Banach space is said to be a projection if and only if $P^2 = P$.

Note : Some authors call a bounded linear operator P on a Banach space X, a projection operator if it is idempotent , that is $P^2 = P$. Since many results can be establised without continuity of P , we do not require a projection operator a bounded linear operator.

Proposition : Let P be a projection on a Banach space X . Then

a. $I - P$ is a projection on X,

b. $R(P) = \{x \in X : Px = x\}$

c. $R(P) = N(I - P)$

d. $X = R(P) \oplus N(I - P)$

e. If P is bounded then $R(P)$ and $R(I - P)$ are closed .

Here $R(P)$ and $N(I - P)$ stand for range and null space of P and $I - P$ respectively .

a. $(I - P)^2 = I - 2P + P^2 = I - P$

b. Clearly $\{x \in X : x = Px\} \subseteq R(P)$. Conversely $y \in R(P)$

implies $y = Px$ for some $x \in X$ and hence

$$Py = P(Px) = P^2x = Px = y \text{ .So } R(P) \subseteq \{x : Px = x\}.$$

c. $x \in R(P) \Leftrightarrow x = Px \Leftrightarrow (I - P)x = 0$

d. $x \in X \Rightarrow x = Px + (I-P)x$

$$\Rightarrow X = R(P) + R(I-P).$$

Also, $x \in R(P) \cap R(I-P)$

$$\Rightarrow x = Px = (I-P)x$$

$$\Rightarrow x = P(I-P)x = Px - P^2x = Px - Px = 0$$

So, $R(P) \cap R(I-P) = \{0\}$.

e. If P is a bounded linear operator,

$R(P) = N(I-P) = (I-P)^{-1}(\{0\})$ is a closed subspace

Since $I-P$ is continuous.

$R(I-P) = N(P) = P^{-1}(\{0\})$ is also closed subspace.

Lemma : Let M and N be linear subspaces of X with $X = M \oplus N$. Then there is a unique projection P on X with $R(P) = M$ and $R(I-P) = N$.

Proof : Let $x \in X$. Then there are unique points $y \in M$ and $z \in N$ with

$x = y + z$. Let $Px = y$. Then $P : X \to X$ is a linear and $R(P) = M$ and

$N(P) = N$. Also $P^2 = P$. Since $x \in X$ implies $Px \in M$, $P(Px) = Px$.

Hence P is a projection on X. We have seen $R(P) = M$ and $N(P) = N$. Also from the preceding result, we have $N(P) = R(I-P)$. So, $R(I-P) = N$.

Uniqueness : Let Q be a projection on X with $R(Q) = M$ and $R(I-Q) = N$. For each $x \in X$, we have $x = Qx + (I-Q)x$, $Qx \in M$ and

$(I-Q)x \in N$. So, by definition of P, we must have $Px = Qx$. This proves that $P = Q$.

Note : By the last two parts of the preceding result, X has a direct sumdecomposition

$$X = R(P) \oplus R(I-P)$$

Where $R(P)$ and $R(I-P)$ are closed linear subspaces of X.

Theorem : Let X be a Banach space and let M and N be closed linear subspaces of X with $X = M \oplus N$. Then there is a unique bounded projection P on X such that $R(P) = M$ and $R(I-P) = N$.

Proof: By the Lemma, it is enough to show that graph of P is closed. Let $\{x_n\}$

be a sequence in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Px_n = y$. Since $Px_n \in M$ and M is closed we have $y \in M$. Also $x_n - Px_n = (I-P)x_n \in N$.

Then $Py=y$ since $R(P)=\{x \in X : Px=x\}$ and $P(x-y)=0$ since $R(P)=N(I-P)$. Consequently, $y = Py = Px$. This shows that graph of P is closed and hence P is continuous.

Projection on Hilbert Space :

Theorem : If P is a bounded projection on H with $R(P) = M$ and $N(P) = N$ then $M \perp N \Leftrightarrow$ P is self adjoint and $N = M^{\perp}$.

Proof : Each vector z in H can be written uniquely in the for $z = x + y$ with x and y in M and N. If $M \perp N$, so that $x \perp y$, then

$$< P^{*} z , z > = < z , P z > = < x + y , x > = < x , x > = < x , x + y > = < Pz , z >$$

$$\Rightarrow < (P^{*} - P) z , z > = 0 \qquad \forall z \in H$$

So, $P^{*} = P$ and P is self adjoint.

If conversely, $P^{*} = P$, then for $x \in M, y \in N$, we have

$$< x , y > = < P x , y > = < x , P^{*} y > = < x , P y > = < x , 0 > = 0$$

This shows that $M \perp N$.

We are left to show that $M \perp N \Rightarrow N = M^{\perp}$.

$$x \in N \Rightarrow < x , y > = 0 \quad \text{for all } y \in M$$

$$\Rightarrow x \in M^{\perp}$$

So, $N \subseteq M^{\perp}$ and $M^{\perp}$ is a Hilbert space. If possible let N be a proper closed subspace of Hilbert space $M^{\perp}$ so, there exists a non-zero vector z_0 in $M^{\perp}$ such that $z_0 \perp N$.

$$z \in H = M \oplus N \Rightarrow < z_0, z > = < z_0, x + y > \text{ with } x \in M, y \in N$$

$$= 0 \quad (\text{since } z_0 \in M^{\perp} \text{ and } z_0 \in N^{\perp})$$

Hence, $z_0 \in H^{\perp} = \{0\}$ so that $z_0 = 0$. This is a contradiction. This proves that $N = M^{\perp}$.

Perpendicular Projection :

A projection on a Hilbert space whose range and null space are orthogonal is sometimes called a **perpendicular projection**. In the following subsection we discuss only perpendicular projections.

Definition : A projection in a Hilbert is defined as a bounded linear idempotent and self adjoint operator. Clearly for a projection P, $< P x , x > \geq 0$. We define $P \geq Q$ if $< P x , x > \geq < Q x , x >$ for all x.

Proposition : If P is a projection on a Hilbert space H with Range (P) = M then $x \in M \Leftrightarrow Px = x \Leftrightarrow \|Px\| = \|x\|$ and $0 \leq P \leq I$

Proof : It is clear that P is a projecton with Range (P) = M $\Leftrightarrow$ I – P is the projection with Range (I – P) = $M^{\perp}$

$$P \in M \Leftrightarrow P x = x \Leftrightarrow \|Px\| = \|x\|.$$

Also $\|x\|^2 = \|Px + (I - P)x\|^2 = \|Px\|^2 + \|(I - P)x\|^2$ ....(*)

So ,$\|Px\| = \|x\| \Rightarrow \|(I - P)x\|^2 = 0 \Rightarrow Px = x$.

Also from (*) $\|x\|^2 \geq \|Px\|^2$ or , $\left\|\frac{Px}{x}\right\| \leq 1$ for all $x \neq 0$.

So, $\|P\| \leq 1$. Also for $x \in H$,

$$< P x , x > = < P P x , x > = < P x , P x > = \|Px\|^2 \geq 0 .$$

So, $P \geq 0$.

Since $P \geq 0$ for any projection P on H , we have $I - P \geq 0$ as $I \geq P$

This proves the result.

Invariant Subspace : Let T be an operator on a Hilbert space H . A closed linear subspace M of H is said to be **Invariant** under T if $T (M) \subseteq M$. In this case the restriction of T to M can be regarded as an operator on M . We say M reduces T if both M and $M^{\perp}$ are invariant under T .

Proposition : A closed linear subspace M of H is invariant under an operator T $\Leftrightarrow$ $M^{\perp}$ as invariant under T^* .

Proof : M is invariant under $T \Rightarrow T (M) \subseteq M$. We have to show that

$T^* (M^{\perp}) \subseteq M^{\perp}$. For $x \in M^{\perp}$ and $y \in M$,

$$< T^* (x) , y > = < x , T y > = 0 \qquad (\text{since } x \in M^{\perp} , Ty \in M)$$

$$\Rightarrow T^* x \in M^{\perp} \ \forall \ x \in M^{\perp}.$$

$$\Rightarrow T^* (M^{\perp}) \subseteq M^{\perp} .$$

So , $M^{\perp}$ is invariant under T^*. The converse follows from $M = M^{\perp\perp}$ and $T = T^{**}$.

Proposition : A closed linear subspace M of H reduces an operator T $\Leftrightarrow$ M is invariant under T and T^*.

Proof : M reduces $T \Leftrightarrow T (M) \subseteq M$ and $T (M^{\perp}) \subseteq M^{\perp}$.

$$\Leftrightarrow T (M) \subseteq M \quad \text{and } T^* (M^{\perp\perp}) \subseteq M^{\perp\perp}$$

$$\Leftrightarrow T (M) \subseteq M \quad \text{and } T^* (M) \subseteq M$$

Proposition : If P is the projection on a closed linear subspace M of H (i.e., R (P) = M), then M is invariant under $T \Leftrightarrow TP = PTP$.

Proof : Suppose $T(M) \subseteq M$ Then for $x \in H$,

$$TP(x) \in M. \text{ So } P(TP(x)) = TP(x).$$

This shows that $PTP = TP$. Conversely if $PTP = TP$ and $x \in M$, then $Tx = TP(x) = PTP(x) \in M$. So, $T(M) \subseteq M$.

Proposition : If P is the projection on a closed linear subspace M of H, then M reduces on operator $T \Leftrightarrow TP = PT$.

Proof : M reduces $T \Leftrightarrow M$ is invariant under T and T^*.

$\Leftrightarrow TP = PTP$ and $T^*P = PT^*P$

$\Leftrightarrow TP = PTP$ and $PT = PTP$ (Taking adjoint in second equation)

$\Leftrightarrow TP = PT$ (Multiplying by P from right and left

of last step to obtain the preceding)

Proposition : If P and Q are the projection on closed linear subspace M and N of H, then

$$M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$$

Proof : $PQ = 0 \Rightarrow (PQ)^* = 0^* \Rightarrow Q^*P^* = 0 \Rightarrow QP = 0.$

Similarly $QP = 0 \Rightarrow PQ = 0$.

Now, $M \perp N \Rightarrow N \subseteq M^{\perp}$. For $x \in H$,

$$(PQ)x = P(Qx) = 0 \quad (\because Q(x) \in N \subset M^{\perp} = \text{null space of } P)$$

Hence $PQ = 0$. Conversely, If $PQ = 0$ then for every $x \in N$, $Px = PQx = 0$. So, $N \in$ null space $P = M^{\perp}$ and $M \perp N$.

Theorem : If $P_1, P_2, \ldots, P_n$ are the projection on closed linear subspaces $M_1, M_2, \ldots, M_n$ of H, then $P = P_1 + P_2 + \ldots + P_n$ is a projection $\Leftrightarrow$ P_i^s are pairwise orthogonal (in the sence that $P_i P_j = 0$ for $i \neq j$) and in this case Range of $P = M = M_1 + M_2 + \ldots + M_n$.

Proof : $P^* = P_1^* + P_2^* + \ldots + P_n^* = P_1 + P_2 + \ldots + P_n = P$. P_i^s are pairwise orthogonal $\Rightarrow P_iP_j = 0$ for $i \neq j$. So, $P^2 = P$. Conversely, suppose P is a projection. So, $P^2 = P$.

Let $x \in$ Range P_i. Then $x = P_ix$ and

$$\|x\|^2 = \|P_ix\|^2 \leq \sum_{i=1}^{n} \|P_ix\|^2 = \sum_{j=1}^{n} < P_jx, x > = < Px.x > = \|Px\|^2 \leq \|x\|^2 \quad \ldots\ldots(*)$$

Hence, $\sum_{j=1}^{n} \| P_j x \|^2 = \| P_i x \|^2$

$\Rightarrow \|P_j x\| = 0 \quad$ for $j \neq i$

$\Rightarrow P_j x = 0 \quad$ for $j \neq i$

$\Rightarrow x \in$ null space of P_j ($j \neq i$)

So, $M_i \subseteq M_j^{\perp}$ ($j \neq i$). Hence $M_i \perp M_j$ and $P_i P_j = 0$ for $i \neq j$.

We are left to show that Range(P) = $M_1 + M_2 + \ldots + M_n$. From (*), we have $\|Px\| = \|x\|$ for every $x \in M_i$ and hence $P x = x$ for $x \in M_i$. This shows that

$M_i \subseteq$ Range (P) for each i.

$\Rightarrow M_1 + M_2 + \ldots + M_n \subseteq$ Range (P)

Also, $x \in$ Range (P) $\Rightarrow x = P x = P_1 x + P_2 x + \ldots + P_n x \in M_1 + M_2 + \ldots + M_n = M$

So, Range (P) = $M = M_1 + M_2 + \ldots + M_n$.

Finite Dimensional Spectral Theory :

Definition : If T is an operator on a Hilbert space H, then a non – zero vector satisfying the equation $Tx = \lambda x$ for some scaler λ is called an eigen vector T and a scalar λ satisfying for some non – zero x is called an eigen value of T.

1. Each eigen value has one or more eigen vectors associated with it
2. To each eigen vector there corresponds precisely one eigen value.

For if $T x = \lambda_1 x = \lambda_2 x$ then $(\lambda_1 - \lambda_2) x = 0$ and $x \neq 0$ implies $\lambda_1 = \lambda_2$.

Proposition : The set M of all eigen vectors corresponding to eigen value λ of bounded linear operator T together with the zero vector is a closed linear subspace of H.

Proof : $M = \{ x \mid T x = \lambda x \} \cup \{ 0 \}$, $\quad x_1, x_2 \in M$ and $\alpha, \beta \in K$ implies

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 = \alpha \lambda x_1 + \beta \lambda x_2 = \lambda(\alpha x_1 + \beta x_2)$$

This shows that $\alpha x_1 + \beta x_2 \in M$ and M is a linear subspace of H.

Let $\{x_n\} \subset M$ and $x_n \to x$. Then $Tx_n = \lambda x_n$ and

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \lambda x_n = \lambda x$$

$\Rightarrow T x = \lambda x \qquad$ (Since T is bounded)

$\Rightarrow x \in M \qquad$ So, M is a closed linear subspace of H.

Definition : The closed linear subspace M of H called eigen space of T corresponding to λ.

Proposition : M is invariant under T .

Proof : To show that $T(M) \subseteq M$. For $x \in M$, $Tx = \lambda x$.

But $\lambda x \in M$. So , $Tx \in M$ and $T(M) \subseteq M$. Hence proved .

Example of an operator in an infinite dimensional Hilbert space which has no eigen value .

Consider $T : \ell_2 \to \ell_2$ defined by $T(\{x_1, x_2, x_3, \ldots, \}) = \{0, x_1, x_2, \ldots\}$

Which is a bounded linear operator on H . Suppose $\exists\, \lambda \neq 0$ and $\{x_n\} \neq 0$ such that

$$T(\{x_1, x_2, \ldots\}) = \lambda\{x_1, x_2, \ldots\}$$

$$\Rightarrow \{0, x_1, x_2, \ldots\} = \{\lambda x_1, \lambda x_2, \ldots\}$$

$$\Rightarrow \lambda x_1 = 0,\ x_1 = \lambda x_2,\ x_2 = \lambda x_3, \ldots$$

$$\Rightarrow x_1 = 0,\ x_2 = 0,\ x_3 = 0, \ldots$$

$\Rightarrow \{x_1, x_2, x_3, \ldots\} = 0$, a contradiction . So , T has no eigen vector .

We can show that a finite dimensional Hilbert space must have eigenvalues.

Theorem : If T is an arbitrary operator an a finite dimensional Hilbert space H then the eigenvalues of T constitute a non empty finite subset of the complex plane . The number of points in this set does not exceed the dimension n of the space H .

Proof : λ is an eigenvalue of T .

$\Leftrightarrow \exists$ a non – zero vector x such that $(T - \lambda I)x = 0$

$\Leftrightarrow T - \lambda I$ is singular

$\Leftrightarrow \det(T - \lambda I) = 0$

The eigenvalues of T are therefore precisely the distinct roots of the equation

$$\det(T - \lambda I) = 0$$

If we choose a basis B for H and find the matrix $[a_{ij}]$ of T relative to B , then the

$$\det(T - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n\lambda} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \qquad \text{---------} \ (*)$$

eigenvalues of T are the roots of the equation (*) .

The equation (*) is a polynomial equation of degree n in the variable λ .By the fundamental Theorem of algebra this equation has exactly n complex roots . Since some of these roots may be equal , the numberof distinct roots does not exceed n .

Definition : The set eigenvalues of T ia called spectrum of the operator T . It is denoted by σ (T) .

Theorem : Suppose H is a Hilbert space of dimension n and $\lambda_1, \lambda_2, \ldots, \lambda_m$

are eigenvalues of $T \in B(H)$. $M_1, M_2, \ldots, M_m$ are eigenspaces of $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and $P_1, P_2, \ldots, P_m$ are projections on $M_1, M_2, \ldots, M_m$. Then the followings are equivalent

i . M_i are pairwise orthogonal and span H

ii . P_i s are pairwise orthogonal , $I = P_1 + P_2 + \ldots + P_m$ and

$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots \lambda_m P_m$

iii . T is normal .

Proof : (i) $\Rightarrow$ (ii) : We have already seen that

$P_i P_j = 0 \Leftrightarrow M_i \perp M_j$.

So , P_i s are pairwise orthogonal .

Every $x \in H$ can be uniquely expressed as

$x = x_1 + x_2 + \ldots + x_m$, with $x_i \in M_i$.

So, $Tx = Tx_1 + Tx_2 + \ldots + Tx_m$

$= \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m$

$= \lambda_1 P_1 x + \lambda_2 P_2 x + \ldots + \lambda_m P_m x = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)x$

So , $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$,

Also, $I x = x = x_1 + x_2 + \ldots + x_n$

$= P_1 x + P_2 x + \ldots + P_m x$

$= (P_1 + P_2 + \ldots + P_m) x$ for every x in H .

So , $I = P_1 + P_2 + \ldots + P_m$

(ii) $\Rightarrow$ (iii) :

$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$ implies

$T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \ldots + \bar{\lambda}_m P_m$

So, $TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m)(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \ldots + \bar{\lambda}_m P_m)$

$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_m|^2 P_m$ and similarly,

$T^* T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_m|^2 P_m$

So, $TT^* = T^*T$ and T is normal.

(iii) ⇒ (i) : Let $T \in B(H)$ be a normal operator. It is easy to check that

$T - \lambda I$ is normed.

$$(T-\lambda I)^* = T^* - \bar{\lambda} I^* = T^* - \bar{\lambda} I$$

So, $(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - \bar{\lambda} I)$

$$= TT^* - \bar{\lambda} T - \lambda T^* - |\lambda|^2 I$$

and $(T-\lambda I)^*(T-\lambda I) = (T^* - \bar{\lambda} I)(T-\lambda I)$

$$= TT^* - \lambda T^* - \bar{\lambda} T + |\lambda|^2 I$$

Since T is normal, so is $T - \lambda I$. We know that for a normal operator T,

$\| T^* x \| = \| Tx \|$ since

$\| T^* x \|^2 = \langle T^* x, T^* x \rangle = \langle TT^* x, x \rangle = \langle Tx, Tx \rangle = \langle T^* Tx, x \rangle = \langle Tx, Tx \rangle = \| Tx \|^2$

Consequently,

$\| T - \lambda I \| = \| (T^* - \bar{\lambda} I) x \|$

This shows that x is eigenvector of T coresponding to λ iff x is also eigenvector of T^* corresponding to $\bar{\lambda}$.

Now we show that M_i s are pairwise orthogonal. Let x_i and x_j be vectors in M_i and M_j for $i \neq j$, so that $Tx_i = \lambda_i x_i$ and $Tx_j = \lambda_j x_j$. Then

$$\lambda_i \langle x_i, x_j \rangle = \langle \lambda_i x_i, x_j \rangle = \langle Tx_i, x_j \rangle = \langle x_i, T^* x_j \rangle = \langle x_i, \bar{\lambda}_j x_j \rangle = \lambda_j \langle x_i, x_j \rangle$$

or $(\lambda_i - \lambda_j)\langle x_i, x_j \rangle = 0 \Rightarrow x_i, x_j = 0$ since $\lambda_i \neq \lambda_j$

Hence $M_i \perp M_j$ for $i \neq j$

Lastly, we have to show that M_i s span H. First, we show that each M_i

reduces T. For this we have to show that M_i is invariant under T and T^*.

$x_i \in M_i \Rightarrow Tx_i = \lambda_i x_i \in M_i$. So, $T^*(M_i) \subset M_i$. Hence M_i reduces T. We know that $P = P_1 + P_2 + \ldots + P_m$ is a projection on $M = M_1 + M_2 + \ldots + M_m$.

M_i reduces T $\Rightarrow TP_i = P_iT$ for each P_i

$$\Rightarrow TP = PT$$

$$\Rightarrow M \text{ reduces } T.$$

Hence, $T(M^{\perp}) \subseteq M^{\perp}$. We know that $H = M \oplus M^{\perp}$. If possible $M^{\perp} \neq \{0\}$. Then since all the eigenvectors of T are contained in M, the restriction of T to $M^{\perp}$ is an operator on a non – trivial finite dimensional Hilbert space which has no eigenvalues. This is not possible. So, $M^{\perp} = \{0\}$ and $H = M = M_1 + M_2 + \ldots + M_m =$ Span of M_i s.

Definition :The expression $T = \lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m$ is called sepctral resolution T. The preceding theorem asserts that in a finite dimensional Hilbert space every normal operator has spectral resolution.

Uniqueness of Spectral resolution :

Proposition : The spectral resolution $T = \lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m$ is unique.

Proof :$T = \lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m$ ------- (1) implies

$$T^2 = (\lambda_1P_1 + \lambda_2P_2 + \ldots + \lambda_mP_m)^2 = \lambda_1^2 P_1 + \lambda_2^2 P_2 + \ldots + \lambda_m^2 P_m$$

In general,

$T^n = \lambda_1^n P_1 + \lambda_2^n P_2 + \ldots + \lambda_m^n P_m$. It is customary to write $T^0 = I = P_1 + P_2 + \ldots + P_m$.
Let $p(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \ldots + \alpha_m z^m$ with $\alpha_i \in C$

and $z \in C$. Then

$$P(T) = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \ldots + \alpha_m T^m = \sum_{i=1}^{m} p(\alpha_i)P_i.$$

Consider particular polynomialn p_j, defined by

$$p_j(z) = \frac{(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_{j-1})(z-\lambda_{j+1})\ldots(z-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\ldots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\ldots(\lambda_j-\lambda_m)}$$

Clearly $p_j(\lambda_i) = 0$ if $i \neq j$ and $p_j(\lambda_j) = 1$.

So, $p_j(T) = \sum_{i=1}^{m} p_j(\lambda_i)p_i = p_j$.

This shows that p_i s are uniquely determined as specific polynominals in T.

Suppose T has another spectral resolution,

$$T = \beta_1Q_1 + \beta_2Q_2 + \ldots + \beta_mQ_m \qquad \text{-----------} (2)$$

Where β_i 's are distinct complex numbers , the Q_i s are non–zero pairwise orthogonal projection , and $I = \sum_{i=1}^{k} Q_1$ .We want to show (1) and (2) are identical. We first show that are β_i s are exactly λ_i 's.

Such that $Q_i x = x$ and $Q_j x = 0$ for $j \neq i$. Then from (2) , we have

$Tx = \beta_i x$ so that β_i is eigenvalue of T . Conversely, suppose λ is an eigenvalue of T , so that $Tx = \lambda x$ for some non–zero x , then

$$Tx = \lambda x = \lambda I x = \lambda \sum_{i=1}^{k} Q_i x = \sum_{i=1}^{k} \lambda Q_i x$$

Also $$Tx = \sum_{i=1}^{k} \alpha_i Q_i x$$

$$\sum_{i=1}^{k} (\lambda - \alpha_i) Q_i x = 0 \; .$$

Since Q_i x 's are pairwise orthogonal , the non–zero vectors among them , there is at least one , for $x \neq 0$ are linearly independent , and this implies that

$\lambda = \alpha_i$ for some i. So λ_i 's and α_i 's are the same. (2) can be written as

$$T = \lambda_1 Q_1 + \lambda_2 Q_2 + \ldots + \lambda_m Q_m$$

Proceding in a similar way, it can be shown that $Q_j = p_j(T)$ for every j. Thus $P_j = Q_j = p_j(T)$. This shows that spectral resolution is unique .

Problem 1. Let T be an operator on a finite dimensional Hilbert space H . Then

a. T is singular $\Leftrightarrow 0 \in \sigma(T)$;

b. If T is non – singular , then $\lambda \in \sigma(T) \Leftrightarrow \lambda^{-1} \in \sigma(T^{-1})$;

c. If A is non – singular , then $\sigma(ATA^{-1}) = \sigma(T)$;

d. If $\lambda \in \sigma(T)$, and if p in any polynomial then $\beta(\lambda) \in \sigma(p(T))$;

e. If $T^k = 0$ for some positive integer k_1 then $\sigma(T) = \{0\}$.

Proof : a. T is singular

$\Leftrightarrow$ T is not one – one

$\Leftrightarrow \exists x \neq 0 \in H$ s. t. $T(x) = 0 = 0 . x$

$\Leftrightarrow 0 \in \sigma(T)$

b. $\lambda \in \sigma(T)$

$\Rightarrow \exists\, x \neq 0$ in H s.t. $Tx = \lambda x$

$\Rightarrow T^{-1}(Tx) = T^{-1}(\lambda x) = \lambda(T^{-1}x)$

$\Rightarrow T^{-1}x = \lambda^{-1}x$

$\Rightarrow \lambda^{-1} \in \sigma(T^{-1})$

Conversely, $\lambda^{-1} \in \sigma(T^{-1})$

$\Rightarrow \exists\, x \neq 0$ in H s.t. $T^{-1}x = \lambda^{-1}x \Rightarrow T\,T^{-1}x = T(\lambda^{-1}x) = \lambda^{-1}T(x)$

$\Rightarrow Ix = \lambda^{-1}Tx \Rightarrow Tx = \lambda x \Rightarrow \lambda \in \sigma(T)$ The rest is left as an exercise .

Problem 2. Show that an operator T on H is normal iff its adjoint T^* is a polynomial in T.

Proof : T is normal

$\Rightarrow T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$

$\Rightarrow T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \ldots + \bar{\lambda}_m P_m$

But $p_j = \beta_j(T)$. Hence $T^* = \bar{\lambda}_1 p_1(T) + \bar{\lambda}_2 p_2(T) + \ldots + \bar{\lambda}_m p_m(T)$

where $p_j(z) = \dfrac{(z-\lambda_1)\cdots(z-\lambda_{j-1})(z-\lambda_{j+1})\cdots(z-\lambda_m)}{(\lambda_j-\lambda_1)\cdots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})\cdots(\lambda_j-\lambda_m)}$

The converse is left as an exercise .

Problem 3. Let T be a normal operator on a finite dimensional space T with spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, prove that

a. T is self – adjoint $\Leftrightarrow$ each λ_i is real

b. T is positive $\Leftrightarrow \lambda_i > 0$ for each i

c. T is unitary $\Leftrightarrow |\lambda_i| = 1$ for each i

Proof : a. $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m$

$\Rightarrow T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m$

$T - T^* \Rightarrow (\lambda_1 - \bar{\lambda}_1)P_1 + (\lambda_2 - \bar{\lambda}_2)P_2 + \cdots + (\lambda_m - \bar{\lambda}_m)P_m = 0$

$\Rightarrow (\lambda_1 - \bar{\lambda}_1)P_1 x_i + (\lambda_2 - \bar{\lambda}_2)P_2 x_i + \cdots + (\lambda_i - \bar{\lambda}_i)P_i x_i + \cdots + (\lambda_m - \bar{\lambda}_m)P_m x_i = 0$

$\Rightarrow (\lambda_i - \bar{\lambda}_i)x_i = 0 \Rightarrow \lambda_i = \bar{\lambda}_i$

Try the converse yourself.

b. $< Tx, x> \ > 0$

$$\Rightarrow < (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_m P_m) x_i, x_i > \ > 0$$

$$\Rightarrow <\lambda_i x_i, x_i> \geq 0 \Rightarrow \| x_i \|^2 > 0 \Rightarrow \lambda_i > 0$$

The converse is left as an exercise.

c. $TT^* = I = T^*T$

$$< T^*Tx, x> = < TT^* x, x> = < Ix, x> = \|x\|^2$$

$$\Rightarrow <Tx, Tx> = \|x\|^2$$

$$\Rightarrow <(\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_i P_i + \ldots + \lambda_m P_m) x_i,$$

$$(\lambda_1 P_1 + \ldots + \lambda_i P_i + \ldots + \lambda_m P_m) x_i > = \|x\|^2$$

$$\Rightarrow |\lambda_i|^2 \|P_i x_i\|^2 = \|x_i\|^2 \Rightarrow |\lambda_i|^2 \|x_i\|^2 = \|x_i\|^2 \Rightarrow |\lambda_i| = 1$$

Try the converse yourself.

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Text Book :

Introduction to Topology and Modern Analysis, G.F.Simmons – Mc GRAW HILL

References :

1. **Functional Analysis – B.V. Limaye (WILEY)**
2. **Introduction to Functional Analysis with Applications – Erwin Kreyszig (John Wiley & Sons)**
3. **First course in Functional Analysis – Goffman and Pedrick (Prentice Hall)**
4. **Functional Analysis – B.K.Lahari. – World Press**
5. **Functional Analysis – Brown and Page (Van Nostand Reinhold)**
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2008

MATHEMATICS

Paper : 203

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

(New Syllabus)

(Functional Analysis—I)

1. Answer any *two* parts : 5×2=10

   (a) Let B be a subset of the vector space Y and let T be a linear operator with range in Y. If B is convex, balanced and absorbing, prove that $T^{-1}(B)$ has the same properties.

   (b) Suppose that A is a balanced subset of a vector space X. Prove that A is absorbing if and only if the following holds :

   "For each x in X, there is a positive number t_x such that $x \in t_x A$."

   (c) If n is a positive integer, then show that the vector space F^n of all n-tuples of scalars is a normed space with the Euclidean norm given by the formula

$$\|(\alpha_1, \alpha_2, \ldots, \alpha_n)\| = \left(\sum_{j=1}^{n} |\alpha_j|^2 \right)^{1/2}$$

2. Answer any *two* parts : 5×2=10

   (a) Let X be a normed space. Prove—

   (i) addition of vectors is a continuous operation from $X \times X$ into X;

(ii) multiplication of vectors by scalars is a continuous operation from $F \times X$ into X (with usual notion).

(b) Prove that a normed space X is a Banach space if and only if each absolutely convergent series in X converges.

(c) Let $\sum_n x_n$ be a formal series in a Banach space. Prove that if the series is absolutely convergent, then it is unconditionally convergent. Show with a counterexample that the converse of the above result is not true.

3. Answer any *one* part : 10

(a) Prove that every bounded linear operator from a Banach space onto a Banach space is an open mapping.

(b) Let $\mathscr{F}$ be a non-empty family of bounded linear operators from a Banach space X into a normed space Y. If $\sup\{\|Tx\| : T \in \mathscr{F}\}$ is finite for each x in X, then show that $\sup\{\|T\| : T \in \mathscr{F}\}$ is finite.

Answer any *one* part : 10

(a) Prove that if M is a closed subspace of a Banach space X, then X/M is also a Banach space. By using the above result or otherwise, examine if the space $\mathbb{R}^2/M$ is a Banach space, where $\mathbb{R}^2$ is the Euclidean two-dimensional space and $M = \{(\alpha, \beta) : \alpha = 0\}$.

(b) Let M be a closed subspace of a normed space X. Prove the following :

(i) The quotient map π from X onto X/M is a bounded linear operator that is also an open mapping and has M as its kernel.

(ii) If $M \neq X$, then $\|\pi\| = 1$.

5. Answer any *two* parts : 5×2=10

(a) Suppose that X is a normed space. Let $f_1, f_2, \ldots, f_n$ be a non-empty finite collection of bounded linear functionals on X, and let $c_1, c_2, \ldots, c_n$ be a corresponding collection of scalars. Prove that the following are equivalent :

(i) There is an x_0 in X such that $f_j(x_0) = c_j$, when $j = 1, 2, \ldots, n$.

(ii) There is a non-negative real number M such that

$$|\alpha_1 c_1 + \ldots + \alpha_n c_n| \leq M \|\alpha_1 f_1 + \ldots + \alpha_n f_n\|$$

for each linear combination

$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_n f_n$ of $f_1, f_2, \ldots, f_n$ that is, for each element $\langle\{f_1, f_2, \ldots, f_n\}\rangle$.

(b) Let X and Y be normed spaces. If $T \in L(X, Y)$, then show that T is bounded if and only if

$$\sup\{|\langle Tx, y^*\rangle| : x \in B_X, y^* \in B_{Y^*}\}$$

is finite. If T is bounded, then its norm equals this supremum.

(c) Let X be a normed space, and let A and B be subsets of X and X^* respectively. Prove that the sets $A^{\perp}$ and ${}^{\perp}B$ are closed subspaces of X^* and X respectively.

6. Answer any *three* parts : 5×3=15

(a) Let $X = C([a, b])$, the linear space of all scalar-valued continuous functions on $[a, b]$. For x and y in X, define

$$\langle x, y\rangle = \int_a^b x(t)\overline{y(t)}\,dt$$

Show that $\langle , \rangle$ is an inner product on X.

(b) If E is an orthonormal subset of a Hilbert space H, then show that

$$\|x - y\| = \sqrt{2} \quad \text{for all } x \neq y \text{ in } E$$

(c) Let H be a Hilbert space, $\{u_1, u_2, \ldots\}$ be a countable orthonormal set in H and $k_1, k_2, \ldots$ belong to the field F. Prove the following :

(i) If $\sum_n k_n u_n$ converges to some x in H; then $\langle x, u_n \rangle = k_n$ for each n and $\sum_n |k_n|^2 < \infty$.

(ii) If $\sum_n |k_n|^2 < \infty$, then $\Sigma k_n u_n$ converges in H.

(d) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H and $x \in H$. Let $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then prove that E_x is a countable set. Further, show that if E_x is deaumerable, say $E_x = \{u_1, u_2, \ldots\}$, then $\langle x, u_n \rangle \to 0$ as $n \to \infty$.

7. Answer any *three* parts : 5×3=15

(a) Let H be a Hilbert space and $A \in BL(H)$. Prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$

$$\langle A(x), y \rangle = \langle x, B(y) \rangle$$

(b) Let $A, B \in BL(H)$. Prove that

$$(A+B)^* = A^* + B^*$$

and $(AB)^* = B^* A^*$

(c) Define a positive operator and give an example. Let $A \in BL(H)$ be self-adjoint. Prove that A or $-A$ is a positive operator if and only if

$$|\langle A(x), y \rangle|^2 \leq \langle A(x), x \rangle . \langle A(y), y \rangle$$

for all $x, y \in H$.

(d) Let $u_1, u_2, \ldots$ constitute an orthonormal basis for H. Suppose that $A \in BL(H)$ is defined by a matrix M with respect to the basis $u_1, u_2, \ldots$. Assume that M is triangular. Prove that A is normal if and only if M is diagonal.

(e) Let $H = K^2$, K a field. Define an operator $A : H \to H$ by $A(x, y) = (ax + by, cx + dy)$, $(x, y) \in H$, a, b, c, d in K. Show that A is unitary if and only if $|a|^2 + |b|^2 = 1 = |c|^2 + |d|^2$ and $a\bar{c} + b\bar{d} = 0$.

2009

MATHEMATICS

Paper : 203

(Functional Analysis—I)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *two* parts : 5×2=10

(a) Let C be the collection of all convergent sequences of scalars with the vector space operations

$$(x_n)+(y_n)=(x_n+y_n),\ \alpha(x_n)=(\alpha x_n)$$

and norm defined by

$$\|(x_n)\|_\infty=\sup_n|x_n|$$

show that C is a Banach space.

(b) If C is a convex subset of X, then both $\overline{C}$ and C° are convex. Prove it for a normed linear space X.

(c) Suppose that T is a linear operator from a Banach space X into a normed space Y. Show that if $T^{-1}(B_Y)$ is closed, then T is continuous at O, where

$$B_Y=\{y\in Y:\|y\|\le 1\}$$

2. Answer any *two* parts : 5×2=10

(a) Let $T:C[0,1]\to C[0,1]$ be a mapping defined by $[T(f)](t)=\int_0^t f(s)\,ds$. Show that T is a bounded linear operator and $\|T\|=1$.

(b) Let X and Y be normed spaces such that X is infinite dimensional $Y\neq\{0\}$. Show that there exists some linear operator from X into Y which is unbounded.

(c) Prove that every countably subadditive seminorm on a Banach space is continuous.

3. Answer any *two* parts : 5×2=10

(a) Let T be a linear operator from a Banach space X into a Banach space Y. Suppose that whenever a sequence (x_n) in X converges to some x in X and (Tx_n) converges to some y in Y, it follows that $y = Tx$. Prove that T is bounded.

(b) Let (T_n) be a sequence of bounded linear operators from a Banach space X into a normed space Y such that $\lim_n T_n x$ exists for each x in X. Define

$$T : X \to Y$$

by the formula $Tx = \lim_n T_n x$. Prove that T is a bounded linear operator from X into Y.

(c) Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are two Banach norms on a vector space X and that the identity map from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$ is continuous. Prove that the two norms are equivalent.

4. Answer any *two* parts : 5×2=10

(a) If M is a closed subspace of a Banach space X, then prove that the quotient space X / M is also a Banach space.

(b) Show that completeness is a three-space property.

(c) Suppose that f_0 is a bounded linear functional on a subspace Y of a normed space X. Prove that there is a bounded linear functional f on all of X such that $\|f\|=\|f_0\|$ and the restriction of f to Y is f_0.

5. Answer any *two* parts : 5×2=10

(a) If x be an element of a normed linear space X, then prove that

$$\|x\| = \sup\{|x^* x| : x^* \in B_{X^*}\}$$

and the supremum is attained at some point of B_{X^*}.

(b) Prove that every finite-dimensional normed space is reflexive.

(c) If the dual space X^* of a normed linear space is separable, then prove that X is also separable. Is the converse true?

6. Answer any *three* parts : 5×3=15

(a) Show that l^p with $p \neq 2$ is not an inner product space.

(b) If Y be any closed subspace of a Hilbert space H, then prove that $H = Y \oplus y^\perp$

(c) Let (e_n) be an orthonormal sequence in an inner product space X. Prove that for every $x \in X$

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

(d) Orthonormalize $x_0 = 1, x_1 = t, x_2 = t^2$ on the interval $[-1, 1]$, where

$$\langle x, y \rangle = \int_{-1}^{1} x(t) y(t)\, dt$$

7. Answer any *three* parts : 5×3=15

(a) Show that every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z \rangle$, where z depends on f and is uniquely determined by f and $\|z\| = \|f\|$.

(b) Show that the Hilbert adjoint operator $T^* : H_2 \to H_1$ of $T : H_1 \to H_2$ defined by

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

exists, is unique and a bounded linear operator with norm $\|T^*\| = \|T\|$.

(c) If $T : C^2 \to C^2$ be defined by $Tx = (z_1 + iz_2, z_1 - iz_2)$, where $x = (z_1, z_2)$, then find T^*.

(d) State and prove the Banach Fixed Point theorem.

(e) State and prove the Krein-Milman theorem.

2010

MATHEMATICS

Paper : 203

(Functional Analysis—I)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer any *two* parts : 5×2=10

(a) Show that the subspace C_0 of l^∞ defined by $C_0 = \{\{x_n\} : x_n \to 0\}$ is a Banach space.

(b) What do you mean by— $\sum_{n=1}^{\infty} x_n$ is absolutely convergent series in a normed linear space $(X, \|\cdot\|)$?
State and prove a necessary and sufficient condition for absolute convergence of a series $\sum_{n=1}^{\infty} x_n$ in a normed linear space X.

(c) Prove that in a finite-dimensional normed linear space, any two norms are equivalent.

2. Answer any *two* parts : 5×2=10

(a) Prove that in a finite-dimensional normed linear space, a closed and bounded subset is compact. Is the converse true for a general normed linear space?

(b) If $X = C[0, 1]$, then prove that the integral operator $T : X \to X$ defined by

$$[T(x)](t) = \int_0^1 k(t, s)\, x(s)\, dt$$

is a continuous linear operator, where $k(t, s): [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. What is the upper limit of the value of $\|T\|$?

(c) If X and Y are normed linear spaces and $T \in B(X, Y)$, then show that

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| \leq 1\} \\ &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\{\|Tx\| : \|x\| < 1\} \\ &= \inf\{\alpha > 0 : \|Tx\| \leq \alpha \|x\| \quad \forall\, x \in X\}\end{aligned}$$

3. Answer any *two* parts : 5×2=10

(a) If X is a normed linear space, then show that $X^* = B(X, \mathbb{R})$, the dual space X, is a Banach space.

(b) State and prove the principle of uniform boundedness.

(c) Let Y be a closed subspace of a normed linear space X. Prove that a sequence $\{x_n + Y\}$ converges to $x + Y$ in X/Y if and only if there is a sequence $\{y_n\}$ in Y such that $\{x_n + y_n\}$ converges to x in X.

4. Answer any *two* parts : 5×2=10

(a) Prove that X is a Banach space if and only if Y and X/Y are Banach spaces. Here Y is a closed subspace of X.

(b) State and prove the first isomorphism theorem for Banach spaces

(c) State Vector Space Version and Normed Space Version of the Hahn-Banach theorem. Show that for $x_0 \neq 0$ in a normed linear space $(X, \|\cdot\|)$, there exists a continuous linear functional x_0^* in dual space X^* such that

$$x_0^*(x_0) = \|x_0\| \text{ and } \|x_0^*\| = 1$$

5. Answer any *two* parts : 5×2=10

(a) Show that a normed space is finite-dimensional if and only if its dual space is finite dimensional.

(b) If $T \in L(X, Y)$, then show that T is bounded if and only if

$$\sup\{|\langle Tx, y^*\rangle| : x \in B_X,\ y^* \in B_{Y^*}\}$$

is finite.

(c) Obtain the dual space of l_p.

Answer any *three* parts : 5×3=15

(a) Define an inner product space. Let X and Y be two inner product spaces. Prove that a linear map $F : X \to Y$ satisfies

$$\langle F(x), F(y)\rangle = \langle x, y\rangle \text{ for all } x, y \in X$$

if and only if it satisfies

$$\|F(x)\| = \|x\| \text{ for all } x \in X$$

where the norms on X and Y are induced by the respective inner products.

(b) Let $\{x_1, x_2, \ldots\}$ be a linearly independent subset of an inner product space X. Define

$$y_1 = x_1,\ u_1 = \frac{y_1}{\|y_1\|},\ u_2 = \frac{y_2}{\|y_2\|},$$

for $n = 2, 3, 4, \ldots$ and

$$y_n = x_n - \langle x_n, u_1\rangle u_1 - \ldots - \langle x_n, u_{n-1}\rangle u_{n-1}$$

Show that $\{u_1, u_2, \ldots\}$ is an orthonormal set in X and

$$\text{span}\{u_1, u_2, \ldots, u_n\} = \text{span}\{x_1, x_2, \ldots, x_n\}$$

for $n = 1, 2, \ldots$.

(c) Suppose $\{u_1, u_2, \ldots\}$ is a countable orthonormal set in an inner product space X, and $x \in X$. Prove that

$$\sum_n |\langle x, u_n\rangle|^2 \le \|x\|^2$$

Further, show that the equality holds if and only if

$$x = \sum \langle x, u_n\rangle u_n$$

(d) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Prove that the following conditions are equivalent :

(i) $\{u_\alpha\}$ is an orthonormal basis for H.

(ii) For every $x \in H$

$$x = \sum_n (x, u_n) u_n$$

where $\{u_1, u_2, \ldots\} = \{u_\alpha : (x, u_\alpha) \neq 0\}$.

(iii) If $x \in H$ and $(x, u_\alpha) = 0$ for all α, then $x = 0$.

(iv) Span $\{u_\alpha\}$ is dense in H

(e) Suppose X is an inner product space and that $E \subset X$ is closed under scalar multiplication, and $x \in X$. Prove that $x \perp E$ if and only if

$$\text{dist}(x, E) = \|x\|$$

7. Answer any *three* parts : 5×3=

(a) Define the adjoint of an operator on a Hilbert space. Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an operator defined by

$$A(x, y) = (2x - y, 5y)$$

Obtain the adjoint of A.

(b) Let H be a Hilbert space, and $A, B \in BL(H)$. Prove that

(i) $(A + B)^* = A^* + B^*$

(ii) $(AB)^* = B^* A^*$

(iii) A is invertible if and only if A^* is invertible

(c) Let H be a Hilbert space and $A \in BL(H)$. Show that A is unitary if and only if $\|Ax\| = \|x\|$ for all $x \in H$. Further, show that if A is surjective, then $\|A\| = \|A^{-1}\| = 1$

(d) Let A and B be normal. If A commutes with B^* and B commutes with A^*, then prove that $A+B$ and AB are normal.

(e) Let $A \in BL(H)$ be self-adjoint. Prove that A or $-A$ is a positive operator if and only if

$$|\langle Ax, y\rangle|^2 \le \langle Ax, x\rangle\langle Ay, y\rangle$$

for all $x, y \in H$.

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