M203

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M.A./M.Sc. in Mathematics
Semester 2

Paper III
Hydrodynamics

Contents:

Course Co-Ordination

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1. Hydrodynamics by M.D.Raisinghania, Published by S. Chand and Company Ltd, New Delhi.
2. Hydrodynamics by Shanti Swarup, Published by Krishna Prakashan Mandir, Meerut
3. Fluid Dynamics by J. K. Goyal and K. P. Gupta, Published by Pragati Prakashan, Meerut
4. A text book on Hydrodynamics by M.Ray, Students' Friends and Company Publishers, Agra.

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Unit-1

Fluid Dynamics is the science of treating the study of fluids in motion. By the term 'fluid' is meant a substance that flows. Fluids may be divided into two kinds-(1) Liquids which are incompressible, i.e. their volumes do not change when the pressure changes, and (2) gases which are compressible fluids suffering change in volume whenever the pressure changes. The term **'hydrodynamics'** (the term was introduced by Daniel Bernoulli (1700-1783)) is often applied to the science of moving incompressible fluids i.e. liquids.

When matter is subjected to examination on microscopic or molecular scale, it is found to consist of moleculesin random motion and separated from one another by distances which are at least comparable with molecular size. In the case of gases, the separation distances are great: in the case of liquids, they are less great and in the case of solids even less so.

For the purpose of macroscopic analysis, however, the molecular structure of matter is, in general, of no interest. It is thus more convenient to treat the fluid as having continuous structure so that at each point we can prescribe a unique velocity, a unique pressure, a unique density, etc. Moreover, for a continuous or ideal fluid we can define a **'fluid particle'** as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

Not so many years ago the dynamics of a frictionless fluid had come to be regarded as an academic subject incapable of practical application owing to the great discrepency between calculated and observed results. The ultimate recognition, however, that Lanchester's theory of circulation in a perfect fluid could explain the lift of an aerofoil, and the adoption of Prandtl's hypothesis that outside the boundary layer the effect of viscosity is negligible, gave a fresh impetus to the subject which has always been necessary to the naval architect and which the advent of modern aeroplane has placed in the front rank.

Historical milestones

The term 'hydrodynamics' was introduced by **Daniel Bernoulli(1700-1783)** to comprise the two sciences of hydrostatics and hydraulics. He also discovered the famous theorem still known by his name.

d'Alembert(1717-1783) investigated resistance, discovered the paradox associated with his name, and introduced the principle of conservation of mass (equation of continuity) in a liquid.

Euler(1707-1783) formed the equations of motion of a perfect fluid and developed the mathematical theory. This work was continued by **Lagrange(1736-1813)**.

Navier(1785-1836) derived the equations of motion of a viscous fluid from certain hypothesis of molecular interaction.

Stokes(1819-1903) also obtained the equations of motion of a viscous fluid. He may be regarded as having founded the modern theory of hydrodynamics.

Rankine (1820-1872) developed the theory of sources and sinks.

Helmholtz (1821-1894) introduced the term 'velocity potential', founded the theory of vortex motion, and discontinuous motion, making fundamental contributions to the subject.

Kirchoff (1824-1887) and **Rayleigh (1842-1919)** continued the study of discontinuous motion and the resistance due to it.

Osborne Reynolds (1842-1912) studied the motion of viscous fluids, introduced the concepts of laminar and turbulent flow, and pointed out the abrupt transition from one to the other.

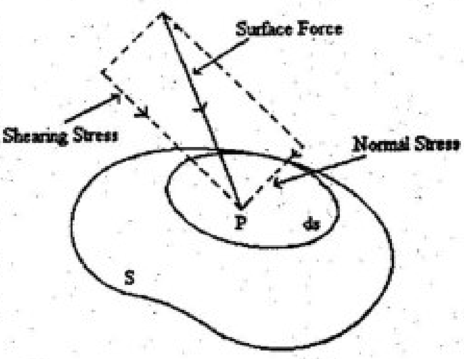
Joukowski (1847-1921) made outstanding contributions to aerofoil design and theory, and introduced the aerofoils known by his name.

Lanchester(1868-1945) made two fundamental contributions to the modern theory of flight; (i) the idea of circulation as the cause of lift, (ii) the idea of tip vortices as the cause of induced drag. He explained his theories to the Birmingham Natural History Society in 1894 but did not publish them till 1907 in his 'Aerodynamics'

Viscous (or real) and Inviscid (non-viscous, frictionless, perfect or ideal) Fluids; Viscosity :

An infinitestimal fluid element is acted upon by two types of forces, namely, **body forces** and **surface forces**. The former is a type of force which is proportional to the mass (or the volume) of the body on which it acts while the latter is one which acts on a surface element and is proportional to the surface area.

Suppose that the fluid element be enclosed by the surface S. Let P be an arbitrary point of S and let dS be the surface element around P. Then the surface force on dS is, in general not in the direction of normal at P to dS. Hence the force may be resolved into components, one normal and the other tangential to the area dS. The normal force per unit area is said to be the **normal stress** or **presure** while the tangential force per unit area is aid to be the **shearing stress.**



A fluid is said to be **viscous** when normal as well as shearing stresses exist. On the other hand, a fluid is to be **inviscid** when it does not exert any shearing stress, whether at rest or in motion. Clearly the presure exerted by an inviscid fluid on any surface is always along the normal to the urface at that point. Due to shearing stress a viscous fluid produces resistance to the body moving through it as well as between the particles of the fluid itself. Water and air are treated inviscid fluids whereas syrup and hevy oil are treated as viscous fluids.

We know that the flow of water and air is much easier than syrup and heavy oil. This demonstrates the existence of a property in the fluid, which controls its rate of flow. This property of fluids is said to be **viscosity** or **internal friction.**

Some Important types of flow :

(i) Laminar (streamline) and Turbulent flows :

A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be **laminar**. On the other hand, a flow, in which each fluid particle does not trace our a definite curve and the curves traced out by fluid particles intersect, is said to be **turbulent**.

(ii) Steady and Unsteady Flows :

A flow, in which properties and conditions (P, say) associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with the time, is said to be **steady**.

Mathematically, we may write $\frac{\partial P}{\partial t} = 0$. Here P may be velocity, density, pressure, temperature etc. On the other hand, a flow, in which properties and conditions associated with the motion of the fluid depend on the time so that the flow pattern varies with time, is said to be **unsteady**.

(iii) Uniform and Non-uniform Flows :

A flow, in which the fluid particles posses equal velocities at each section of the channel or pipe is called **uniform**. On the other hand, a flow, in which the fluid particles posses different velocities at each section of the channel or pipe is called **non-uniform.** These terms are usually used in connection with flow in channels.

(iv) Rotational and Irrotational Flows :

A flow, in which the fluid particles go on rotating about their own axes, while flowing, is said to be **rotational**. On the other hand, a flow in which the fluid particles do not rotate about their own axes, while flowing, is said to be **irrotational**.

(v)Barotropic Flow :

The flow is aid to be barotropic when the pressure is a function of the density.

Methods of describing Fluid Motion :

There are two methods for studying fluid motion mathematically. These are **Lagrangian** and **Eulerian (Flux)** methods and refer to **'individual time-rate of change'** and **'local time rate of change'** respectively.

(I) Lagrangian Method :

In this method we study the history of each fluid particle, i.e., any fluid particle is selected and is pursued on its onward course observing the changes in velocity, presure and density at each point and at each instant. Let (x_0, y_0, z_0) be the coordinates of the choosen particle at a given time $t = t_0$. At a later time, $t = t$, let the co-ordinates of the same particle be (x, y, z). Since the choosen particle is any particle in the fluid, the coordinates (x, y, z) will be functions of t and also of their vlues (x_0, y_0, z_0), so that

$$x = f_1(x_0, y_0, z_0, t),\ y = f_2(x_0, y_0, z_0, t),\ z = f_3(x_0, y_0, z_0, t). \quad \text{......(1)}$$

Let u, v, w and a_x, a_y, a_z be the components of velocity and accleration respectivly. Then we have

$$u = \frac{\partial x}{\partial t}, v = \frac{\partial y}{\partial t}, w = \frac{\partial z}{\partial t} \quad \text{......(2)}$$

and
$$a_x = \frac{\partial^2 x}{\partial t^2}, a_y = \frac{\partial^2 y}{\partial t^2}, a_z = \frac{\partial^2 z}{\partial t^2} \quad \text{......(3)}$$

Remark 1. :

The fundamental equations of motion in Lagrangian form are non-linear and hence it leads to many difficulties while solving a problem. In fact, the present method is employed with an advantage only in some one-dimensional (involving one space coordinate) problems. Hence we need to think another method of describing fluid motion.

Remark 2. :

This method resembles that of dynamics of a particle in so far as (x, y, z) are dependent on t. However, in Lagrangian method of fluid dynamics (x, y, z) are dependent on four independent vriables x_0, y_0, z_0, t.

(II) Eulerian Method :

In this method we select any point fixed in space occupied by the fluid and study the changes which take place in velocity, presure and density as the fluid passes through this point. Let u, v, w be the components of velocity at the point (x, y, z) at time t. Then we have

$$u = F_1(x, y, z, t),\ v = F_2(x, y, z, t),\ w = F_3(x, y, z, t) \quad \text{......(4)}$$

For a particular value of t, (4) exhibits the motion at all points in the fluid at that time. Again for a particular point (x, y, z), u, v, w are functions of t, which define the mode of variations of velocity at that point.

Remark :

The point under consideration being fixed, x, y, z and t are independent variables and hence

$\frac{dx}{dt}, \frac{d^2x}{dt^2}$ etc. have no meaning in this method.

Relationship between the Lagrangian and Eulerian Methods :

In order to establish a relation between the two methods, we investigate a relation between the particle parameters and space parameters.

(i) Lagrangian to Eulerian :

Suppose $\phi(x_0, y_0, z_0, t)$ be some physical quantity involving Lagrangian description

$$\phi = \phi(x_0, y_0, z_0, t) \quad ......(5)$$

Since Lagrangian description is given, (1) holds.Solving (1) for x_0, y_0, z_0, we have

$$x_0 = g_1(x, y, z, t), y_0 = g_2(x, y, z, t), z_0 = g_3(x, y, z, t) \quad ......(6)$$

Using (6), (5) reduces to

$$\phi = \phi[g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t), t] \quad ......(7)$$

which express ϕ in terms of Eulerian desription.

(ii) Eulerian to Lagrangian :

Suppose $\Psi(x, y, z, t)$ be some physical quantity involving Eulerian description

$$\Psi = \Psi(x, y, z, t) \quad ......(8)$$

Since Eulerian description is given, (4) holds. Again, (2) holds for the proposed Lagrangian description. Hence (2) and (4) yield

$$\frac{\partial x}{\partial t} = F_1(x, y, z, t), \frac{\partial y}{\partial t} = F_2(x, y, z, t), \frac{\partial z}{\partial t} = F_3(x, y, z, t) \quad ...(9)$$

The integration of (9) involves three consants of integration which may be taken as initial coordinates x_0, y_0, z_0 of the fluid prticle. Thus the integration of (9) leads to the well known equations of Lagrange (1). Using (1), (8) reduces to

$$\Psi = \Psi[f_1(x_0, y_0, z_0, t), f_2(x_0, y_0, z_0, t), f_3(x_0, y_0, z_0, t), t] \quad ...(10)$$

which express Ψ in terms of Lagrangian description.

Exercise 1. :

The velocity components for a two-dimensional fluid system can be given in the Eulerian system by $u = 2x + 2y + 3t,\ v = x + y + \frac{1}{2}t$. Find the displacement of a fluid particle in the Lagrangian system.

Solution :

Given $u = 2x + 2y + 3t,\ v = x + y + \frac{1}{2}t \quad ......(1)$

In terms of the displacement x and y, the velocity components u an v may also be represented by

$$u = \frac{dx}{dt}, v = \frac{dy}{dt} \quad ......(2)$$

From (1) and (2), we have

$$\frac{dx}{dt} = 2x + 2y + 3t, \frac{dy}{dt} = x + y + \frac{1}{2}t \quad ......(3)$$

Let $D \equiv \frac{d}{dt}$. The equations (3) becomes

$$(D-2)x - 2y = 3t \qquad \text{......(4)}$$

$$-x + (D+1)y = \frac{1}{2}t \qquad \text{......(5)}$$

Operating (5) by (D - 2), we have

$$-(D-2)x + (D-2)(D-1)y = \frac{1}{2}(D-2)t$$

or $$-(D-2)x + (D^2 - 3D + 2)y = \frac{1}{2} - t \qquad \text{......(6)}$$

Adding (4) and (6), we have

$$(D^2 - 3D)y = \frac{1}{2} + 2t \qquad \text{......(7)}$$

Auxiliary equation of (7) is $D^2 - 3D = 0$. Solving for D, it gives D = 0, 3. Hence complementary function (C.F.) is given by

$$\text{C.F.} = c_1 + c_2 e^{3t}$$

Next, the particular integral (P.I.) is given by

$$\text{P.I.} = \frac{1}{D^2 - 3D}\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left(1 - \frac{1}{3}D\right)^{-1}\left(\frac{1}{2} + 2t\right) = -\frac{1}{3D}\left(1 + \frac{1}{3}D + \dots\right)\left(\frac{1}{2} + 2t\right)$$

$$= \frac{1}{3D}\left(\frac{1}{2} + 2t + \frac{1}{3} \times 2\right) \doteq -\frac{1}{3}\frac{1}{D}\left(2t + \frac{7}{6}\right)$$

$$= -\frac{1}{3}\left(2 \times \frac{t^2}{2} + \frac{7}{6} \times t\right) = -\frac{t^2}{3} - \frac{7t}{18}$$

Hence the general solution of (7) is

$$y = c_1 + c_2 e^{3t} - \left(\frac{1}{3}\right)t^2 - \left(\frac{7}{18}\right)t \qquad \text{......(8)}$$

From (8), $$\frac{dy}{dt} = 3c_2 e^{3t} - \left(\frac{2}{3}\right)t - \left(\frac{7}{18}\right) \qquad \text{......(9)}$$

Re-writing the second equation of (3), we get

$$x = \frac{dy}{dt} - y - \frac{1}{2}t \qquad \text{......(10)}$$

Putting the values of y and $\frac{dy}{dt}$ given by (8) and (9) in (10), we get

$$x = 3c_2e^{3t} - \frac{2}{3}t - \frac{7}{18} - c_1 - c_2e^{3t} + \frac{1}{3}t^2 + \frac{7}{18}t - \frac{1}{2}t$$

or $$x = -c_1 + 2c_2e^{3t} + \left(\frac{1}{3}\right)t^2 - \left(\frac{7}{9}\right)t - \left(\frac{7}{18}\right) \qquad ......(11)$$

We now use the following initial conditions :

$$x = x_0, y = y_0 \text{ when } t = t_0 = 0 \qquad ......(12)$$

Using (12), (8) and (11) reduce to

$$y_0 = c_1 + c_2 \text{ and } x_0 = -c_1 + 2c_2 - \left(\frac{7}{18}\right) \qquad ......(13)$$

Solving (13) for c_1 and c_2, we have

$$c_1 = \frac{2y_0 - x_0}{3} - \frac{7}{54},\; c_2 = \frac{x_0 + y_0}{3} + \frac{7}{54} \qquad ......(14)$$

Using (14), (11) and (8) give

$$x = \frac{1}{3}x_0 - \frac{2}{3}y_0 + \frac{1}{3}\left(2x_0 + 2y_0 + \frac{7}{9}\right)e^{3t} - \frac{7}{9}t + \frac{1}{3}t^2 - \frac{7}{27} \qquad ......(15)$$

$$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 + \frac{1}{3}\left(x_0 + y_0 + \frac{7}{18}\right)e^{3t} - \frac{7}{18}t - \frac{1}{3}t^2 - \frac{7}{54} \qquad ......(16)$$

(15) and (16)give the desired displacements x and y in the Lagrangian system involving the initial positions x_0 and y_0 and the time t.

Velocity of a fluid particle :

Let the fluid particle be at P at any time t and let it be at Q at time $t + \delta t$ such that

$$\overrightarrow{OP} = \bar{r} \text{ and } \overrightarrow{OQ} = \bar{r} + \overline{\delta r}.$$

Then in the interval δt the movement of the prticle is $\overline{PQ} = \overline{\delta r}$ and hence the prticle velocity $\bar{q}$ at P is

$$\bar{q} = \lim_{\delta t \to 0}\left(\frac{\overline{\delta r}}{\delta t}\right) = \frac{\overline{dr}}{dt},$$

assuming such a limit to exist uniquely. Taking the fluid as continuous, the above asumption is justifed. Clearly $\bar{q}$ is a function of $\bar{r}$ and t and hence it can be expresed as $\bar{q} = f(\bar{r}, t)$. If u, v, w are the components of $\bar{q}$ along the axes, we have $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$.

Material, local and convective derivatives :

Suppose a fluid particle moves form $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$.Further suppose $f(x, y, z, t)$ be a scalar function asociated with some property of the fluid (e.g. the presure or density). Let the total change of f due to movement of the fluid particle from P to Q be δf. Then we have,

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t$$

$$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial f}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial f}{\partial t} \qquad .....(1)$$

Let

$$\left.\begin{array}{l} \lim\limits_{\delta t\to 0}\dfrac{\delta f}{\delta t} = \dfrac{Df}{Dt} \text{ or } \dfrac{df}{dt},\ \lim\limits_{\delta t\to 0}\dfrac{\delta x}{\delta t} = \dfrac{dx}{dt} = u \\ \lim\limits_{\delta t\to 0}\dfrac{\delta y}{\delta t} = \dfrac{dy}{dt} = v \text{ and } \lim\limits_{\delta t\to 0}\dfrac{\delta z}{\delta t} = \dfrac{dz}{dt} = w \end{array}\right\} \qquad .....(2)$$

where $q = (u, v, w)$ is th velocity of the fluid particle at P. Making $\delta t \to 0$ and uing (2), (1) reduces to

$$\frac{Df}{Dt} = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \qquad ......(3)$$

But $\quad \bar{q} = u\hat{i} + v\hat{j} + w\hat{k} \qquad ......(4)$

and $\quad \nabla = \hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z} \qquad ......(5)$

From (4) and (5),

$$\bar{q}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad ......(6)$$

Using (6), (3) reduces to

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + (q.\nabla)f \qquad ......(7)$$

Again, suppose $\bar{g}(x, y, z, t)$ be a vector function associated with some property of the fluid (e.g. velocity etc.). Then proceeding as above, we have

$$\frac{D\bar{g}}{Dt} = \frac{\partial \bar{g}}{\partial t} + (\bar{q}\cdot\nabla)\bar{g} \qquad ......(8)$$

From (7) and (8), we have, for both scalar and vector functions

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \bar{q}\cdot\nabla \qquad ......(9)$$

$\dfrac{D}{Dt}\left(\text{or } \dfrac{d}{dt}\right)$ is called the **material (or particle or substantial) derivative**. It is also spoken of as **differentiation following the motion of the fluid.** The first term on R.H.S. of (9), namely $\frac{\partial}{\partial t}$, is called the **local derivative** and it is associated with time variation at a fixed position. The second term on R.H.S. of (9), namely $\bar{q}\cdot\nabla$, is called the **convetive derivative** and it is asociated with the change of a physical quantity f or $\bar{g}$ due to motion of the fluid particle.

Note : The operator $\frac{D}{Dt}$ signifies that we are calculating the rate of change of a physical quantity f or $\bar{g}$ associated with the same fluid particle as it moves about.

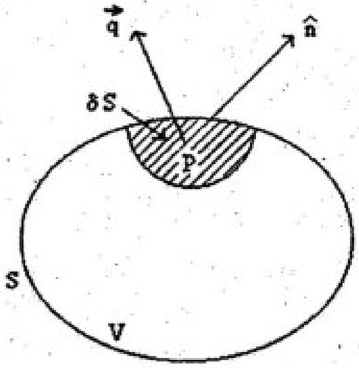
Significance of the Equation of Continuity, or conserversion of mass :

The law of conservation of mass states that fluid mass can neither be created nor destroyed. The equation of continuity axioms at expressing the law of conservtion of mass in a mathematical form.

Thus, in continuous motion, the equation of continuity expresses the fact that the increases in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the exces of the mass that flows in over the mas that flows out.

The equation of continuity (Vector form) by Euler's method :

Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space. Let P(x, y, z) be any point of S and let $\rho(x, y, z, t)$ be the fluid density at P at any time t. Let δS denote element of the surface S enclosing P. Let $\hat{n}$ be the unit outward-drawn normal at δS and let $\bar{q}$ be the fluid velocity at P. Then the normal component of q measured outwards from V is $\bar{n}.\bar{q}$. Thus,



Rate of mass flow across $\delta S = \rho(\hat{n}.\bar{q})\delta S$

$\therefore$ Total rate of mass flow across S

$$= \int_S \rho(\bar{n}.\bar{q})dS$$

$$= \int_V \nabla\cdot(\rho\bar{q})dV$$

(By Gauss divergence theorem)

$\therefore$ Total rate of mass flow into $V = -\int_V \nabla.(\rho\bar{q})dV$ ......(1)

Again, the mass of the fluid within S at time $t = \int_V \rho dV$.

$\therefore$ Total rate of mass increase within S

$$= \frac{\partial}{\partial t}\int_V \rho dV = \int_V \frac{\partial \rho}{\partial t}dV \quad ......(2)$$

Suppose that the region V of the fluid contains neither sources nor sinks (i.e. there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from (1) and (2), we have

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla \cdot (\rho \overline{q}) dV \quad \text{or} \quad \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{q}) \right] dV = 0$$

which holds for arbitrary small volumes V, if $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{q}) = 0$ ......(3)

Equation (3) is **called the equation of continuity**, or **the conservation of mass** and it holds at all points of fluid free from source and sinks.

Corollary 1. :

Since $\nabla \cdot (\rho \overline{q}) = \rho \nabla \cdot \overline{q} + \nabla \rho \cdot \overline{q}$, other forms of (3) are

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \overline{q} + \nabla \rho \cdot \overline{q} = 0 \qquad .....(4)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \overline{q} = 0 \qquad ......(5)$$

and $$\frac{d}{dt}(\log \rho) + \nabla \cdot \overline{q} = 0 \qquad ......(6)$$

Corollary 2. :

For an **incompressible** fluid the density of any fluid particle is invariable with time so that $\frac{D\rho}{Dt} = 0$. In such a fluid there could be a variation of ρ from particle to particle as in the case of a non-homogeneous and incompressible fluid. For a homogeneous and incompressible fluid ρ is constant throughout the entire field. In either case (5) shows that the equation of continuity is $\nabla . \overline{q} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ if $\overline{q} = u\hat{i} + v\hat{j} + w\hat{k}$. From now onwards, unless otherwise stated the term 'incompressible fluid' will be taken to imply one which is not only incompressible but also homogeneous. If in addition the flow is of **potential kind**, then there exists a velocity potential ϕ s.t. $\overline{q} = -\nabla \phi$. In that case the equation of continuity $\nabla . \overline{q} = 0$ reduces to $\nabla^2 \phi = 0$ which is **Laplac's equation.**

Corollary 3. :

If the motion is steady, then $\frac{\partial \rho}{\partial t} = 0$. Then (3) gives $\nabla \cdot (\rho \overline{q}) = 0$ and if ρ is constant then $\nabla \cdot \overline{q} = 0$.

From the above investigation it appears that a fluid can not move according to an arbitrary assigned law of distribution of velocity. For the motion to be possible it is evidently necessary that the equation of continuity should be satisfied. In particular, **possible irrotational motion of a liquid are subject to the condition that the velocity potential ϕ shall satisfy Laplac's equation.**

Equation of continuity in different coordinate systems:

The general form of the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{q}) = 0$$

Now, the expression of $\nabla \cdot (\rho \bar{q})$ in orthogonal system (λ, μ, ν) is given by

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \lambda}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial \mu}(\rho q_2 h_3 h_1) + \frac{\partial}{\partial \nu}(\rho q_3 h_1 h_2)\right]$$

where h_1, h_2, h_3 are the scale factors.

Hence $\nabla \cdot (\rho \bar{q})$ in cartesian, cylendrical and spherical systems becomes

In cartesian

$h_1 = 1, h_2 = 1, h_3 = 1$

$\lambda = x, \mu = y, \nu = z$

$q_1 = u, q_2 = v, q_3 = w$

Hence $$\nabla \cdot (\rho \bar{q}) = \frac{1}{1\cdot 1\cdot 1}\left[\frac{\partial}{\partial x}(\rho u \cdot 1 \cdot 1) + \frac{\partial}{\partial y}(\rho v \cdot 1 \cdot 1) + \frac{\partial}{\partial z}(\rho w \cdot 1 \cdot 1)\right]$$

$$= \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

In cylendrical (r, θ, z)

$h_1 = 1, h_2 = r, h_3 = 1$

$\lambda = r, \mu = \theta, \nu = z$

$q_1 = q_r, q_2 = q_\theta, q_3 = q_z$

Hence $$\nabla \cdot (\rho \bar{q}) = \frac{1}{1\cdot r\cdot 1}\left[\frac{\partial}{\partial r}(\rho \cdot q_r \cdot r \cdot 1) + \frac{\partial}{\partial \theta}(\rho \cdot q_\theta \cdot 1 \cdot 1) + \frac{\partial}{\partial z}(\rho \cdot q_z \cdot 1 \cdot r)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho r q_z)\right]$$

In spherical (r, θ, ϕ)

$h_1 = 1, h_2 = r, h_3 = r\sin\theta$

$\lambda = r, \mu = \theta, \nu = \phi$

$q_1 = q_r, q_2 = q_\theta, q_3 = q_\phi$

Hence $\nabla \cdot (\rho \bar{q}) = \frac{1}{1 \cdot r \cdot r \sin\theta}\left[\frac{\partial}{\partial r}(\rho \cdot q_r \cdot r \cdot r \sin\theta)\right.$

$$\left. + \frac{\partial}{\partial \theta}(\rho \cdot q_\theta \cdot r \sin\theta \cdot 1) + \frac{\partial}{\partial \phi}(\rho \cdot q_\phi \cdot 1 \cdot r)\right]$$

Hence the equation of continuity becomes

In cartesian

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

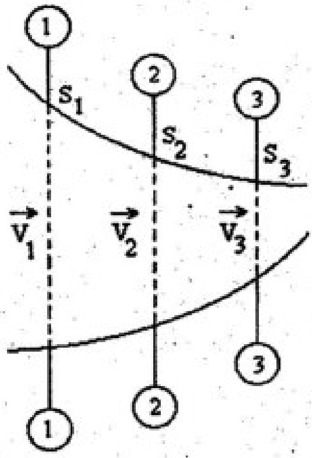
In cylendrical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho r q_z)\right] = 0$$

In spherical

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(\rho q_r r^2 \sin\theta\right) + \frac{\partial}{\partial \theta}\left(\rho r q_\theta \sin\theta\right) + \frac{\partial}{\partial \phi}\left(\rho r q_\phi\right)\right] = 0$$

Equation of Continuity of Luquid Flow Through a Channel or a Pipe :



Let an incompressible liquid continuously flow through a channel or pipe whose cross-sectional area may or may not be fixed. Then the quantity of liquid passing per second is the same at all sections.

Suppose some liquid is flowing through a tapering pipe as shown in figure. Let S_1, S_2, S_3 be the area of the pipe at sections 1-1, 2-2, 3-3 respectively. Further, let V_1, V_2 and V_3 be velocities of the liquid at sections 1-1, 2-2, 3-3 respectively. Let Q_1, Q_2 and Q_3 be the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 respectively. Then

$$Q_1 = S_1 V_1,\ Q_2 = S_2 V_2,\ Q_3 = S_3 V_3 \qquad \text{......(1)}$$

From the law of conservation of mass, the total quantity of liquid flowing across the sections 1-1, 2-2, 3-3 must be the same.

Hence $Q_1 = Q_2 = Q_3$ ... and so on.

Thus,

$S_1 V_1 = S_2 V_2 = S_3 V_3$ is the equation continuity.

Boundary Conditions (Kinematical) :

When fluid is in contact with a rigid solid surface (or with another unmixed fluid), the following boundary condition must be satisfied in order to maintain contact.

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.

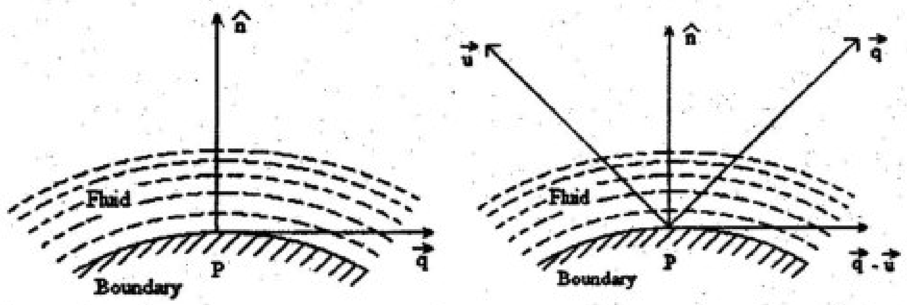
Let $\hat{n}$ denote a normal unit vector drawn at the point P of the surface of contact and let $\overline{q}$ denote the fluid velocity at P. When the rigid surface of contact is at rest, we must have $\overline{q}.\hat{n} = 0$ at each point of the surface. This expresses the condition that the normal velocities are both zero and hence the fluid velocity is tangential to the surface at its each point as shown in figure.

Next, let the rigid surface be in motion and let $\overline{u}$ be its velocity at P. Then we must have

$$\overline{q}.\hat{n} = \overline{u}.\hat{n}$$

or $$(\overline{q} - \overline{u}).\hat{n} = 0$$

which expresses the fact that there must be no normal velocity at P between boundary and fluid, that is, the velocity of the fluid relative to the boundary is tangential to the boundary at its each point.



Remark :

For inviscid fluid the above condition must be satisfied at the boundary. However, for viscous fluid (in which there is no slip), the fluid and the surface with which contact is maintained must also have the same tangential velocity at P.

Boundary Condition (Physical) :

The above mentioned kinematical boundary conditions must hold independently of any particular physical hypothesis.

Let S denote the surface of separation of two fluids (which do not mix). Then the following additional condition must be satisfied :

The pressure must be continuous at the boundary as we pass from one side of S to the other.

Boundary Surface :

We propose to derive the differential equation satisfied by a boundary surface of a fluid. Thus, we discuss the following problem :

To find the condition that the surface $F(\overline{r}, t) = 0$ or $F(x, y, z, t) = 0$ may be a boundary surface.

For figure, refer figure (ii).

Let P be a point on the moving boundary surface

$$F(\bar{r}, t) = 0 \qquad ......(1)$$

where the fluid velocity is $\bar{q}$ and the velocity of the surface is $\bar{u}$.

Now in order to preserve contact, the fluid and the surface with which contact is to be maintained must have the same velocity normal to the surface. Thus, we have

$$\bar{q}.\hat{n} = \bar{u}.\hat{n}$$

or $$(\bar{q} - \bar{u}).\hat{n} = 0 \qquad ......(2)$$

where $\bar{n}$ in the unit normal vector drawn at P on the boundary surface (1). We know that the direction ratios on $\hat{n}$ are $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$.

Again, $$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k} \qquad ......(3)$$

which shows that $\hat{n}$ and ∇F are parallel vectors and hence we may write $\hat{n} = k\nabla F$. With the value of $\hat{n}$, (2) reduces to

$$(\bar{q} - \bar{u}) \cdot k\nabla F = 0$$

so that $$\bar{q} \cdot \nabla F = \bar{u} \cdot \nabla F \qquad ......(4)$$

Let $P(\bar{r}, t)$ move to a point $Q(r + \delta\bar{r}, t + \delta t)$ on time δt. Then Q must satisfy the equation of the boundary surface (1), at time $t + \delta t$, namely

$$F(\bar{r} + \delta\bar{r}, t + \delta t) = 0$$

Expanding by Taylor's theorem, the above equation gives

$$F(\bar{r}, t) + \delta\bar{r} \cdot \nabla F + \delta t\left(\frac{\partial F}{\partial t}\right) = 0$$

or $$\frac{\partial F}{\partial t} + \frac{\delta\bar{r}}{\delta t} \cdot \nabla F = 0 \qquad \text{(using (1))} \qquad ......(5)$$

Proceeding to the limits as $\delta r \to 0$, $\delta t \to 0$ and noting that

$$\lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt} = u,$$

(5) gives $$\frac{\partial F}{\partial t} + \bar{u} \cdot \nabla F = 0 \qquad ......(6)$$

or $$\frac{\partial F}{\partial t} + \bar{q} \cdot \nabla F = 0 \qquad \text{(using (4))} \qquad ......(7)$$

which is the required condition for $F(\bar{r}, t)$ to be a boundary surface.

Remark 1. :

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$. Then (7) may be re-writtem as

$$\frac{\partial F}{\partial t} + \left(u\hat{i} + v\hat{j} + w\hat{k}\right) \cdot \left(\frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}\right) = 0$$

or $$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

or $$\frac{DF}{Dt} = 0 \qquad \text{......(8)}$$

(8) presents the required condition in cartesian co-ordinates for $F(x, y, z, t) = 0$ to be a boundary surface.

Remark 2. :

The normal velocity of the boundary

$$= \bar{u} \cdot \hat{n} = \bar{u} \cdot \frac{\nabla F}{|\nabla F|} = -\frac{\frac{\partial F}{\partial t}}{\left|\frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}\right|} \quad \text{(by (3) and (6))}$$

$$= \frac{-\frac{\partial F}{\partial t}}{\sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right\}}} \qquad \text{......(9)}$$

$$= \frac{u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}}{\sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right\}}} \quad \text{(by (8))} \qquad \text{......(10)}$$

Remark 3. :

When the boundary surface is at rest, then $\frac{\partial F}{\partial t} = 0$ and hence the condition (8) reduces to

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \qquad \text{......(11)}$$

Exercise 1. :

Show that the surface

$$\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1$$

is a possible form of boundary of a liquid at time t.

Solution :

The given is surface

$$F(x,y,z,t)=\frac{x^2}{a^2k^2t^4}+kt^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)-1=0 \qquad ......(1)$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\frac{\partial f}{\partial t}+u\left(\frac{\partial F}{\partial x}\right)+v\left(\frac{\partial F}{\partial y}\right)+w\left(\frac{\partial F}{\partial z}\right)=0 \qquad ......(2)$$

and the same values of u, v, w satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad ......(3)$$

From (1),

$$\frac{\partial F}{\partial t}=-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^4},\ \frac{\partial F}{\partial y}=\frac{2kt^2y}{b^2},\ \frac{\partial F}{\partial z}=\frac{2kt^2z}{c^2}$$

With these values, (2) reduces to

$$-\frac{4x^2}{a^2k^2t^5}+2kt\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+\frac{2xu}{a^2k^2t^4}+\frac{2kt^2yv}{b^2}+\frac{2kt^2zw}{c^2}=0$$

or $$\frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2kyt}{b^2}(y+vt)+\frac{2ktz}{c^2}(z+wt)=0$$

which is identically satisfied if we take

$$u=\frac{2x}{t},v=-\frac{y}{t},w=-\frac{z}{t} \qquad ......(4)$$

From (4),

$$\frac{\partial u}{\partial x}=\frac{2}{t},\frac{\partial v}{\partial y}=-\frac{1}{t},\frac{\partial w}{\partial z}=-\frac{1}{t} \qquad ......(5)$$

Using (5), we find that (3) is also satisfied by the above values of u, v and w. Hence (1) is a possible boundary surface with velocity components given in (4).

Exercise 2. :

Show that $\left(\frac{x^2}{a^2}\right)\tan^2 t+\left(\frac{y^2}{b^2}\right)\cot^2 t=1$ is a possible form for the boundary surface of a liquid, and find an expression for the normal velocity.

Solution :

For the present two dimensional motion ($\frac{\partial F}{\partial z}=0$ and $\frac{\partial w}{\partial z}=0$), the surface

$$F(x,y,t)=\left(\frac{x^2}{a^2}\right)\tan^2 t+\left(\frac{y^2}{b^2}\right)\cot^2 t-1=0 \qquad \text{.......(1)}$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\frac{\partial F}{\partial t}+u\left(\frac{\partial F}{\partial x}\right)+v\left(\frac{\partial F}{\partial y}\right)=0 \qquad \text{......(2)}$$

and the same values of u and v satisfies the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{......(3)}$$

From (1), $\frac{\partial F}{\partial t}=\frac{x^2}{a^2}2\tan t\sec^2 t-\frac{y^2}{b^2}2\cot t\,\mathrm{cosec}^2 t$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}\tan^2 t,\quad \frac{\partial F}{\partial y}=\frac{2y}{b^2}\cot^2 t$$

With these values, (2) reduces to

$$\frac{x\tan t}{a^2}\left(x\sec^2 t+u\tan t\right)+\frac{y\cot t}{b^2}\left(-y\,\mathrm{cosec}^2 t+v\cot t\right)=0$$

which is identically satisfied if we take

$$x\sec^2 t+u\tan t=0 \quad\text{and}\quad -y\,\mathrm{cosec}^2 t+v\cot t=0$$

i.e., $$u=-\frac{x}{\sin t\cos t}\quad\text{and}\quad v=\frac{y}{\sin t\cos t} \qquad \text{....(4)}$$

From (4), $$\frac{\partial u}{\partial x}=\frac{-1}{\sin t\cos t}\quad\text{and}\quad\frac{\partial v}{\partial y}=\frac{1}{\sin t\cos t} \qquad \text{.....(5)}$$

Using (5), we find that (3) is also satisfied by the above values of u and v. Hence (1) is a possible bounding surface ith velocity components given in (4).

Using result (equation 10) of page 22 (with $\frac{\partial F}{\partial z}=0$ here), the normal velocity

$$= \frac{u\left(\frac{\partial F}{\partial x}\right) + v\left(\frac{\partial F}{\partial y}\right)}{\sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right\}}}$$

$$= \frac{-\frac{x}{\sin t \cos t}\frac{2x\tan^2 t}{a^2} + \frac{y}{\sin t \cos t}\frac{2y\cot^2 t}{b^2}}{\left\{\left(\frac{2x\tan^2 t}{a^2}\right)^2 + \left(\frac{2y\cot^2 t}{b^2}\right)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{a^2 y^2 \cot t \operatorname{cosec}^2 t - b^2 x^2 \tan t \sec^2 t}{\sqrt{\left(x^2 b^4 \tan^4 t + y^2 a^4 \cot^4 t\right)}}$$

Stream line or line of flow :

A stream line is a curve drawn in the fluid so that its tangent at each point is in the direction of motion (i.e.fluid velocity) at that point.

Let $\bar{r}$ be the position vector of a point P on a stream line and let $\bar{q}$ be the fluid velocity at P. Then $\bar{q}$ is parallel to $d\bar{r}$ at P on the stream line. Thus, the equation of stream lines is given by

$$\bar{q} \times d\bar{r} = 0 \qquad \text{(i)}$$

i.e. $\left(q_1\hat{e}_1 + q_2\hat{e}_2 + q_3\hat{e}_3\right) \times \left(h_1\hat{e}_1 d\lambda + h_2\hat{e}_2 d\mu + h_3\hat{e}_3 d\nu\right) = 0$

where

$\bar{q} = q_1\hat{e}_1 + q_2\hat{e}_2 + q_3\hat{e}_3$ and $d\bar{r} = h_1\hat{e}_1 d\lambda + h_2\hat{e}_2 d\mu + h_3\hat{e}_3 d\nu$ in orthogonal system (λ, μ, ν)

$$\Rightarrow (q_2h_3d\nu - q_3h_2d\mu)\hat{e}_1 + (q_3h_1d\lambda - q_1h_3d\nu)\hat{e}_2 + (q_1h_2d\mu - q_2h_1d\lambda)\hat{e}_3 = 0$$

$$\Rightarrow q_2h_3d\nu - q_3h_2d\mu = 0 \Rightarrow \frac{h_3 d\nu}{q_3} = \frac{h_2 d\mu}{q_2}; \quad \text{and} \quad q_3h_1d\lambda - q_1h_3d\nu = 0 \Rightarrow \frac{h_3 d\nu}{q_3} = \frac{h_1 d\lambda}{q_1}$$

combining the above two we get $\frac{h_1 d\lambda}{q_1} = \frac{h_2 d\mu}{q_2} = \frac{h_3 d\nu}{q_3}$ (ii)

Stream line in different coordinate systems: From (ii) we can deduce the equations for stream line in different coordinate systems.

In cartesian system $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

In cylindrical system $\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{dz}{q_z}$

In cylindrical system $\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{dz}{q_z}$

In cylindrical system $\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\phi}{q_\phi}$

The equations (ii) have a double infinite set of solutions. Through each point of the flow field where u(x, y, z, t), v(x, y, z, t) and w(x, y, z, t) do not all vanish, there passes one and only one stream line at a given instant. This fact can be verified by employing the well known existence theorem for the system of equations (2). If the velocity vanishes at a given point, various singularities occure there. Such a point is known as a **critical point or stagnation point.**

Path line or path of a particle :

A path line is the curve or trajectory along which a particular fluid particle travels during its motion. The differential equations of path lines are

$$\frac{d\bar{r}}{dt} = \bar{q} \qquad \text{......(i)}$$

i.e.,
$$\frac{dx}{dt} = u, \ \frac{dy}{dt} = v, \ \frac{dz}{dt} = w \qquad \text{......(2)}$$

where $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ and $\bar{r} = x\hat{i} + y\hat{j} + w\hat{k}$.

Remark :

Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then the path line is determined from equations

$$\left.\begin{aligned} \frac{dx}{dt} &= u(x, y, z, t) \\ \frac{dy}{dt} &= v(x, y, z, t) \\ \frac{dz}{dt} &= w(x, y, z, t) \end{aligned}\right\} \qquad \text{......(3)}$$

with initial conditions

$$x(t_0) = x_0, \ y(t_0) = y_0, \ z(t_0) = z_0 \qquad \text{......(4)}$$

Difference between the stream lines and path lines :

It is important to note that stream lines are not, in general, the same as the path lines. Stream lines show how **each particle** is moving **at a given instant of time** while the path lines presents the motion of **the particles at each instant**. Except in the case of steady motion, u, v, w are always functions of the time and hence the stream lines go on changing with the time, and the actual path of any fluid particle will not in general coincide with a stream line.

Stream tube (or tube of flow) and stream filament :

If we draw the stream lines from each point of a closed curve in the fluid, we obtain a tube called the stream tube.

A stream tube of infinitestimal cross-section is known as a stream filament.

Remark :

Since there is no movement of fluid across a stream line, no fluid can enter or leave the stream tube except at the ends. So in the case of the steady motion, a stream tube behaves like an actual solid tube through which the fluid is flowing. Due to steady flow, the walls of the tube are fixed in space and hence the motion through the stream tube would remain unchanged on replacing the walls of the tube by a rigid boundary.

Exercise :

The velocity components in a three-dimensional flow field for an incompresible fluid are (2x, - y, - z). Is it a posible field? Determine the equations of the stream line passing through the point (1, 1, 1).

Solution :

Here $u = 2x$, $v = -y$, $w = -z$.

Stream lines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

i.e., $$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z} \qquad \text{......(1)}$$

Taking the first two members of (1), we have

$$\frac{dx}{2x} = \frac{dy}{-y} \qquad \text{or} \qquad \frac{dx}{x} + 2\frac{dy}{y} = 0$$

Integrating, $\log x + 2\log y = \log c_1$ or $xy^2 = c_1$ ......(2)

Again, taking the first and third members of (1) and proceeding as above, we get

$$xz^2 = c_2 \qquad \text{......(3)}$$

Here c_1 and c_2 are arbitrary constants. The stream lines are given by the curves of intersection of (2) and (3). The required stream line passes through (1, 1, 1) so that $c_1 = 1$ and $c_2 = 1$. Thus, the desired stream line is given by the intersection of $xy^2 = 1$ and $xz^2 = 1$.

We also have

$$\frac{\partial u}{\partial x} = 2,\ \frac{\partial v}{\partial y} = -1,\ \frac{\partial w}{\partial z} = -1$$

so that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$,

showing that the equation of continuity is satisfied for the given flow field for an incompresible fluid. Hence the given **velocity components correspond to a possible field.**

Exercise 2. :

Find the strean lines and paths of the particles when

$$u = \frac{x}{(1+t)},\ v = \frac{y}{(1+t)},\ w = \frac{z}{(1+t)}$$

Solution :

Stream lines are given by $\frac{dx}{du} = \frac{dy}{dv} = \frac{dz}{dw}$

i.e., $$\frac{dx}{\frac{x}{(1+t)}} = \frac{dy}{\frac{y}{(1+t)}} = \frac{dz}{\frac{z}{(1+t)}}$$

i.e., $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \quad \text{.......(1)}$$

Taking the first two member of (1), $\frac{x}{y} = c_1$ ......(2)

Taking the last two member of (1), $\frac{y}{z} = c_2$ ......(3)

The desired stream lines are given by the intersection of (2) and (3).

The paths of the particles are given by

$$\frac{dx}{dt} = u,\ \frac{dy}{dt} = v,\ \frac{dz}{dt} = w$$

i.e, $$\frac{dx}{dt} = \frac{x}{(1+t)},\ \frac{dy}{dt} = \frac{y}{(1+t)},\ \frac{dz}{dt} = \frac{z}{(1+t)}$$

giving $$\frac{dx}{x} = \frac{dt}{1+t},\ \frac{dy}{y} = \frac{dt}{1+t},\ \frac{dz}{z} = \frac{dt}{1+t}$$

Integrating,

$$x = c_1(1+t),\ y = c_2(1+t),\ z = c_3(1+t),$$

which give the desired paths of the particles.

The Velocity Potential or Velocity Function :

Suppose that the fluid velocity at time t is $\overline{q} = (u, v, w)$. Further suppose that at the considered instant t, there exists a scalar function $\phi(x, y, z, t)$, uniform throughout the entire field of flow and such that

$$-d\phi = udx + vdy + wdz \qquad ......(1)$$

i.e.,
$$-\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = udx + vdy + wdz \qquad ......(2)$$

Then the expression on the R.H.S. of (1) is an exact differential and we have

$$u = -\frac{\partial\phi}{\partial x}, v = -\frac{\partial\phi}{\partial y}, w = -\frac{\partial\phi}{\partial z} \qquad ......(3)$$

$$\therefore \quad \overline{q} = -\nabla\phi = -\text{grad}\ \phi \qquad ......(4)$$

ϕ is called the **velocity potential**. The negative sign in (4) is a convention. It ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for (4) to hold is

$$\nabla \times \overline{q} = 0 \quad \text{i.e.,} \quad \text{curl}\ q = 0 \qquad ......(5)$$

or
$$i\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + j\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + k\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad ......(6)$$

Relation between velocity components and the velocity potential :

From equation (4) we have $\overline{q} = -\nabla\phi$. Writing the equation in orthogonal system we get

$$q_1\hat{e}_1 + q_2\hat{e}_2 + q_3\hat{e}_3 = -\nabla\phi = -\left[\frac{\hat{e}_1}{h_1}\frac{\partial\phi}{\partial\lambda} + \frac{\hat{e}_2}{h_2}\frac{\partial\phi}{\partial\mu} + \frac{\hat{e}_3}{h_3}\frac{\partial\phi}{\partial\nu}\right]$$

Comparing the coefficients of $\hat{e}_1, \hat{e}_2, \hat{e}_3$ we get

$$q_1 = -\frac{1}{h_1}\frac{\partial\phi}{\partial\lambda}, q_2 = -\frac{1}{h_2}\frac{\partial\phi}{\partial\mu}, q_3 = -\frac{1}{h_3}\frac{\partial\phi}{\partial\nu}.$$

They give the relationship between the velocity components and the velocity potential. Using the above equations we can find the relationship between the velocity components and the velocity potential in all the coordinate systems.

Thus,

In cartesian system $u = -\dfrac{\partial\phi}{\partial x}, v = -\dfrac{\partial\phi}{\partial y}, w = -\dfrac{\partial\phi}{\partial z}.$

In cylindrical system $q_r = -\frac{\partial\phi}{\partial r}, q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, q_z = -\frac{\partial\phi}{\partial z}$.

In spherical system $q_r = -\frac{\partial\phi}{\partial r}, q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, q_\omega = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega}$

Remark 1. :

The surface

$$\phi(x, y, z, t) = \text{constant} \quad \text{......(7)}$$

are called the **equipotentials**. The stream lines

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{......(8)}$$

are cut at right angles by the surfaces given by the differential equation

$$udx + vdy + wdz = 0 \quad \text{......(9)}$$

and the condition for the existence of such orthogonal surface is the condition that (9) may posses a solution of the form (7) at the considered instant t, the analytical condition being

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad \text{......(10)}$$

When the velocity potential exists, (3) holds. Then

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -\frac{\partial^2\phi}{\partial y\partial z} + \frac{\partial^2\phi}{\partial z\partial y} = 0$$

i.e., $$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad \text{......(11)}$$

Similarly,

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{......(12)}$$

Using (11) and (12), we find that the condition (10) is satisfied. Hence surfaces exist which cut the stream lines orthogonally. We also conclude that all points of field of flow, the equipotentials are cut orthogonally by the stream lines.

Remark 2. :

When (5) holds, the flow is known as the **potential kind**. It is also known as **irrototional**. For such flow the field of $\overline{q}$ is conservative.

Remark 3. :

The equation of continuity of an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{......(13)}$$

Suppose that the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\frac{\partial \phi}{\partial x}, v = -\frac{\partial \phi}{\partial y}, w = -\frac{\partial \phi}{\partial z} \qquad ......(14)$$

Using (14), (13) reduces to

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad ......(15)$$

showing that ϕ is a **harmonic function** satisfying the **Laplace equation** $\nabla^2\phi = 0$, where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad ......(16)$$

The Vorticity vector :

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid veclocity such that curl $\bar{q} \neq 0$. Then the vector

$$\bar{\Omega} = \text{curl}\,\bar{q} \qquad ......(1)$$

is called the **vorticity vector**.

Let $\Omega_x, \Omega_y, \Omega_z$, be the components of Ω in cartesian co-ordinates.

The (1) reduces to

$$\Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k} = \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

so that $\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Note : Some authors use ξ, η, ζ, for $\Omega_x, \Omega_y, \Omega_z$ and define $\bar{\Omega} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k} = \frac{1}{2}\text{curl}\,\bar{q}$. Thus, we have

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Note: Writing the equation $\bar{\Omega} = \text{curl}\,\bar{q}$ in orthogonal system we can find the components in all the three coordinate systems.

Vortex Line :

A vortex line is a curve drawn in the fluid such that the tangent to it every point is in the direction of the vorticity vector Ω.

Let $\bar{\Omega} = \Omega_x\hat{i} + \Omega_y\hat{j} + \Omega_z\hat{k}$ and let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a vortex line. Then Ω is parallel to $d\bar{r}$ at P on the vortex line. Hence the equation of vortex lines is given by

$$\bar{\Omega}\times d\bar{r}=0$$

i.e. $$\left(\Omega_x\hat{i}+\Omega_y\hat{j}+\Omega_z\hat{k}\right)\times\left(dx\hat{i}+dy\hat{j}+dz\hat{k}\right)=0$$

or $$\left(\Omega_y dz-\Omega_z dy\right)\hat{i}+\left(\Omega_z dx-\Omega_x dz\right)\hat{j}+\left(\Omega_x dy-\Omega_y dx\right)\hat{k}=0$$

whence $\Omega_y dz-\Omega_z dy=0,\ \Omega_z dx-\Omega_x dz=0,\ \Omega_x dy-\Omega_y dx=0$

so that $$\frac{dx}{\Omega_x}=\frac{dy}{\Omega_y}=\frac{dz}{\Omega_z} \qquad .....(1)$$

Note: Writing the equation $\bar{\Omega}\times d\bar{r}=0$ in orthogonal system we can find the equations in all the three coordinate systems.

Vortex Tube and Vortex Filament :

If we draw the vortex lines from each point of a closed curve in the fluid, we obtain a tube called the **vortex tube.**

A vortex tube of infinitestmal cross-section is known as **vortex filament** or simply a **vortex.**

Remark :

It wil be shown that vortex lines and tubes cannot originate or terminate at internal points in a fluid. They can only form closed curves or terminate on boundaries.

Rotational and Irrotational Motion :

The motion of a fluid is said to be **irrotational** when the vorticity vector Ω of every fluid particle is zero. When the vorticity vector is different from zero, the motion is said to be **rotational.**

Since $\bar{\Omega}=\text{curl}\ \bar{q}$

$$\Rightarrow\bar{\Omega}=\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)i+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)j+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)k,$$

we conclude that the motion is irrotational if

$$\text{curl}\ \bar{q}=0$$

or $$\frac{\partial w}{\partial y}=\frac{\partial v}{\partial z},\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x},\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}.$$

When the motion is irrotational i.e., when $\bar{\Omega}=\text{curl}\ \bar{q}=0$, then $\bar{q}$ must be of the form (- grad ϕ) for some scalar point function ϕ (say) because curl grad $\phi=0$. **Thus velocity potential exists whenever the fluid motion is irrotational. Again notice that when velocity potential exists, the motion is irrotational because** $\bar{q}=-\nabla\phi\Rightarrow$ curl q = - curl grad $\phi=0$.

Rotational motion is also said to be **vortex motion.** Again by definition it follows that there are no vortex lines in an irotational fluid motion.

Exercise : Test whether the motion specified by $\bar{q} = \frac{k^2(x\hat{j} - y\hat{i})}{x^2 + y^2}$ (k = constant),

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Solution :

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$. Then, here

$$u = \frac{-k^2 y}{x^2 + y^2}, \quad v = \frac{k^2 x}{x^2 + y^2}, w = 0 \qquad \text{......(1)}$$

The equation of continuity for an incompresible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{......(2)}$$

From (1), $\frac{\partial u}{\partial x} = \frac{2k^2 xy}{(x^2 + y^2)^2}, \frac{\partial v}{\partial y} = -\frac{2k^2 xy}{(x^2 + y^2)^2}, \frac{\partial w}{\partial z} = 0$

Hence (2) satisfied and so the motion specified by given $\bar{q}$ is possible.

The equations of the stream lines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{i.e.,} \quad \frac{dx}{-\frac{k^2 y}{(x^2 + y^2)}} = \frac{dy}{\frac{k^2 x}{(x^2 + y^2)}} = \frac{dz}{0} \qquad \text{......(3)}$$

Taking the last fraction, $dz = 0$ so that $z = c_1$. ......(4)

Taking the first two fractions simplifying, we get

$$\frac{dx}{(-y)} = \frac{dy}{x} \quad \text{or} \quad 2x\,dx + 2y\,dy = 0$$

Integrating, $x^2 + y^2 = c_2$ ......(5)

(4) and (5) together give the stream lines. Clearly, the stream lines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

$$\text{curl}\,q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{k^2 y}{x^2 + y^2} & \frac{k^2 x}{x^2 + y^2} & 0 \end{vmatrix}$$

$$= k^2 \left\{ \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \right\} \hat{k} = 0$$

Hence the flow is of the potential kind and we can find velocity potential $\phi(x, y, z)$ such that $q = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \qquad ......(6)$$

$$\frac{\partial\phi}{\partial y} = -v = \frac{k^2 x}{x^2+y^2} \qquad ......(7)$$

$$\frac{\partial\phi}{\partial z} = -w = 0 \qquad ......(8)$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x, y)$.

Integratiag (6),

$$\phi(x, y) = k^2 \tan^{-1}\left(\frac{x}{y}\right) + f(y) \qquad ......(9)$$

when f(y) in an arbitrary function of y.

From (9), $$\frac{\partial\phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2} \qquad ......(10)$$

Comparing (7) and (10) we have

$f'(y) = 0$ so that $f(y) = \text{constant}$.

Since the constant can be omitted while writing velocity potential, the required velocity potential cn be taken as

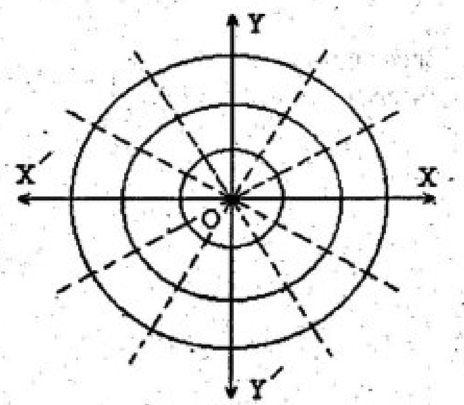
$$\phi(x, y) = k^2 \tan^{-1}\left(\frac{x}{y}\right) \qquad \text{(using (9))} \qquad ......(11)$$

The equipotentials are given by

$$k^2 \tan^{-1}\left(\frac{x}{y}\right) = \text{cons tan t}$$

or $\quad x = cy,$

which are planes through the z-axis. They are intersected by the stream lines as shown in the figure. Dotted lines represent equipotentials and ordinary lines represent stream lines.



Exercise :

At a point in an incompressible fluid having spherical polar co-ordinates (r, θ, ϕ), the velocity components are $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, where M is a constant. Show that the velocity is of the potential kind. Find the velocity potential and the equations of the stream lines.

Solution :

Here $q_r = 2Mr^{-3}\cos\theta$, $q_\theta = Mr^{-3}\sin\theta$, $q_\phi = 0$.

Then, we have

$$q = 2Mr^{-3}\cos\theta\,\hat{r} + Mr^{-3}\sin\theta\,\hat{\theta} + 0.(\hat{\phi})$$

and $$\text{curl}\,q = \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ q_r & q_\theta & q_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin^2\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 2Mr^{-3}\cos\theta & Mr^{-3}\sin\theta & 0 \end{vmatrix}$$

$= 0$ (on simplification).

Hence the flow is of the potential kind. We have used F for velocity potential to avoid confusion. Then by definition

$$-\frac{\partial F}{\partial r} = q_r = 2Mr^{-3}\cos\theta, -\frac{\partial F}{r\partial\theta} = q_\theta = Mr^{-3}\sin\theta$$

and $$\frac{\partial F}{r\sin\theta\partial\phi} = q_\phi = 0$$

$$\therefore dF = \frac{\partial F}{\partial r}dr + \frac{\partial F}{\partial\theta}d\theta + \frac{\partial F}{\partial\phi}d\phi = -(2Mr^{-3}\cos\theta)dr - (Mr^{-2}\sin\theta)d\theta + 0.d\phi$$

$$= d(Mr^{-2}\cos\theta)$$

Integrating, $F = Mr^{-2}\cos\theta$ (omit constant of integration, for it has no significance in F).

Finally, the stream lines given by

$$\frac{dr}{q_r} = \frac{rd\theta}{q_\theta} = \frac{r\sin\theta d\phi}{q_\phi}$$

i.e., $$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{rd\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta d\phi}{0}$$

given $$d\phi = 0 \text{ and } 2\cot\theta d\theta = \left(\frac{1}{r}\right)dr.$$

Integrating, the equation of the stream lines are given by

$$\phi = C_1$$

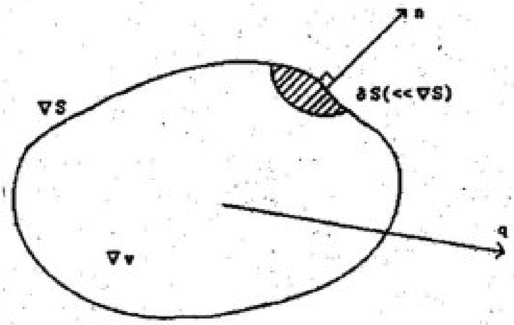
and $r = C_2\sin^2\theta$.

The equation ϕ = constant shows that the required stream lines lie in a plane which pass through the axis of symmetry $\theta = 0$.

Equations of Motion of Inviscid Fluids :

Euler's Equations of Motion :

This mathematical formulation uses Newton's 2nd law of motion which states that the total force acting on a mass of fluid is equal to the rate of change in linear momentum.



At time t, suppose that ΔS denotes the closed surface of a particle of a moving fluid, the particle moving along with the fluid with velocity $\overline{q}$. We take the particle to be of fixed mass $\rho\Delta v$, ρ being the density and Δv the volume. Ultimately, ΔS and Δv are to be made vanishingly small.

Suppose that δS ($<< \Delta S$) is a surface element of ΔS and that $\hat{n}$ is the unit normal vector to δS, drawn outwards from the particle. If p denotes the fluid pressure at δS, then since ΔS is moving locally with the fluid, the force on it due to the fluid outside the paticle is $-p\delta S\,\hat{n}$. Hence the total surface force on the fluid particle due to the actions of the surrounding fluid is

$$-\int_{\Delta S} \hat{n} p\, dS = -\int_{\Delta v} \nabla p\, dv$$

Let $\overline{F}$ be the body force per unit mass at any volume element $\delta v(<<\Delta v)$ of the particle. Then $\rho\,\overline{F}\,\delta v$ is the body force on the element δv and so the total body force on the entire particle $\int_{\Delta v} \rho \overline{F} dv$.

Since the mass of the volume element δv is $\rho\delta v$, that of the entire fluid particle is $\int_{\Delta v} \rho dv$. But this mass is constant. Hence the equation of motion of the fluid particle is

$$\int_{\Delta v} (\rho\overline{F} - \nabla p) dv = \frac{d\overline{q}}{dt} \int_{\Delta v} \rho dv$$

The terms on the L.H.S. represent the total force acting on the fluid particle. $\frac{d\overline{q}}{dt}$ is the accleration of the fluid particle following its motion. To the first order this equation approximates to

$(\rho\overline{F} - \nabla p)\Delta v = \frac{d\overline{q}}{dt} \rho dv$ so that in the limit $\Delta v \to 0$, we obtain

$$\frac{d\overline{q}}{dt} = \overline{F} - \frac{1}{\rho}\nabla p \qquad \text{......(1)}$$

which is called the **Euler's equation of motion.**

But $$\frac{D\bar{q}}{Dt}=\frac{\partial\bar{q}}{\partial t}+(\bar{q}.\nabla)\bar{q} \qquad ......(2)$$

Using (2), (1) may be re-written as

$$\frac{\partial\bar{q}}{\partial t}+(\bar{q}.\nabla)\bar{q}=\bar{F}-\frac{1}{\rho}\nabla p \qquad ......(3)$$

Again, $\nabla(\bar{q}\cdot\bar{q})=2[\bar{q}\times\text{curl}\,\bar{q}+(\bar{q}\cdot\nabla)\cdot\bar{q}]$

so that $$(\bar{q}.\nabla)\bar{q}=\frac{1}{2}\nabla q^2-\bar{q}\times\text{curl}\,\bar{q} \qquad ......(4)$$

Using (4), (3) takes the form

$$\frac{\partial\bar{q}}{\partial t}+\nabla\left(\frac{1}{2}q^2\right)-\bar{q}\times\text{curl}\,\bar{q}=\bar{F}-\frac{1}{\rho}\nabla p \qquad ....(5)$$

or $$\frac{\partial\bar{q}}{\partial t}-\bar{q}\times\text{curl}\,\bar{q}=\bar{F}-\frac{1}{\rho}\nabla p-\frac{1}{2}\nabla q^2 \qquad ......(6)$$

Conservative Field of Force :

In a conservative field of force, the work done by the force $\bar{F}$ of the field in taking a unit mass from one point to the other is independent of the path of motion.

Thus, if $\bar{F}=X\hat{i}+Y\hat{j}+Z\hat{k}$, then a scalar point function V(x, y, z) exists such that

$$Xdx+Ydy+Zdz=-dV \quad \text{or} \quad \bar{F}=-\nabla V \qquad (*)$$

so that $$X=-\frac{\partial V}{\partial x},Y=-\frac{\partial V}{\partial y},Z=-\frac{\partial V}{\partial z}$$

V is said to be **force potential** and it measures the potential energy of the field.

Note: Writing the equation (*) in orthogonal system we can find out the relation between the force components and the force potential in all the three coordinate systems.

Euler's equations of motion in cylindrical co-ordinates :

Euler's equation of motion is $$\frac{D\bar{q}}{Dt}=\bar{F}-\frac{1}{\rho}\nabla p \qquad ......(1)$$

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the components of external force in r, θ, z directions. Then know that

$$\frac{D\bar{q}}{Dt}=\left(\frac{Dq_r}{Dt}-\frac{q_\theta^2}{r},\frac{Dq_\theta}{Dt}+\frac{q_rq_\theta}{r},\frac{Dq_z}{Dt}\right)$$

$$\overline{F} = (F_r, F_\theta, F_z), \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z} \right)$$

Substituting in (1) and equating the co-efficients of $\hat{r}, \hat{\theta}, \hat{k}$, we obtain Euler's equations of motion in cylidrical co-ordinates as

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_z}{dt} &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad \text{......(2)}$$

where $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z \frac{\partial}{\partial z} \qquad \text{......(3)}$$

Euler's equations of motion in spherical co-ordinates :

Euler's equation of motion is

$$\frac{D\overline{q}}{Dt} = F - \frac{1}{\rho}\nabla p \qquad \text{......(1)}$$

Let (q_r, q_θ, q_ϕ) be the velocity components and (F_r, F_θ, F_ϕ) be the components of external force r, θ, ϕ directions. Then we know that

$$\frac{Dq}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r}, \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi \cot\theta}{r} \right)$$

$$\overline{F} = (F_r, F_\theta, F_\phi), \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} \right)$$

Substituting in (1) and equating the co-efficients of $\hat{r}, \hat{\theta}, \hat{k}$, we obtain Eule'sequations of motion in spherical polar co-ordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi \cot\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} \end{aligned}\right\} \qquad \text{......(2)}$$

where $$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + q_r \frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \qquad ......(3)$$

Integrals of Equation of Motion :

Bernoulli's Equation :

Let us suppose that **(a)** the body forces are conservative **(b)** that the flow is of the potential kind and (c)the fluid is barotropic i.e pressure is a function of density only so that $\int \frac{dp}{\rho}$ exists. Then due to (a) and (b) there exists scalar functions Ω, ϕ such that

$$\overline{F} = -\nabla\Omega, \overline{q} = -\nabla\phi$$

also due to (c) $\int \frac{dp}{\rho}$ exists and there must be a function P [$P = \int \frac{dp}{\rho}$] such that $\nabla P = \left(\frac{1}{\rho}\right)\nabla p$.

Then equation (5) of Euler's equation of motion becomes

$$-\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}q^2\right) = -\nabla\Omega - \nabla P$$

Let $\overline{r}$ be the position vector of the fluid particle at time t and let $d\overline{r}$ be an instantaneous displacement made in the position of the particle at this instant t. Then scalar multiplying the last equation through by $d\overline{r}$ and using $d\overline{r} \cdot \nabla\Omega = d\Omega$, etc., we obtain

$$-d\left(\frac{\partial\phi}{\partial t}\right) + d\left(\frac{1}{2}q^2\right) = -d\Omega - dP$$

subject to t being constant. Rearranging and integrating gives

$$\frac{1}{2}\overline{q}^2 + \Omega + \int\frac{dp}{\rho} - \frac{\partial\phi}{\partial t} = f(t) \qquad ......(1)$$

where f(t) is an arbitrary function of t arising from the integration in which t is being kept constant. **(1) is Bernoulli's equation in its most general form or the pressure equation.**

The pressure equation is of paramount importance, for once we know the velocity potential ϕ, the velocity is determined by $\overline{q} = -\nabla\phi$, and the pressure is then found from the pressure equation and the relation $p = f(\rho)$. Note that $\frac{\partial\phi}{\partial t}$ is calculated by varying t only and refers to a point fixed in space.

Other forms of Bernoulli's equation, often of considerable use in the solution of problems, may be derived from (1). Thus for steady motion $\frac{\partial\phi}{\partial t}=0$ and f(t) is constant, so that

$$\frac{1}{2}\bar{q}^2+\Omega+\int\frac{dp}{\rho}=\text{constant} \qquad ......(2)$$

If, further, the fluid is homogeneous and incompressible so that ρ is constant. (2) becomes

$$\frac{1}{2}\bar{q}^2+\Omega+\left(\frac{p}{\rho}\right)=\text{constant} \qquad ......(3)$$

It follows that in principle the solution of any problem of irrotational motion of a liquid is reduced to finding the velocity potential ϕ which satisfy Laplac's equation $\nabla^2\phi=0$ and the other conditions of the problem. The calculation of fluid thrust on a surface is then reduced to an integration.

An important Theorem :

If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the stream lines and vortex lines.

Proof :

Euler's equation in vector form is given by (Refer equation (6), Euler's equ^n of motion.)

$$\frac{\partial\bar{q}}{\partial t}-\bar{q}\times\text{curl}\,\bar{q}=\bar{F}-\frac{1}{\rho}\nabla p-\frac{1}{2}\nabla q^2 \qquad ......(i)$$

For steady flow, $\frac{\partial\bar{q}}{\partial t}=0$.

Since the external forces are conservative, there exists force potential V such that $\bar{F}=-\nabla V$.

Further, density being a function of presure p only, $\int\frac{dp}{\rho}$ exists and there must be a function P [$P=\int\frac{dp}{\rho}$]

such that $\nabla P=\left(\frac{1}{\rho}\right)\nabla p$. Using these facts, (i) reduces to

$$\nabla\left(V+P+\frac{1}{2}q^2\right)=\bar{q}\times\text{curl}\,\bar{q} \qquad ......(ii)$$

Let $\bar{\Omega}=\text{curl}\,\bar{q}$ = vorticity vector.

Then $\nabla\left(V+P+\frac{1}{2}q^2\right)=\bar{q}\times\Omega$ ......(iii)

Let $\hat{n}=\nabla\left(V+P+\frac{1}{2}q^2\right)$ ......(iv)

Then (iii) reduces to $\hat{n} = \bar{q} \times \bar{\Omega}$. ......(v)

From (v)

$$\hat{n} \cdot \bar{q} = (\bar{q} \times \bar{\Omega}) \cdot \bar{q} = (\bar{q} \times \bar{q}) \cdot \bar{\Omega} = 0$$

and $$\hat{n} \cdot \bar{\Omega} = (\bar{q} \times \bar{\Omega}) \cdot \bar{\Omega} = \bar{q} \cdot (\bar{\Omega} \times \bar{\Omega}) = 0$$

These results show that $\hat{n}$ is perpendicular to both $\bar{q}$ and $\bar{\Omega}$.

Since ∇f is perpendicular everywhere to the surface f = constant, (iv) shows that $\hat{n}$ is perpendicular to the family of surface

$$V + P + \frac{1}{2}q^2 = C \qquad \text{......(vi)}$$

Thus $\bar{q}$ and $\bar{\Omega}$ are both tangential to the surfaces (vi). Hence (vi) contains the stream lines and vortex lines.

Another Form : Prove that for steady motion of an inviscid isotropic fluid $P = f(\rho), \int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega =$ constant over a surface containing the stream lines and vortex lines.

Exercise :

Asphere of radius R, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, show that the presure at the surface of the sphare at time t is

$$\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$$

If $R = a(2 + \cos nt)$, show that, to prevent cavitation in the fluid, Π must not be less than $3\rho a^2 n^2$.

Solution :

Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v will be radial and hence v will be function or r' (the radial distance from the centre of the sphere which is taken as origin), and time t only.Let P be pressure at a distance r'. Let P be the pressure on the surface of the sphere of radius R and V be the velocity there. Then the equation of continuity is

$$r'^2 v' = R^2 V = F(t) \qquad \text{......(1)}$$

From (1), $\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}$ ......(2)

Again equation of motion is $\dfrac{\partial v'}{\partial t}+v'\dfrac{\partial v'}{\partial r'}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$

or $$\frac{F'(t)}{r'^2}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{[using (2)]} \qquad ......(3)$$

Integrating with respect to r', (3) reduces to

$$-\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{\rho}+C$$

When $r'=\infty$, then $v'=0$ and $p=\Pi$ so that $C=\dfrac{\Pi}{\rho}$

$$\therefore \quad -\frac{F'(t)}{r'}+\frac{1}{2}v'^2=-\frac{\Pi-p}{\rho}$$

or $$p=\Pi+\frac{1}{2}\rho\left[2\frac{F'(t)}{r'}-v'^2\right] \qquad ......(4)$$

But $p=P$ and $v'=V$ when $r'=R$. Hence (4) gives

$$p=\Pi+\frac{1}{2}\rho\left[\frac{2}{R}\{F'(t)\}_{r'=R}-V^2\right] \qquad ......(5)$$

Also $V=\dfrac{dR}{dt}$. Hence using (1), we have

$$\{F'(t)\}_{r'=R}=\frac{d}{dt}\left(R^2V\right)=\frac{d}{dt}\left(R^2\frac{dR}{dt}\right)=\frac{d}{dt}\left(\frac{R}{2}\frac{dR^2}{dt}\right)$$

$$=\frac{R}{2}\frac{d^2R^2}{dt^2}+\frac{1}{2}\frac{dR^2}{dt}\frac{dR}{dt}=\frac{R}{2}\frac{d^2R^2}{dt^2}+R\left(\frac{dR}{dt}\right)^2$$

Using above values of V and $\{F'(t)\}_{r'=R}$, (5) reduces to

$$p=\Pi+\frac{1}{2}\rho\left[\frac{2}{R}\left\{\frac{R}{2}\frac{d^2R^2}{dt^2}+R\left(\frac{dR}{dt}\right)^2\right\}-\left(\frac{dR}{dt}\right)^2\right]$$

or $$p=\Pi+\frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2}+\left(\frac{dR}{dt}\right)^2\right] \qquad ......(6)$$

Second Part :

From $r'^2v'=$ constant, we conclude that v' is maximum where r' is minimum i.e., $r'=R$. Hence pressure is minimum on $r'=R$ by using Bernoulli's theorem.

Given $R = a(2 + \cos nt)$

$$\therefore \quad \frac{dR}{dt} = -an \sin nt$$

and $$\frac{dR^2}{dt} = 2a^2(2 + \cos nt)(-n \sin nt)$$

$$\therefore \quad \frac{d^2R^2}{dt^2} = -2a^2n^2(2 + \cos nt)\cos nt + 2a^2n^2 \sin^2 nt$$

with the above values, (6) reduces to

$$p = \Pi + \left(\frac{3}{2}\right)\rho a^2n^2 \sin^2 nt - a^2n^2\rho\left(2\cos nt + \cos^2 nt\right) \qquad ...(8)$$

From (7), R varies from 3a to a. Thus the sphere has the greatest radius 3a when $nt = 0$ or $2m\pi$. Clearly as the sphere shrinks from $R = 3a$, there is a possibility of a cavitation there becuse pressure would be minimum there.Hence the minimum value of pressure P' (say) on the surface of the sphere is given by replacing $t = 0$ or $nt = 2m\pi$ in (8). We thus obtain

$$P' = \Pi - 3\rho a^2n^2 \qquad ......(9)$$

To prevent cavitation in the fluid. P' given by (9) must be positive i.e., Π must not less that $3\rho a^2n^2$.

Exercise :

Liquid is contained between two parallel planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated; prove that if Π be the pressure at the outer surface, the initial pressure at any point on the liquid distant r from the centre is

$$\Pi \frac{\log r - \log b}{\text{laga} - \log b}$$

Solution :

Here the motion of the liquid will take place in uch a manner so that each element of the liquid moves towards the axis of the cylinder $|z| = b$. Hence the free surface would be cylindrical. Thus the liquid velocity v' will be radical and v' will be function of r' (the radial distance from the centre of the cylinder $|z| = b$ which is taken as origin and time t only.Let p be the pressure at a distance r'. Then the equation of continuity is

$$r'v' = F(t) \qquad ......(1)$$

From (1), $$\frac{\partial v'}{\partial t'} = \frac{F'(t)}{\partial r'} \qquad ......(2)$$

The equation of motion is $$\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{F'(t)}{r}+\frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right)=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$ [using (2)]

Integrating, $$F'(t)\log r'+\frac{1}{2}v'^2=-\frac{p}{\rho}+C \qquad ......(3)$$

Initially when $t=0$, $v'=0$, $p=P$.

$$\therefore \quad F'(0)\log r'=-\frac{p}{\rho}+C \qquad ......(4)$$

Again, $P=\Pi$ when $r'=a$ and $P=0$ when $r'=b$

$$\therefore \quad F'(0)\log a=-\frac{\Pi}{\rho}+C \text{ and } F'(0)\log b=C \qquad ......(5)$$

Solving (5) for $F'(0)$ and C, we have

$$C=-\log b\frac{\Pi}{\rho\log\left(\frac{a}{b}\right)}, F'(0)=\frac{-\Pi}{\rho\log\left(\frac{a}{b}\right)}$$

Putting these values in (4), we get

$$\frac{P}{\rho}=\frac{\Pi}{\rho\log\left(\frac{a}{b}\right)}\log r'-\frac{\Pi\log b}{\rho\log\left(\frac{a}{b}\right)}$$

or $$P=\Pi\frac{\log r'-\log b}{\log\left(\frac{a}{b}\right)}=\Pi\frac{\log r'-\log b}{\log a-\log b} \qquad ......(6)$$

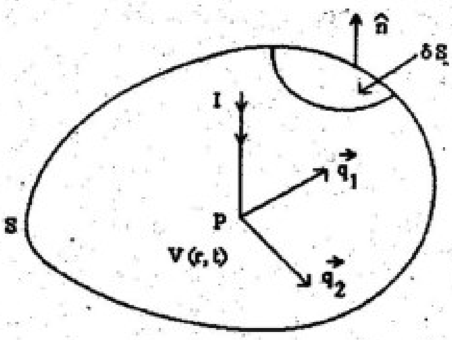
For the required result, replace r' by r in (6).

Impulsive Action :

Let sudden velocity changes be produced at the boundaries of an incompressible fluid or that impulsive forces be made to act to its interior. Then it is known that the impulsive pressure at any point is the same in every direction. Moreover the disturbances produced in both cases are propagated instantaneously throughout the fluid.

Equation of motion under Impulsive Forces (Vector Form) :

Let S be an arbitrary small closed surface drawn in the incompresible fluid enclosing a volume V. Let $\bar{I}$ be the implusive body force per unit mass. Let this impulse change the velocity at $P(\bar{r}, t)$ of V instantaneously fro $\bar{q}_1$ to $\bar{q}_2$ and let it produce impulsive pressure on the boundary S. Let $\tilde{\omega}$ denote the impulsive pressure on the element δS of S. Let $\hat{n}$ be the unit outward drawn normal at δS. Let ρ be density of the fluid.



We now apply Newton's second law of impulsive motion to the fluid enclosed by S, namely

Total impuse applied = Change of momentum

$$\therefore \int_V \bar{I}\rho dV - \int_V \hat{n}\tilde{\omega} dS = \int_V \rho(\bar{q}_2 - \bar{q}_1) dV \quad \text{......(1)}$$

But $\int_V \hat{n}\tilde{\omega} dS = \int_V \nabla\tilde{\omega} dV$

(by Gauss divergence theorem)

$$\therefore \quad \text{From (1)}, \int_V \left[\bar{I}\rho - \nabla\tilde{\omega} - \rho(\bar{q}_2 - \bar{q}_1)\right] dV = 0 \quad \text{......(2)}$$

Since V is an arbitrary small volume, (2) gives

$$\bar{I}\rho - \nabla\tilde{\omega} - \rho(\bar{q}_2 - \bar{q}_1) = 0$$

$$\text{or} \quad \bar{q}_2 - \bar{q}_1 = \bar{I} - \left(\frac{1}{\rho}\right)\nabla\tilde{\omega} \quad \text{......(3)}$$

This is the **general equation of impulsive motion.**

Corollary 1. :

Let $\bar{I} = 0$ (i.e. external impulsive body forces are absent) whereas impulsive pressures be present. Then (3) reduces to

$$\bar{q}_2 - \bar{q}_1 = -\left(\frac{1}{\rho}\right)\nabla\tilde{\omega} \quad \text{......(4)}$$

$$\text{From (4)}, \nabla\cdot(\bar{q}_2 - \bar{q}_1) = \nabla\cdot\left[-\left(\frac{1}{\rho}\right)\nabla\tilde{\omega}\right]$$

$$\text{or} \quad \nabla\cdot\bar{q}_2 - \nabla\cdot\bar{q}_1 = -\left(\frac{1}{\rho}\right)\nabla^2\tilde{\omega} \quad \text{......(5)}$$

For the incompressible fluid, the equation of continuity gives

$$\nabla\cdot\bar{q}_2 = \nabla\cdot\bar{q}_1 = 0 \quad \text{......(6)}$$

Making use of (6), (5) reduces to

$$\nabla^2\tilde{\omega} = 0 \quad \text{(Laplace's equation)} \quad \text{......(7)}$$

Corollary 2. : Let $\bar{q}_1 = 0$ and $\bar{I} = 0$ so that the motion is started from rest by the application of impulsive pressure at the boundaries but without use of external impulsive body forces. Then, writing $\overline{q}_2 = \overline{q}$, (3) reduces to

$$\overline{q} = -\nabla\left(\frac{\tilde{\omega}}{\rho}\right) \qquad ......(8)$$

showing that there exists a velocity potential $\phi = \frac{\tilde{\omega}}{\rho}$ and that the motion is irrotational.

Corollary 3. :

Let $\overline{I} = 0$, i.e., let there be no extraneous impulses. Further, let ϕ_1 and ϕ_2 denote the velocity potential just before and just fter the impulsive action. Then

$$\overline{q}_1 = -\nabla\phi_1$$

and $$\overline{q}_2 = -\nabla\phi_2 \qquad ......(9)$$

Then (3) reduces to $-\nabla\phi_2 + \nabla\phi_1 = -\left(\frac{1}{\rho}\right)\nabla\tilde{\omega}$

or $$\nabla\tilde{\omega} = \rho\nabla(\phi_2 - \phi_1)$$

Integrating, when ρ is constant

$$\tilde{\omega} = \rho(\phi_2 - \phi_1) + C$$

The constant C may be omitted by regarding as an extra pressure and constant throughout the fluid

$$\therefore \quad \tilde{\omega} = \rho\phi_2 - \rho\phi_1 \qquad ......(10)$$

Corollary 4. Physical meaning of velocity potential :

Take $\phi_1 = 0$ and $\rho = 1$ in corollary 3. Then we find that any actual motion, for which a single valued velocity potential exists, could be produced instantaneously from rest by applying appropriate impulses. We then also note that the velocity potential is the impulsive presure at any point.

It is also easily seen that when a state of rotational motion exists in a fluid, the motion could neither be created nor destroyed by impulsive pressures.

Exercise :

A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is sudenly annihilated. Show that the pressure at a distance r from the centre immediately falls to $\Pi\left(1 - \frac{a}{r}\right)$.

Solution : Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p the pressure there. Then the equation of continuity is

$$r'^2 v' = F(t) \qquad ......(1)$$

From (1), $\dfrac{\partial v'}{\partial t'} = \dfrac{F'(t)}{r'^2}$ ......(2)

The equation of motion is

$$\frac{\partial v'}{\partial t'} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $\dfrac{F'(t)}{r'^2} + \dfrac{\partial}{\partial r'}\left(\dfrac{1}{2}v'^2\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r'}$ [using (2)]

Integrating, $-\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = -\dfrac{P}{\rho} + C$

When $r' = \infty$, then $p = \Pi$ and $v' = 0$ so that $C = \dfrac{\Pi}{\rho}$.

$\therefore$ $-\dfrac{F'(t)}{r'} + \dfrac{1}{2}v'^2 = \dfrac{\Pi - p}{\rho}$ ......(3)

When the sphere is sudenly annihilated, we have

$t = 0, r' = a, v' = 0$ and $p = 0$

$\therefore$ From (3), $-\dfrac{F'(0)}{a} = \dfrac{\Pi}{\rho}$ so that $F'(0) = -\dfrac{a\Pi}{\rho}$

Hence immediately after the annihilation of the sphere (with $t = 0, v' = 0$), (3) reduces to

$\dfrac{a\Pi}{\rho r'} + 0 = \dfrac{\Pi - p}{\rho}$ or $p = \Pi\left(1 - \dfrac{a}{r'}\right)$ ......(4)

Thus at the time of annihilation, when $r' = r$, the pressure is given by

$$p = \Pi\left(1 - \frac{a}{r}\right) \qquad ......(5)$$

The Energy Equation :

Statement : The rate of change of total energy (kinetic, potential) of any portion of an incompressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the extraneous forces is supposed to be independent of time.

Proof :

Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dV be the volume element surrounding P. Let $q(\bar{r}, t)$ be the velocity of P. Then the Euler's equation of motion is

$$\rho \frac{d\bar{q}}{dt} = -\nabla p + \rho \bar{F} \qquad \text{......(1)}$$

(We use here $\frac{d}{dt}$ for $\frac{D}{Dt}$ so that $\frac{d}{dt} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q} \cdot \nabla$)

Let the external forces be conservative so that there exists a force potential Ω which is independent of time. Thus, $\bar{F} = -\nabla \Omega$ and $\frac{\partial \Omega}{\partial t} = 0$ ......(2)

Using (2) and then multiplying both sides of (1) scalarly by $\bar{q}$, we get

$$\rho\left(\bar{q} \cdot \frac{d\bar{q}}{dt}\right) = -\bar{q} \cdot \nabla p - \rho[q \cdot \nabla \Omega]$$

or $$\rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right) + (\bar{q} \cdot \nabla)\Omega\right] = -\bar{q} \cdot \nabla p \qquad \text{......(3)}$$

But $$\frac{d\Omega}{dt} = \frac{\partial \Omega}{\partial t} + (\bar{q} \cdot \nabla)\Omega = (\bar{q} \cdot \nabla)\Omega \qquad \text{[using (2)]}$$

$\therefore$ From (3), $\rho \frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) = -\bar{q} \cdot \nabla p$

Integrating both sides over V, we get

$$\int_V \rho \frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) dV = -\int_V (\bar{q} \cdot \nabla p)\, dV$$

or $$\frac{d}{dt}\left[\int_V \frac{1}{2}\rho q^2 dV + \int_V \rho \Omega \, dV\right] = -\int_V (\bar{q} \cdot \nabla p)\, dV \qquad \text{......(4)}$$

wherein we have used the equation of continuity.

i.e., $$\frac{d}{dt}(\rho dV) = 0 \qquad \text{......(5)}$$

Let T, W denote the kinetic, potential energies respectively. Then, by definitions

$$T = \int_V \frac{1}{2}\rho q^2 dV, W = \int_V \rho \Omega \, dV \qquad \text{......(6)}$$

Since $\nabla \cdot (p\bar{q}) = p\nabla \cdot \nabla + \bar{q} \cdot \nabla p$, we have

$$q \cdot \nabla p = \nabla \cdot (p\bar{q}) - p\nabla \cdot q$$

$\therefore$ R.H.S. of (4) $= -\int_V \nabla \cdot (p\bar{q}) dV + \int_V p \nabla \cdot \bar{q} \, dV$

$$= -\int_S p\bar{q}\cdot\hat{n}dS + \int_V p\nabla\cdot\bar{q}dV$$

[by Gauss divergence theorem]

where $\hat{n}$ is unit inward normal and dS is the element of the fluid surface S.

The second theorem in the above expression vanishes due to equation of continiuity $\nabla\cdot\bar{q} = 0$

Hence R.H.S of (4) $= -\int_S p\bar{q}\cdot\hat{n}ds$ ......(7)

Again the rate of work done by the fluid pressure on an element δS of S is $p\delta S\ \hat{n}\cdot\bar{q}$ Hence the net rate at which work is being done by the fluid presure is

$$\int_S p\bar{q}\cdot\hat{n}dS = R \text{ (say)} \quad ......(8)$$

Hence using (6), (7) and (8) in (4) we get

$$\frac{d}{dt}(T+W) = R,$$

which is the desired energy equation.

Remark :

This principle is used to shorthen the solution of some problems.

The energy equation is stated as follows :

The rate of increase of energy in the system is equal to the rate at which work is done on the system.

Exercise :

An infinite mass of fluid is acted on by force $\frac{\mu}{r^{\frac{3}{2}}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5}\mu\right)^{\frac{1}{2}} \cdot c^{\frac{5}{4}}$.

Solution :

At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2 v' = r^2 v \quad ......(1)$$

When the radius of the cavity is r, then kinetic energy is

$$= \int_r^\infty \frac{1}{2}\left(4\pi r'^2 \rho dr'\right) v'^2 \qquad \left[\because \quad K.E. = \frac{1}{2}mv^2\right]$$

$$= 2\pi\rho r^4 v^2 \int_r^{\infty} \frac{dr'}{r'^2} \qquad \text{[using (1)]}$$

$$= 2\pi\rho r^3 v^2$$

The initial kinetic energy is zero.

Let V be the work function (or force potential) due to external forces.

Then we have

$$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{\frac{3}{2}}} \text{ so that } V = \frac{2\mu}{r'^{\frac{1}{2}}}$$

$\therefore$ the work done $= \int_r^c V dm$, dm being the elementary mass

$$= \int_r^C \left(\frac{2\mu}{r'^{\frac{1}{2}}} \right) 4\pi r'^2 dr' \rho = 8\pi\mu\rho \int_r^C r'^{\frac{3}{2}} dr' = \left(\frac{16}{5} \right) \pi\rho\mu \left(c^{\frac{5}{2}} - r^{\frac{5}{2}} \right)$$

We now use to energy equation, namely Increase in energy = work done

$$2\pi\rho r^3 v^2 - 0 = \left(\frac{16}{5} \right) \pi\rho\mu \left(c^{\frac{5}{2}} - r^{\frac{5}{2}} \right)$$

$$\therefore \quad v = \frac{dr}{dt} = -\left(\frac{8\pi}{5} \right)^{\frac{1}{2}} \frac{\left(c^{\frac{5}{2}} - r^{\frac{5}{2}} \right)^{\frac{1}{2}}}{r^{\frac{3}{2}}} \qquad \text{......(2)}$$

wherein negative sign is taken because r decreases as t increases.

Let T be the time of filling up the cavity. Then (2) gives

$$\int_0^T dt = -\left(\frac{5}{8\pi} \right)^{\frac{1}{2}} \int_C^0 \frac{r^{\frac{3}{2}} dr}{\sqrt{\left(c^{\frac{5}{2}} - r^{\frac{5}{2}} \right)}}$$

$$T = \left(\frac{5}{8\pi} \right)^{\frac{1}{2}} \int_0^C \frac{r^{\frac{3}{2}} dr}{\sqrt{\left(c^{\frac{5}{2}} - r^{\frac{5}{2}} \right)}}$$

Put $r^{\frac{5}{2}} = c^{\frac{5}{2}} \sin^2\theta$ so that $\left(\frac{5}{2} \right) r^{\frac{3}{2}} dr = 2c^{\frac{5}{2}} \sin\theta \cos\theta d\theta$.

$$\therefore \quad T=\left(\frac{5}{8\pi}\right)^{\frac{1}{2}}\int_C^{\frac{\pi}{2}}\frac{4}{5}c^{\frac{5}{4}}\sin\theta d\theta=\left(\frac{2}{5\pi}\right)^{\frac{1}{2}}c^{\frac{5}{4}}$$

Exercise :

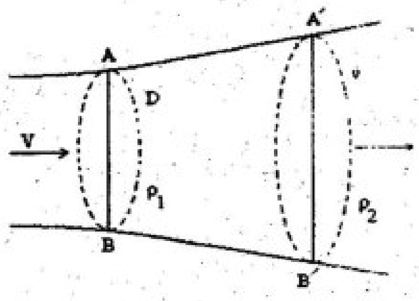
A stream is rushing from a boiler through a conical pipe. the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion be suspended to be that of the divergence from the vertex of the cone, prove that

$$\frac{v}{V}=\frac{D^2}{d^2}e^{\frac{(v^2-V^2)}{2k}}$$

where k is the pressure divided by the density and supposed constant.

Solution :

Let AB and A′B′ be the ends of the conical pipe such that A′B′ = d and AB = D. Let ρ_1 and ρ_2 be densities of the stream at AB and A′B′. Hence the equation of continuity is



$$\pi\left(\frac{d}{2}\right)^2 v\rho_1=\pi\left(\frac{D}{2}\right)^2 V\rho_2$$

$$\text{so that } \frac{v}{V}=\frac{D^2}{d^2}\times\frac{\rho_2}{\rho_1} \qquad \text{......(1)}$$

By Bernoulli's heorem (in absence of external forces like gravity), we hve

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2=C \qquad \text{......(2)}$$

But $\frac{p}{\rho}=k$ so that $dp=kd\rho$ ......(3)

$\therefore$ (2) gives $k\int\frac{d\rho}{\rho}+\frac{1}{2}q^2=C$ [using (3)]

Integrating,

$$k\log\rho+\frac{1}{2}q^2=C \qquad \text{......(4)}$$

When $q = v$, $\rho=\rho_1$ and when $q = V$, $\rho=\rho_2$.

$$\therefore \quad k\log\rho_1+\frac{1}{2}v^2=C \text{ and } k\log\rho_2+\frac{1}{2}V^2=C$$

Subtarcting,

$$k(\log\rho_2-\log\rho_1)+\frac{1}{2}(V-v^2)=0$$

or $\quad \log\frac{\rho_2}{\rho_1} = \frac{\left(v^2 - V^2\right)}{2k}$

or $\quad \frac{\rho_2}{\rho_1} = e^{\frac{\left(v^2 - V^2\right)}{2k}}$ .....(5)

Using (5), (1) reduces to $\frac{v}{V} = \frac{D^2}{d^2} e^{\frac{\left(v^2 - V^2\right)}{2k}}$

Exercise :

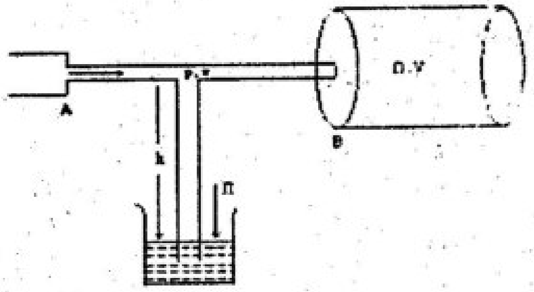
A stream in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A is delivered at atmospheric pressure at a place, where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth

$$\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$$

below the pipe, s being the delivery per second.

Solution :

Let v be the velocity in the tube of smaller section A and p the pressure at that section. Further, let V and Π be the corresponding quantities at the bigger section B of the figure. Then by Bernoulli's Theorem (in absence of external forces like gravity) for incompresible fluid, namely



$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant}$$

we obtain $\quad \frac{p}{\rho} + \frac{1}{2}v^2 = \frac{\Pi}{\rho} + \frac{1}{2}v^2$

or $\quad \frac{1}{\rho}(\Pi - p) = \frac{1}{2}\left(v^2 - V^2\right)$ ......(1)

Let h be the height through which water is sucked up. Then

$g\rho h$ = difference of pressure = $\Pi - p$ ......(2)

The equation ofcontinuity is

$Av = BV = s$ (delivery per second)

so that $\quad v = \frac{s}{A}$ and $V = \frac{s}{B}$ ......(3)

Using (2) and (3), (1) reduces to

$$\frac{1}{\rho}\times g\rho h=\frac{1}{2}\left(\frac{s^2}{A^2}-\frac{s^2}{B^2}\right)$$

or $$h=\frac{s^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)$$

Exercise :

Amass of homogeneous liquid is moving so that the velocity at any point is proportional to the time and that the pressure is give by

$$\frac{p}{\rho}=\mu xyz-\frac{1}{2}t^2\left(y^2z^2+z^2x^2+x^2y^2\right)$$

Prove that this motion may have been generated from rest by natural forces independent of the time and show that, if the direction of motion at every point coincides with the direction of the acting force, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Solution :

Given that velocity q is proportional to time.

$\therefore \quad q=\lambda t$ ......(1)

Also $\frac{p}{\rho}=\mu xyz-\frac{1}{2}t^2\left(y^2z^2+z^2x^2+x^2y^2\right)$ ......(2)

Suppose that the motion is produced by finite natural forces (conservative forces) which are derivativable from the potential function V. Then by Bernoulli's equation, we get

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\frac{1}{2}q^2+V=F(t)$$

or $\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\frac{1}{2}\lambda^2t^2-V+F(t)$ [using (1)] ......(3)

Snce (2) and (3) must be identical, equating the coekkicients of t^2 on R.H.S. of (2) and (3) gives

$\lambda^2=y^2z^2+z^2x^2+x^2y^2$ ......(4)

Using (4), (1) reduces to $q^2=t^2(y^2z^2+z^2x^2+x^2y^2)$ ......(5)

But $q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$ ......(6)

Compairing (5) and (6), an appropriate value of ϕ given by

$\phi=t\,xyz$ ......(7)

Uing (7) and (4), (3) reduces to

$$\frac{p}{\rho} = xyz - \frac{1}{2}t^2\left(y^2z^2 + z^2x^2 + x^2y^2\right) - V + F(t) \qquad \text{......(8)}$$

Compairing (2) and (8), we find

$$F(t) = 0 \text{ and } xyz - V = \mu\, xyz$$

$$\therefore \quad V = xyz(1 - \mu) \qquad \text{......(9)}$$

IF u, v, w are the components of velocities and X, Y, Z are the components of forces, then

$$u = -\frac{\partial\phi}{\partial x} = -tyz, v = \frac{\partial\phi}{\partial y} = -txz, w = -\frac{\partial\phi}{\partial z} = -txy$$

and $\quad X = -\frac{\partial V}{\partial x} = (\mu - 1)yz, Y = \frac{\partial V}{\partial y} = (\mu - 1)xz, Z = -\frac{\partial V}{\partial z} = (\mu - 1)xy$

Given tht the direction of motion coincides with that of the acting forces.Hence, we have

$$\frac{u}{X} = \frac{v}{Y} = \frac{w}{Z}$$

Again, the equations of the path becomes

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \text{ or } \frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}$$

i.e., $\quad \frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \qquad$ ......(10)

taking the first two members of (10), we get

$$xdx - ydy = 0$$

so that $\quad x^2 - y^2 = C_1 \qquad$ ......(11)

taking that last two member of (10), we get

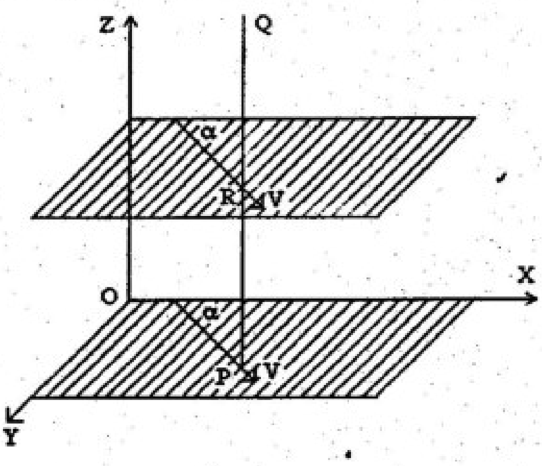
$ydy - zdz = 0$ so that $y^2 - z^2 = C_2 \qquad$ ......(12)

Thus each particle of the fluid will be on the curve which is the intersection of two hyperbolic cylinders $x^2 - y^2 = C_1$ and $y^2 - z^2 = C_2$.

Unit-2

Motion in two-dimensions :

Let a fluid move in such a way that at any given instant the flow pattern in a certain plane (say XOY) is the same as that in all other parallel planes within the fluid.Then the fluid is said to have two-dimensional motion. If (x, y, z) are co-ordinates of any point in the fluid, then all physical quantities (velocity, density, presure etc.) associated with the fluid are independent of z. Thus u, v are functions of x, y and t and w = 0 for such a motion.



To make the concept of two-dimensional motion more clear, suppose the plane under comsideration by xy-plane. Let P be an arbitrary point on that plane. Draw a straight line PQ parallel to OZ (or perpendicular to the xy-plane). Then all points on the line PQ are said to correspond to P. Draw a plane (in the fluid) parallel to xy-plane and metting PQ in R. Then, if the velocity at P is V in the xy-plane in a direction making an angle α with OX, the velocity at R is also V in magnitude and parallel in direction to the velocity at P as shown in the figure. It follows that thevelocity at corresponding points is a function of x, y and the time t, but not of z.

In order to maintain physical reality, we assume that the fluid in two-dimensional motion is confined between two planes parallel to the plane of motion and at a unit distance apart. The reference plane of motion is taken parallel to and midway between the assumed fixed planes. Thus while studying the flow of a fluid past a cylinder in two-dimensional motion in planes perpendicular to the axis of the cylinder, it is useful to restrict attention to a unit length of cylinder confined between the said planes in place of worrying over the cylinder of infinite lenght.

Suppose we are dealing with a two-dimensional motion in xy-plane. Then by flow accross a curve in this plane, we mean the flow across unit length of a cylinder whose trace on the plane xy is the curve under consideration, the generators of the cylinder being parallel to the z-axis. By a point in a flow, we mean a line through that point parallel to z-axis.

Stream function or Current function :

Let u and v be the components of velocity in two-dimensional motion. Then the differential equation of lines of flow is

$$\frac{dx}{u} = \frac{dy}{v} \text{ or } udx - udy = 0 \qquad \text{......(1)}$$

and the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \text{ or } \frac{\partial v}{\partial y}=\frac{\partial(-u)}{\partial x} \qquad \text{......(2)}$$

(2) shows that L.H.S. of (1) must be an exct differential, $d\Psi$ (say).

Then we have

$$vdx-udy=d\Psi=\left(\frac{\partial \Psi}{\partial x}\right)dx+\left(\frac{\partial \Psi}{\partial y}\right)dy \qquad \text{......(3)}$$

so that $$u=\frac{-\partial \Psi}{\partial y} \text{ and } v=\frac{\partial \Psi}{\partial x} \qquad \text{......(4)}$$

The function Ψ is known as the stream function. Then using (1) and (3), the stream lines are given by $d\Psi=0$ i.e., by the equation $\Psi=c$, where c is an arbitrary constant. Thus the stream function is contant along a stream line. Clearly the current function exists by virtue of the equation of continuity and incompressibility of the fluid. **Hence the current function exists in all types of two-dimensional motion whether rotational or irrotational.**

Note: (4) gives the relation between the velocity components and the stream function in cartesian system. Proceeding in the same way we can find the relations in polar system also.

Physical significance of stream function :

Let LM be any curve in the x-y plane and let Ψ_1 and Ψ_2 be the stream functions at L and M respectively. Let P be an arbitrary point on LM such that arc LP = s and let Q be a neighbouring point on LM such that arc LQ = s + δs. Let θ be the angle between tangent at P and the x-axis. If u and v be the velocity components at P, then velocity at P along inward drawn normal PN.

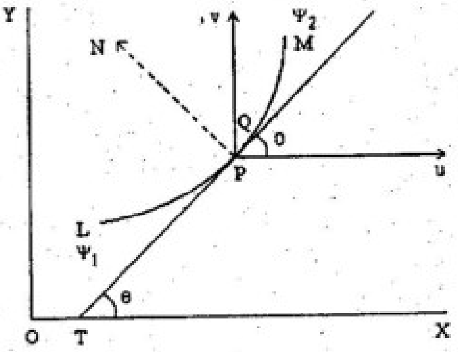
$$= v\cos\theta - u\sin\theta \qquad \text{......(1)}$$

When Ψ is the stream function, then we have

$$u=\frac{\partial \Psi}{\partial y} \text{ and } v=\frac{\partial \Psi}{\partial x} \qquad \text{......(2)}$$

Also from calculus,

$$\cos\theta=\frac{dx}{ds} \text{ and } \sin\theta=\frac{dy}{ds} \qquad \text{......(3)}$$



Using (1), flux across PQ from right to left

$$= (v\cos\theta - u\sin\theta)ds$$

∴ Total flux across curve LM from right to left

$$=\int_{LM}(v\cos\theta-u\sin\theta)ds=\int_{LM}\left(\frac{\partial \Psi}{\partial x}\frac{dx}{ds}+\frac{\partial \Psi}{\partial y}\frac{dy}{ds}\right)ds$$

(by (2) & (3))

$$= \int_{LM}\left(\frac{\partial\Psi}{\partial x}dx + \frac{\partial\Psi}{\partial y}dy\right) = \int_{\Psi_1}^{\Psi_2} d\Psi = \Psi_2 - \Psi_1$$

Thus a property of the current function is that the difference of its values at two points reprsents the flow across any line joining the points.

Remark 1. :

The current function Ψ at any point can also be defined as the flux (i.e., rate of flow of fluid) across a curve LP where L is some fixed point in the plane.

Remark 2. :

Since the velocity normal δs will contribute to the flux across δs wheras the velocity along tangent to δs will not contribute towards flux acros δs,

$\therefore$ flux aross $\delta s = \delta s \times$ normal velocity

or $(\Psi + \delta\Psi) - \Psi = \delta s \times$ velocity from right to left acros δs

or velocity from right to left cross $\delta s = \dfrac{\partial\Psi}{\partial s}$ ......(4)

Remark 3. :

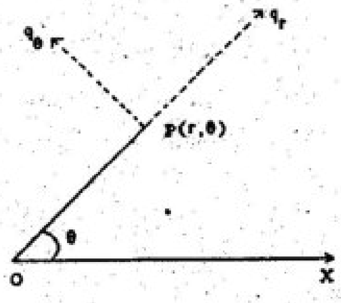
Velocity components in terms of Ψ in plane-polar co-ordinates (r, θ) can be obtained by using the method outlined in remark 2 above. Let q_r and q_θ be velocity components in the directions of r and θ increasing respectively. Then

$-q_r =$ velocity from right to left across $r\delta\theta$

$$= \lim_{\delta\theta \to 0} \frac{\partial\Psi}{r\delta\theta} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}$$

and $q_\theta =$ velocity from right to left acros δr

$$= \lim_{\delta r \to 0} \frac{\partial\Psi}{\delta r} = \frac{\partial\Psi}{\partial r}$$



Thus, $q_r = -\dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta}$ and $q_\theta = \dfrac{\partial\Psi}{\partial r}$ ......(5)

which gives the relationship between the velocity components and the stream function in polar system.

Some aspects of elementrary theory of functions of a complex variables :

Suppose that $z = x + iy$ and that

$$w = f(z) = \phi(x, y) + i\Psi(x, y),$$

where x, y, ϕ, Ψ are all real and $i = \sqrt{-1}$. Also, suppose that ϕ and Ψ and their first derivatives are everywhere continuous within a given region. If at any point of the region specified by z the derivative $\dfrac{dw}{dz} = f'(z)$ is unique, then w is said to be **analytic or regular at that point**. If the derivative is unique throughout the region then w is said to be **analytic or regular throughout the region.** It can be shown that the necessary and sufficient conditions for w to be analytic at z are

$$\frac{\partial\phi}{\partial x}=\frac{\partial\Psi}{\partial x} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\Psi}{\partial x},$$

which are known as the **Cauchy-Riemann equations.** The functions ϕ, Ψ are known as conjugate functions.

Irrotational motion in two-dimensions :

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u=-\frac{\partial\phi}{\partial x}, v=-\frac{\partial\phi}{\partial y} \quad \text{.......(1)}$$

In two-dimensional flow the stream function Ψ always exists such that

$$u=-\frac{\partial\Psi}{\partial y}, v=\frac{\partial\Psi}{\partial x} \quad \text{.......(2)}$$

From (1) and (2), we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\Psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\Psi}{\partial x} \quad \text{......(3)}$$

which are well known Cauchy-Riemann's equations.Hence $\phi + i\Psi$ is an analytic fuction of $z = x + iy$. Moreover ϕ and Ψ are known as **conjugate functions.**

On multiplying and re-writing, (3) given

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\Psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\Psi}{\partial y}=0$$

showing that the families of curves given by ϕ = constant and Ψ = constant interesect orthogonally. **Thus the curves of equi-velocity potential and the stream lines intersect orthogonally.**

Differentiating the equations given in (3) with respect to x and y respectively, we get

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\Psi}{\partial x\partial y} \text{ and } \frac{\partial^2\phi}{\partial y^2}=\frac{\partial^2\Psi}{\partial y\partial x} \quad \text{......(5)}$$

Since $\frac{\partial^2\Psi}{\partial x\partial y}=\frac{\partial^2\Psi}{\partial y\partial x}$, (5) gives $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$ ......(6)

Again, differentiating the equations given in (3) with respect to y and x respectively, we get

$$\frac{\partial^2\phi}{\partial y\partial x}=\frac{\partial^2\Psi}{\partial y^2} \text{ and } \frac{\partial^2\phi}{\partial x\partial y}=\frac{-\partial^2\Psi}{\partial x^2}$$

Subtracting these, $\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=0$ ......(7)

Equations (6) and (7) show that ϕ and Ψ satisfy Laplace's equation when a two-dimensional irrotational motion is considered.

Complex potential :

Let $w = \phi + i\Psi$ be taken as a function of $x + iy$ i.e., z. Thus, suppose that $w = f(z)$ i.e.

$$\phi + i\Psi = f(x+iy) \qquad ......(1)$$

Differentiating (1) w.r.t. x and y respectively, we get

$$\frac{\partial\phi}{\partial x}+i\frac{\partial\Psi}{\partial x}=f'(x+iy) \qquad ......(2)$$

and $$\frac{\partial\phi}{\partial y}+i\frac{\partial\Psi}{\partial y}=if'(x+iy)$$

or $$\frac{\partial\phi}{\partial y}+i\frac{\partial\Psi}{\partial y}=i\left(\frac{\partial\phi}{\partial x}+i\frac{\partial\Psi}{\partial x}\right) \qquad \text{[by (2)]}$$

Equating real and imaginary parts, we get

$$\frac{\partial\phi}{\partial x}=\frac{\partial\Psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\Psi}{\partial x} \qquad ......(3)$$

which are Cauchy-Riemann equations. Then w is an analytical function of z and w is known as the complex potential.

Conversely, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two-dimensional motion.

Remarks :

If $\phi + i\Psi = f(x+iy)$, then $i\phi - \Psi = if(x+iy)$.

$\therefore \quad \Psi - i\phi = -if(x+iy) = g(x+iy)$, say.

Hence proceeding as before, we get (3). hEnce another irrotational motion is also possible in which lines of equi-velocity potential are given by Ψ = constant and the stream lines by ϕ = constant.

Magnitude of velocity :

Let $w = f(z)$ be the complex potential. THen

$$w = \phi + i\Psi \text{ and } z = x+iy \qquad ......(1)$$

Also $$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\Psi}{\partial x} \qquad ......(2)$$

From (1),

$$\frac{dw}{dz}\frac{\partial z}{\partial x}=\frac{\partial \phi}{\partial x}+i\frac{\partial \Psi}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial x}=1$$

$$\therefore \quad \frac{dw}{dz}=\frac{\partial \phi}{\partial x}-i\frac{\partial \phi}{\partial y} \qquad \text{[using (2)]} \qquad ......(3)$$

or $$\frac{dw}{dz}=-u+iv \qquad ......(4)$$

which is called the **complex velocity**.

From (3) and (4), we see that the magnitude of velocity $\overline{q}$ at any point in a two-dimensional irrotational motion is given by $\left|\frac{dw}{dz}\right|$, where

$$\left|\frac{dw}{dz}\right|=\left\{\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2\right\}=\left(u^2+v^2\right)^{\frac{1}{2}}=q \qquad ......(5)$$

Remarks :

The points where velocity is zero are known as **stagnation points**.

Cauchy-Riemann equations in polar form :

Let $\phi+i\Psi=f(z)=f(re^{i\theta})$ ......(1)

Differentiating (1) w.r.t. and θ, we get

$$\frac{\partial \phi}{\partial r}+i\frac{\partial \Psi}{\partial r}=f'\left(re^{i\theta}\right)\cdot e^{i\theta} \qquad ......(2)$$

and $$\frac{\partial \phi}{\partial \theta}+i\frac{\partial \Psi}{\partial \theta}=f'\left(re^{i\theta}\right)\cdot rie^{i\theta} \qquad ......(3)$$

From (2) and (3), we easily obtain

$$\frac{\partial \phi}{\partial \theta}+i\frac{\partial \Psi}{\partial \theta}=ir\left(\frac{\partial \phi}{\partial r}+i\frac{\partial \Psi}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial \theta}=-r\frac{\partial \Psi}{\partial r} \quad \text{and} \quad \frac{\partial \Psi}{\partial \theta}=r\frac{\partial \phi}{\partial r}$$

$$\therefore \quad \frac{\partial \phi}{\partial r}=\frac{1}{r}\frac{\partial \Psi}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta}=-\frac{\partial \Psi}{\partial r} \qquad ......(4)$$

which are Cauchy-Riemann equations in polar form. They give the relationship between the potential and the stream function in polar form.

Exercise :

In irrotational motion in two dimensions, prove that

$$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\nabla^2 q$$

Solution :

Since the motion is irrotational, the velocity potential ϕ exists such that

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad \text{......(1)}$$

Again, $q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2$ ......(2)

Differentiating (2) partially w.r.t. x and y respectively, we get

$$q\frac{\partial q}{\partial x}=\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} \qquad \text{......(3)}$$

$$q\frac{\partial q}{\partial y}=\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} \qquad \text{......(4)}$$

Differentiating (3) and (4) partially w.r.t. x and y respectively, we get

$$q\frac{\partial^2 q}{\partial x^2}+\left(\frac{\partial q}{\partial x}\right)^2=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x^3}+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial x^2\partial y} \qquad \text{...(5)}$$

$$q\frac{\partial^2 q}{\partial y^2}+\left(\frac{\partial q}{\partial y}\right)^2=\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x\partial y^2}+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2+\frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial y^3} \qquad \text{...(6)}$$

Adding (5) and (6) and simplifying, we get

$$q\nabla^2 q+\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2$$

$$+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

$$=2\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 \qquad \text{......(7)}$$

$$\left[\Theta\ \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \text{ so that } \frac{\partial^2\phi}{\partial x^2}=-\frac{\partial^2\phi}{\partial y^2}\ \therefore \left(\frac{\partial^2\phi}{\partial x^2}\right)^2=\left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$

Next, equaring and adding (3) and (4), we get

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right]=\left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right]$$

$$+\left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]+2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$$

$$=\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right] \quad \text{[using (1)]}$$

$$=q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right] \quad \text{[using (2)]}$$

Thus, $$\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right]=\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2 \quad \text{......(8)}$$

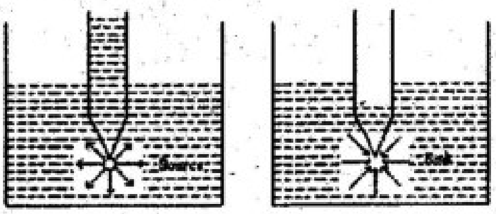
From (7) and (8), we find

$$q\nabla^2 q+\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=2\left[\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right]$$

or $$q\nabla^2 q=\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2$$

Sources and sinks :

If the motion of a fluid consists of symmetrical radial flow in all directions proceeding from a point, the point is known as a **simple source.** If, however, the flow is such that the fluid is directed radially inwards to a point from all directions in a symmetrical manner, then the point is known as a **simple sink.**



Obviously a source implies the creation of fluid at a point whereas a sink implies the annihilation of fluid at a point. Sources and sinks are not readily obtained by some dynamical effects of the motion of fluid but may occur due to some external causes. For example, consider a simple source in a tank filled with a fluid. This source may be created by taking a long tube of very small cross-section and injecting fluid through it into the tank as shown in figure (i). In such a suituation, we find that the fluid is coming out from the tube raduially into the tank. Again, a sink can be created by taking a long tube of very small cross-section and sucking fluid through the tube from the tank as shown in the figure (ii).

Consider a source at the origin. Then the mass m of the fluid coming out from the origin in a unit time is known as the **strength of the source**. Similarly, in a tank at the origin, the amount of fluid going into the sink in a unit time is called the **strength of the sink**.

Remark :

Since the velocity is unique at a point, so usually no two streamlines intersect each other. But some flow fields may have singularities, where the velocity vector is not unique. Sources and sinks are examples of singularities of a flow field because infinitely many stream lines meet at such points as indicated in the figures (i) and (ii).

Source and sinks in two-dimensions :

In two-dimensions a source of strength m is such that the flow across any small curve surrounding in $2\pi m$. Sink is regarded as a source of strength — m.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad ......(1)$$

or $$q_r = -\frac{\partial \phi}{\partial r} \qquad \left[\Theta \quad \frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right] \qquad ......(2)$$

Then the flow across the circle is $2\pi r q_r$. Hence we have

$$2\pi r q_r = 2\pi m \text{ or } r q_r = m \qquad ......(3)$$

or $$r\left(-\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) = m \qquad [\text{by } (1)]$$

Integrating and omitting constant of integration, we get

$$\Psi = -m\theta \qquad ......(4)$$

Using (2) and (3), we obtain as before

$$\phi = -m \log r \qquad ......(5)$$

Equation (4) shows that the stream lines are θ = constant, i.e., straight linea radiating from the source. Again (5) shows that the curves of equi-velocity potential are r = constant, i.e., concentric circles with centre at the source.

Complex potential due to a source :

Let thet be a source of strength m at origin. Then

$$w = \phi + i\Psi = -m\log r - im\theta = -m\log(re^{i\theta}) = -m\log z.$$

If, however, the source is at z', then the complex potential is given by $w = -m\log(z - z')$.

The relation between w and z for sources of strength $m_1, m_2, m_3, \ldots$ situated at points $z = z_1, z_2, z_3, \ldots$ is

$$w = -m_1\log(z - z_1) - m_2\log(z - z_2) - m_3\log(z - z_3) \ldots,$$

leading to $\phi = -m_1\log r_1 - m_2\log r_2 - m_3\log r_3 - \ldots$

and $\Psi = -m_1\theta_1 - m_2\theta_2 - m_3\theta_3 - \ldots$

where $r_n = |z - z_n|$ and $\theta_n = \arg(z - z_n)$, $n = 1, 2, 3, \ldots$

Doublet :

A combination of a source of strength m and a sink of strength - m at a small distance δs apart, where in the limit m is taken infinitely great and δs infinitely small but so that the product $m\delta s$ remains finite and equal to μ, is called a doublet of strength μ; and the line δs taken in the sense from - m to + m is taken as the axis of the doublet.

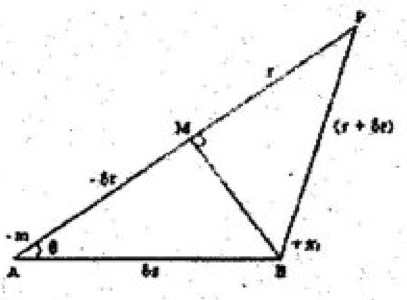
Complex potential due to a doblet in two-dimensions :

Let A, B denote the positions of the sink and source and P be any point. Let AP = r, BP = r + δr and $\angle PAB = \phi$. Let ϕ be the velocity potential due to this doublet.

Then $\phi = m\log r - m\log(r + \delta r)$

$$= -m\log\frac{r+\delta r}{r}$$

or $$\phi = m\log\left(1 + \frac{\delta r}{r}\right)$$



$\therefore$ $$\phi = -m\frac{\delta r}{r} \quad \ldots\ldots(1)$$

[The first order of approximation].

Let BM be perpendicular drawn from B on AP. Then, we get

$$AM = AP - MP = r - (r + \delta r) = -\delta r$$

$\therefore$ $$\cos\theta = \frac{AM}{AB} = -\frac{\delta r}{\delta s} \text{ so that } \delta r = -\delta s\cos\theta$$

$\therefore$ From (1), $$\phi = m\delta s\frac{\cos\theta}{r} = \frac{\mu\cos\theta}{r} \quad \ldots\ldots(2)$$

where $\mu = m\delta s$ = strength of doublet.

From (2), $\frac{\partial\phi}{\partial r} = \frac{-\mu\cos\theta}{r^2}$

or $\quad \frac{1}{r}\frac{\partial\Psi}{\partial\theta} = -\frac{\mu\cos\theta}{r^2} \quad \left[\because \quad \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\right]$

or $\quad \frac{\partial\Psi}{\partial\theta} = -\frac{\mu\cos\theta}{r}$

Integrating it with respect to θ, we get

$$\Psi = -\frac{\mu\sin\theta}{r} + f(r) \qquad \text{......(3)}$$

Now, $\quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\Psi}{\partial r} \qquad \text{......(4)}$

Using (2) and (3), (4) reduces to

$$\frac{1}{r}\left(-\frac{\mu\sin\theta}{r}\right) = -\left[\frac{\mu\sin\theta}{r^2} + f'(r)\right]$$

or $f'(r) = 0$ so that $f(r)$ = constant. Hence omitting the additive constant, (3) reduces to

$$\Psi = -\frac{\mu\sin\theta}{r} \qquad \text{......(5)}$$

$\therefore$ Complex potential due to a doublet is given by

$$w = \phi + i\Psi = \frac{\mu}{r}(\cos\theta - i\sin\theta) \qquad \text{[using (2) and (5)]}$$

$$= \frac{\mu}{r}e^{-i\theta} = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z}$$

Note I. :

If the doublet makes an angle α with x-axis, we have to write $\theta - \alpha$ for θ so that

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu e^{i\alpha}}{re^{i\theta}} = \frac{\mu e^{i\alpha}}{z}$$

If the doublet be at point $A(x', y')$ where $z' = x' + iy'$ [in place of A being origin (0, 0)] then we have

$$\omega = \frac{\mu e^{i\alpha}}{z - z'}$$

Note II. :

If doublets of strength $\mu_1, \mu_2, \mu_3, \ldots$ situated at $z = z_1, z_2, z_3, \ldots$ and their axes makes angles $\alpha_1, \alpha_2, \alpha_3, \ldots$ wit x-axis, then the complex potential due to the above system is given by

$$w=\frac{\mu_1 e^{i\alpha_1}}{z-z_1}+\frac{\mu_2 e^{i\alpha_2}}{z-z_2}+\frac{\mu_3 e^{i\alpha_3}}{z-z_3}+\ldots$$

Exercise :

What arrangement of sorces and sinks will give rise to the function $w=\log\left(z-\frac{a^2}{z}\right)$. Prove that two of the stream lines subdivided into the circle r = a and axis of y.

Solution :

Given $w=\log\left(z-\frac{a^2}{z}\right)=\log\left[\frac{(z-a)(z+a)}{z}\right]$

or $w=\log(z-a)+\log(z+a)-\log z$

which shows that there are two sinks of unit strength at the points z = a and z = - a and a source of unit strength at origin. Since $w=\phi+i\Psi$ and $z=x+iy$, we obtain

$$\phi+i\Psi=\log(x+iy-a)+\log(x+iy+a)-\log(x+iy)$$

$$\therefore\quad \phi+i\Psi=\log[(x-a)+iy]+\log[(x+a)+iy]-\log(x+iy)$$

Equating imaginary parts on both sides, we have

$$\Psi=\tan^{-1}\frac{y}{x-a}+\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x}$$

$$\left[\because\quad \log(\alpha+i\beta)=\frac{1}{2}\log\left(\alpha^2+\beta^2\right)+i\tan\left(\frac{\beta}{\alpha}\right)\right]$$

$$=\tan^{-1}\frac{\frac{y}{x-a}+\frac{y}{x+a}}{1-\frac{y}{x-a}\frac{y}{x+a}}-\tan^{-1}\frac{y}{x}$$

$$=\tan^{-1}\frac{2xy}{x^2-y^2-a^2}-\tan^{-1}\frac{y}{x}$$

$$=\tan^{-1}\frac{\frac{2xy}{x^2-y^2-a^2}-\frac{y}{x}}{1+\frac{2xy}{x^2-y^2-a^2}\frac{y}{x}}$$

$$=\tan^{-1}\frac{y\left(x^2+y^2+a^2\right)}{x\left(x^2+y^2-a^2\right)}$$

The desired stream lines are given by $\Psi=\text{constant}=\tan^{-1}(C)$, i.e.,

$$\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}=C \qquad \text{......(1)}$$

When $C = 0$, (1) reduces to $y = 0$. Thus x-axis is a strem line. Again, when $C \to \infty$, (1) reduces to $x(x^2 + y^2 - a^2) = 0$, i.e., $x = 0$ and $x^2 + y^2 = a^2$ or $r = a$, which are strem lines.

Exercise :

Two sources, each of strength m are placed at points (- a, 0), (a, 0) and a sink of strength 2m at the origin. Show that the stream lines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$ where λ is a variable parameter.

Show also that the fluid speed at any point is $\frac{(2ma^2)}{(r_1r_2r_3)}$, where r_1, r_2, r_3 are the distances of the points from the sources and the sink.

Solution :

First Part :

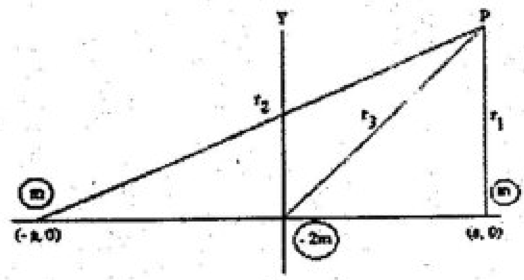
The complex potential w at ant point P(z) is given by

$$w = -m\log(z-a) - m\log(z+a) + 2m\log z \qquad \text{......(1)}$$

or $\quad w = m[\log z^2 - \log(z^2 - a^2)]$

or $\quad \phi + i\Psi = m[\log(x^2 - y^2 + 2ixy) - \log(x^2 - y^2 - a^2 + 2ixy)]$

Equating the imaginary parts, we have



$$\Psi = m\left[\tan^{-1}\left\{\frac{2xy}{(x^2-y^2)}\right\} - \tan^{-1}\left\{\frac{2xy}{x^2-y^2-a^2}\right\}\right]$$

$$\therefore \qquad \Psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right], \text{on simplification.}$$

The desired stream lines are given by $\Psi = \text{constant} = m\tan^{-1}\left(-\frac{2}{\lambda}\right)$. Then we obtain

$$\left(-\frac{2}{\lambda}\right) = \frac{(-2a^2xy)}{[(x^2+y^2)^2 - a^2(x^2-y^2)]}$$

or $\quad (x^2+y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$

Second Part : From (1), we have

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)(z+a)}$$

$$\therefore \quad q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

where $\quad r_1 = |z-a|, r_2 = |z+a|$ and $r_3 = |z|$.

Images :

If in a liquid a surface S can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface. Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it, the motion on the other side will remain unchanged.

As there is no flow across the surface, it must be stream line. Thus the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

Images in two dimensions :

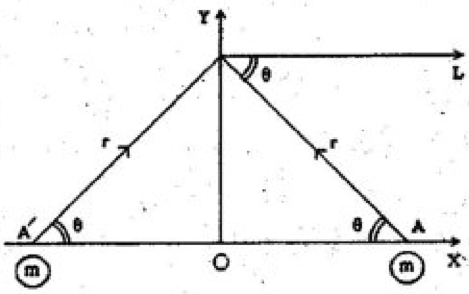
If in a liquid a curve C can be drawn across which there is no flow, then any system of sources, sinks and doublets on opposite sides of this cutve is known as the image of the system with regard to the curve.

Advantages of images in fluid dynamics :

The method of images is used to determine the complex potential due to sources, sinks and doublets in the presence of rigid boundaries. Suppose we wish to determine the flow field outside a rigid boundary due to sources, sinks, doublet lying outside the boundary. To this end we assume the existence of some hypothetical image sources, sinks, doublets within the boundary in such a manner so that the boundardy behaves as a stream line or surface. Then the given system of sources, sirks and doublets together with the hypothetical one will be equivalent to the given sources and the rigid boundaries for the region outside the boundary.

Image of a source with respect to a line :

Suppose that image of source m at A(a, 0) on x-axis is required with respect to OY. Take an equal source at A′(- a, 0). Let P be any point on OY such that AP = A′P = r. Then the velocity at P due to source A′ is $\frac{m}{r}$ along A′P.



Let PL be perpendicular to OY. Then, we see that resultant velocity at P due to sources at A and A′ along PL

$$=\left(\frac{m}{r}\right)\cos\theta-\left(\frac{m}{r}\right)\cos\theta=0$$

showing that there will no flow across OY. Hence by definition, the image of a simple source with respect to a line in two-dimensions is an equal source equidistant from the line opposite to the source.(Proceeding similarly the result is found to be true for sink also)

Corollary : Image of a doublet with respect to a line :

Let PQ be a doublet with its axis inclined at an angle α to OX. Then by using the above result for finding the images of sources and sink with respect to OY, we see that the image of the doublet PQ is again an equal doublet P′Q′ symmetrically placed as shown in figure.

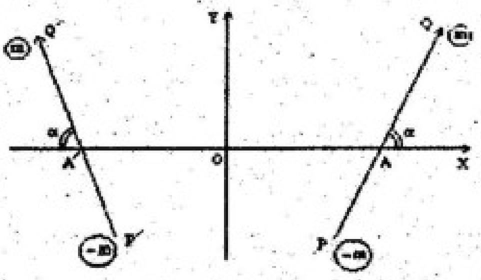
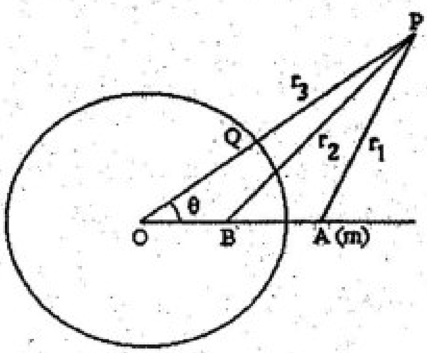


Image of a source with regard to a circle :

Let us determine the image of a source of strength m at a point A with respect to the circle with O as centre. Let OA = f and let B be inverse point of A with respect to the circle. If a be the radius of the circle, then OA.OB = a^2 so that $OB=\frac{a^2}{f}$.



Let there be a source of strengt m at B. If w be the complex potential due to sources at A and B, then we get

$$w=-m\log(z-f)-m\log\left(z-\frac{a^2}{f}\right).$$

$$=-m\left[\log(r\cos\theta-r+ir\sin\theta)+\log\left(r\cos\theta-\frac{a^2}{f}+ir\sin\theta\right)\right]$$

$$(\because \quad z=re^{i\theta}=r\cos\theta+ir\sin\theta)$$

Writing $w=\phi+i\Psi$ and equating real parts, we get

$$\phi=-\,m\frac{m}{2}\left[\log\left\{\left(r\cos\theta-f^2\right)+(r\sin\theta)^2\right\}\right.$$

$$\left.+\log\left\{\left(r\cos\theta-\frac{a^2}{f}\right)^2+(r\sin\theta)^2\right\}\right]$$

$$= -\frac{m}{2}\left[\log\left(r^2+f^2-2fr\cos\theta\right)+\log\left(r^2+\frac{a^4}{f^2}-\frac{2ra^2}{f}\cos\theta\right)\right]$$

$$\therefore \quad \frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[-\frac{2(r-f\cos\theta)}{r^2+f^2-2fr\cos\theta}+\frac{2\left\{r-\left(\frac{a^2}{f}\right)\cos\theta\right\}}{r^2+\frac{a^4}{f^2}-2r\left(\frac{a^2}{f}\right)\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle

$$= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = m\left[\frac{a-f\cos\theta}{a^2+f^2-2fa\cos\theta}+\frac{\left(\frac{a}{f}\right)(f-a\cos\theta)}{\left(\frac{a^2}{f^2}\right)\left(f^2+a^2-2fa\cos\theta\right)}\right]$$

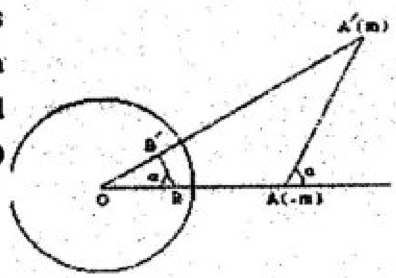
$$= m\left[\frac{a-f\cos\theta+\frac{f^2}{a}-f\cos\theta}{a^2+f^2-2fa\cos\theta}\right] = \frac{m}{a}$$

Now, if we place a source of strength - m at O, the normal velocity due to it at Q will be $-\left(\frac{m}{a}\right)$ and hence the normal velocity of the system will reduce to zero.

Hence the image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle.

Image of doublet with regard to a circle :

Let us determine the image of doublet AA′ with its axis its axis making an angle α with OA, outside the circle, there being a sink - m at A and a source m at A′. Join OA and OA′. Let B and B′ be the inverse points of A and A′ with regard to the circle with O as centre. Then



$OA \, . \, OB = OA' \, . \, OB' = a^2$ ......(1)

where a is the radius of the circle.

Now the image of source m at A′ consists of a source m at B′ and a sink - m at O. Similarly, the image of sink - m at A consists of a sink at B and a source m at O. Compounding these, we see that source m and sink - m at O cancel each other and hence the image of the given dioublet AA′ is another doublet BB′.

Let the strength of the given doublet AA′ be μ.

Then $\mu = \lim_{A \to A'} (m \cdot AA')$ ......(2)

From (1), $\frac{OA}{OA'} = \frac{OB}{OB'}$ ......(3)

showing that triangles OAA′ and OB′B are similar. From these similar triangles, we have

$$\frac{BB'}{AA'} = \frac{OB'}{OA'} = \frac{OB'}{OA} \cdot \frac{OA'}{OA'} = \frac{a^2}{OA \cdot OA'} \quad \text{......(4)}$$

$\therefore$ μ = strength of doublet B′B

$$= \lim_{B' \to B} (m \cdot B'B)$$

$$= \lim_{A \to A'} \frac{a^2}{OA \cdot OA'} (m \cdot AA') \qquad \text{[by (4)]}$$

$$= \frac{\mu a^2}{f^2}, \text{ using (2) and taking } OA = OA' = f$$

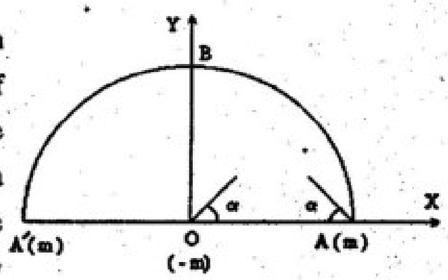
Thus the image of two-dimensional doublet at A with regard to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with the radius OBA.

Example 1. :

In the region bounded by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii. Show that the stream line leaving either end at an angle α with radius is

$$r^2 \sin(\alpha + \theta) = a^2 \sin(\alpha - \theta)$$

Solution :



Let AOB be the circular quarant of radius a wuth OA and OB as bounding radii. Consider a source of strength m at A and a sink of strength - m at O. Then the image system consists of (i) a source m at A(a, 0) (ii) a source m at A′(- a, 0) and (iii) a sink - m at O(0, 0). Hence the complex potential w for the motion of the fluid at any point $P(z = x + iy = re^{i\theta})$ is given by

$$w = - m\log(z - a) - m\log(z + a) + m\log z$$

or $\quad w = -m\log\frac{z^2 - a^2}{z} = -m\log\left(z - a^2 z^{-1}\right)$

or $\quad w = - m\log(re^{i\theta} - a^2 r^{-1} e^{-i\theta}) \qquad [\because \quad z = re^{i\theta}]$

or $\quad w = - m\log[r(\cos\theta + i\sin\theta) - a^2 r^{-1}(\cos\theta - i\sin\theta)]$

$$\phi + i\Psi = -m\log\left[\left(r - \frac{a^2}{r}\right)\cos\theta + i\left(r + \frac{a^2}{r}\right)\sin\theta\right]$$

Equating imaginary parts, we obtain

$$\Psi = -m\tan^{-1}\frac{\left(r + \frac{a^2}{r}\right)\sin\theta}{\left(r - \frac{a^2}{r}\right)\cos\theta} = -m\tan^{-1}\left\{\frac{r^2 + a^2}{r^2 - a^2}\tan\theta\right\}$$

The stream line leaving the end A and O at an angle α is given by

$$\Psi = -m(\pi - \alpha)$$

i.e. $$-m\tan^{-1}\left\{\frac{r^2 + a^2}{r^2 - a^2}\tan\theta\right\} = -m(\pi - \alpha)$$

or $$\frac{(r^2 + a^2)\sin\theta}{(r^2 - a^2)\cos\theta} = \tan(\pi - \alpha) = -\tan\alpha = -\frac{\sin\alpha}{\cos\alpha}$$

or $$(r^2 + a^2)\sin\theta\cos\alpha = -(r^2 - a^2)\cos\theta\sin\alpha$$

or $$r^2\sin(\alpha + \theta) = a^2\sin(\alpha - \theta)$$

Exercise 2. :

In the case of the two-dimensional fluid motion produced by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS meet the circle, prove that its magitude at these points is $2m\frac{OS}{(OS^2 - a^2)}$.

Solution :

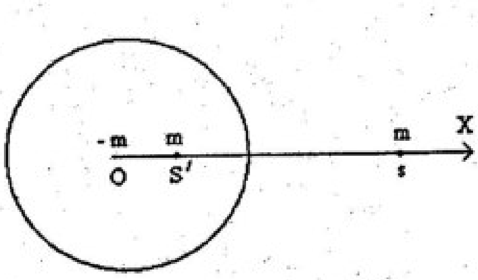
Let S′ be the inverse point of S with respect to the circular disc, with O at its centre. Let OS = c. Then OS.OS′ = a^2 so that $OS' = \frac{a^2}{c}$.

The equivalent image system consists of

(i) a source of strength m at S(c, 0),

(ii) a source of strength m at $S' = \left(\frac{a^2}{c}, 0\right)$,

(iii) a sink of strength - m at O(0, 0).



Let OS be taken as x-axis. Then the complex potential for the motion of the fluid at any point $z(= x+iy = re^{i\theta})$ is given by

$$w = -m\log(z-c) - m\log\left(z - \frac{a^2}{c}\right) + m\log z$$

$$\therefore \quad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-\frac{a^2}{c}} + \frac{m}{z}$$

Let $q = \left|\frac{dw}{dz}\right|$ be the velocity at any point z. Then

$$q = m\left|\frac{1}{z-c} + \frac{m}{z-\frac{a^2}{c}} - \frac{1}{z}\right| = m\left|\frac{(z-a)(z+a)}{z(z-c)\left(z-\frac{a^2}{c}\right)}\right|$$

Hence the velocity at any point $z = ae^{i\theta}$ on the boundary of the circular disc is given by

$$q = m\left|\frac{(ae^{i\theta}-a)(ae^{i\theta}+a)}{ae^{i\theta}(ae^{i\theta}-c)\left(ae^{i\theta}-\frac{a^2}{c}\right)}\right|$$

$$= m\left|\frac{c(e^{i\theta}-1)(e^{i\theta}+1)}{e^{i\theta}(ae^{i\theta}-c)(ce^{i\theta}-a)}\right|$$

$$= mc\left|\frac{(1-e^{i\theta})(1+e^{i\theta})}{(ae^{i\theta}-c)(ce^{i\theta}-a)}\right|$$

$$\therefore \quad q = \frac{2mc\sin\theta}{a^2+c^2-2ac\cos\theta} \quad \text{......(1)}$$

For maximum q, $\frac{dq}{d\theta} = 0$. Hence (1) gives

$$2mc\frac{(a^2+c^2-2ac\cos\theta)\cos\theta - \sin\theta(2ac\sin\theta)}{(a^2+c^2-2ac\cos\theta)^2} = 0$$

or $\quad (a^2+c^2)\cos\theta - 2ac = 0$ or $\cos\theta = \frac{2ac}{a^2+c^2}$ ......(2)

Since $\theta = 0$ gives the minimum velocity [q becomes zero at $\theta = 0$ by (1)], the value of θ given by (2), must correspond to the maximum value of velocity q. Moreover (2) gives the same angles which the diameter through the point where the line joining S to the end of the diameter at right angles to OS cuts the circle, will make with OS.

From (2), $\sin\theta = \sqrt{1-\cos^2\theta} = \dfrac{c^2-a^2}{c^2+a^2}$ ......(3)

Using (1), (2) and (3), the maximum value of q is given by

$$q = \frac{2mc\left(\dfrac{c^2-a^2}{c^2+a^2}\right)}{a^2+c^2-\dfrac{4a^2c^2}{a^2+c^2}} = \frac{2mc\left(c^2-a^2\right)}{\left(a^2+c^2\right)^2-4a^2c^2}$$

or $\quad q = \dfrac{2mc}{c^2-a^2} = \dfrac{2m.OS}{OS^2-a^2}$ Since the boundary of the circular disc is a stream line, the velocity on the boundary is the velocity of the slip.

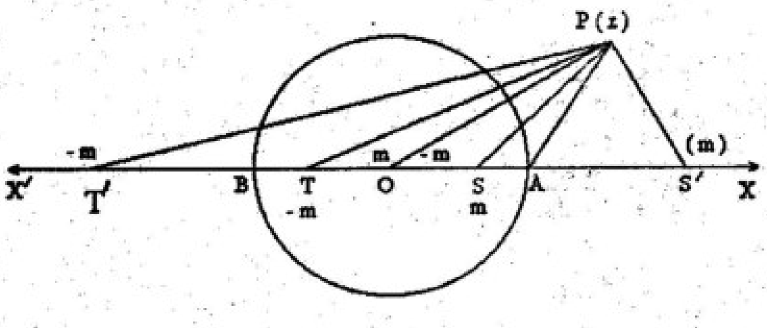
Exercise 3. :

A source S and a sink T of equal strenghts m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, show that the velocity of the liquid at any point P is

$2m\dfrac{OS^2+OA^2}{OS}\dfrac{PA.PB}{PS\cdot PS'\cdot PT\cdot PT'}$ where S' and T' are the inverses of S and T with respect to the circle.

Solution :

Let OS = OT = c. Then, we have OA = a, OS . OS' = a^2 and OT . OT' = a^2 so that



$$OS' = \frac{a^2}{c} \text{ and } OT' = \frac{a^2}{c} \qquad ......(1)$$

Now the image system of source m at S consists of a source m at S' and a sink - m at O. Again the image system of a sink - m at T' consists of a sink -m at T' and a source m at o. Compounding these, we find that source m and sink - m at O cancel each other. Hence the equivalent image system finally consists of

(i) a source of strength m at S(c, 0)

(ii) a source of strength m at $S\left(\frac{a^2}{c},0\right)$

(iii) a sink of strength - m at T(- c, 0)

(iv) a sink of strength - m at $T'\left(-\frac{a^2}{c},0\right)$

Taking OS as the x-axis, the complex potential at any point z(= x + iy) is given by

$$w = -m\log(z-c) - m\log\left(z-\frac{a^2}{c}\right) + m\log(z+c) + m\log\left(z+\frac{a^2}{c}\right)$$

$$\therefore \quad \frac{dw}{dz} = -\frac{m}{z-c} - \frac{m}{z-\frac{a^2}{c}} + \frac{m}{z+c} + \frac{m}{z+\frac{a^2}{c}}$$

The velocity $q = \left|\frac{dw}{dz}\right|$ at any point is given by

$$q = m\left|-\frac{2c}{z^2-c^2} - \frac{2\frac{a^2}{c}}{z^2-\frac{a^4}{c^2}}\right|$$

$$= 2m\frac{c^2+a^2}{c}\left|\frac{z^2-a^2}{\left(z^2-c^2\right)\left(z^2-\frac{a^4}{c^2}\right)}\right|$$

$$= 2m\frac{c^2+a^2}{c}\frac{|z-a||z+a|}{|z-c||z||z+c|\left|z-\frac{a^2}{c}\right|\left|z+\frac{a^2}{c}\right|}$$

$$= 2m\frac{OS^2+OA^2}{OS}\frac{PA\cdot PB}{PS\cdot PS'\cdot PT\cdot PT'}$$

Exercise 4. :

With a rigid boundary in the form of the circle $(x+\alpha)^2+(y-4\alpha)^2=8\alpha^2$, there is a liquid motion due to a doublet of strenght μ at the point (0, 3a) with its axis along of y. Show that the velocity potential is

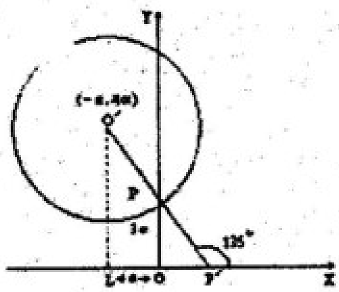
$$\mu\left\{\frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2}+\frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right\}$$

Solution :

The given circle has centre $O'(-\alpha, 4\alpha)$ and radius $= \sqrt{(8\alpha^2)} = 2\sqrt{2}\alpha$. Let the given doublet be at $P(0, 3\alpha)$.

Gradient of $O'P = \dfrac{3\alpha-4\alpha}{0-(-\alpha)} = -1 = \tan\dfrac{3\pi}{4}$.

Hence $O'P$ makes an angel $\dfrac{\pi}{4}$ with OY. Let P' be the image of P. Then the axis



of P' will make an angle 45^0 with PP' and hence it will be parallel to x-axis.

We now show that P' lies on x-axis. We have

$$O'P \,.\, O'P' = 8\alpha^2$$

or $\quad O'P(O'P + PP') = 8\alpha^2$

But $\quad O'P = \sqrt{(-\alpha-0)^2+(4\alpha-3\alpha)^2} = \alpha\sqrt{2}$ ......(2)

$\therefore \quad$ From (2), $\alpha\sqrt{2}\left(\alpha\sqrt{2}+PP'\right) = 8\alpha^2$

or $\quad PP' = 3\alpha\sqrt{2} = 3\alpha\sec 45^0 = OP\sec 45^0$.

This shows that P' lies on the x-axis and that co-ordinates of P' are $(3\alpha, 0)$. Let μ be the strength of P. Then the strength of P' is

$$=\mu\frac{(\text{radius})^2}{(O'P)^2} = \mu\frac{8\alpha^2}{2\alpha^2} = 4\mu.$$

Thus the equivalent image system consists of doublet at P and P'. Hence the complex potential of motiom at point $z(= x + iy)$ is given by

$$w = \frac{\mu e^{\frac{\pi i}{2}}}{z-3i\alpha}+\frac{4\mu e^{0.i}}{z-3\alpha}$$

or $$\phi + i\Psi = \mu\left[\frac{4}{x+iy-3\alpha} + \frac{i}{x+iy-3i\alpha}\right]$$

$$= \mu\left[4\frac{(x-3\alpha)-iy}{(x-3\alpha)^2+y^2} + \frac{i\{x-i(y-3\alpha)\}}{x^2+(y-3\alpha)^2}\right]$$

Equating real parts, we get

$$\phi = \mu\left[\frac{4(x-3\alpha)}{(x-3\alpha)^2+y^2} + \frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right]$$

Connectivity : Definition :

A region of space is said to be connected if a continuous curve joining any points of the region lies entirely in the given region.

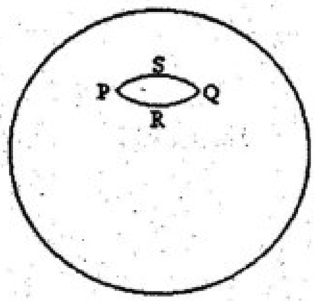
Thus the region interior to a sphere, or the region between two coaxial infinitly long cylinders are connected.

Reducible and Irreducible Circuits :

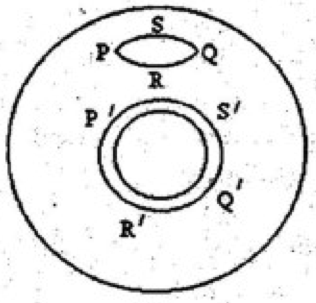
Definition :

A closed circuit, all of whose points lie in the region, is said to be reducible if it can be shrunk to a point of the region without passing outside of the region.

The circuit PRQS in figures (i) and (ii) are reducible; the circuit P'R'Q'S' in figure (ii) is irreducible, for it cannot be made smaller than the circumference of the inner cylinder.



(i)



(ii)

Simply Connected Reigion :

Definition :

A region in which every circuit is reducible is known as simply connected.

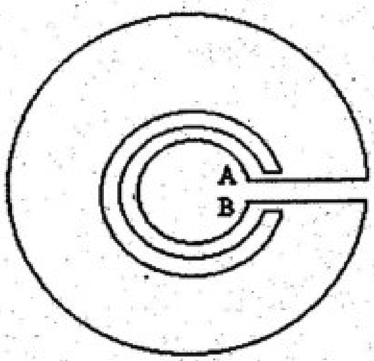
Thus the region interior to a sphere, the region exterior to a sphere, the region between two concentric spheres, unbounded space etc. are simply connected regions. The region between the concentric cylinders in figure (ii) above is not simply connected, for it contains irreducible circuits. This region can be made simply connected by inserting one boundary or barrier which may not be crossed, such as AB containing a generating line of each cylinder as shown in figure (ii).

With the insertion of this barrier each circuit in the modified region becomes reducible and hence the modified region is simply connected.

Double Connected and n-ply Connected Regions :

Definition :

A region is said to be doubly connected, if it can be made simply connected by the insertion of one barrier. Similarly, a region is said to ne n-ply connected, if can be made simply conneceted by the insertion of n - 1 barriers.



Thus the region between two coaxial infinitely long long cylinders, the region exterior to an infinitely long cylinder, the region interior (or exterior) to an anchor ring etc. are doubly connected regions.

Flow and Circulation :

If A, P be any two points in a fluid the value of the integral

$$\int_A^P (udx + vdy + wdz),$$

taken olong any path from A to P, is called the **flow** along that path from A to P.

When a velocity potential ϕ exists, the flow from A to P is

$$= -\int_A^P \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \int_A^P d\phi = \phi_A - \phi_P$$

The flow round a closed curve is known as the **circulation** round the curve. Let C be closed curve and Γ be the circulation, then we have

$$\Gamma = \int_C (udx + vdy + wdz) = \int_C \bar{q} \cdot d\bar{r}$$

where the line integral is taken round C in a counter clockwise direction and $\bar{q}$ is the velocity vector.

Remark :

When a single-valued velocity potential exists the circulation round any closed curve is clearly zero. Again, itis seen that if the velocity potential is many-valued then there exists curves for which the circulation is zero, though it is not zero for all such paths.

Stokes's Theorem :

Let $\bar{q}$ be the velocity vector, Ω the velocity vector and S be a surface bounded by a closed curve

C. Then $\int_C \bar{q}\cdot d\bar{r} = \int_C \text{curl}\,\bar{q}\cdot \hat{n}dS$ i.e., $\Gamma = \int_C \Omega\cdot\hat{n}dS$

where Γ is the circulation round C and the unit normal vector $\hat{n}$ at any point of S is drawn in the sense in which a right-handed screw would move when rotated in the sense of description of C.

Kelvin's Circulation Theorem :

When the external forces are conservative and derivable from a single valued potential function and the density is a function of pressure only, the circulation in any closed circuit moving with the fluid is constant for all time.

Proof :

Let C be a closed circuit moving with the fluid so that C always consists of the same fluid particles. Let $\bar{q}$ be the fluid velocity at point P of the circuit and let $\bar{r}$ be its position vector. Then the circulation along the closed circuit C is given by

$$\Gamma = \int_C \bar{q}\cdot d\bar{r}$$

or $$\frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_C \bar{q}\cdot d\bar{r}$$

Since the above integration is performed at constant time, reversing the order of integration and differentiation is justified. Then (1) may be re-written as

$$\frac{D\Gamma}{Dt} = \int_C \frac{D}{Dt}(\bar{q}\cdot d\bar{r}) \quad ......(2)$$

But $$\frac{D}{Dt}(\bar{q}\cdot d\bar{r}) = \frac{D\bar{q}}{Dt}.d\bar{r} + \bar{q}.\frac{D}{Dt}d\bar{r}$$

$$= \frac{D\bar{q}}{Dt}.d\bar{r} + \bar{q}\cdot d\bar{q} \quad ......(3)$$

The Euler's equation of motion is

$$\frac{D\bar{q}}{Dt} = \bar{F} - \frac{1}{\rho}\nabla p \quad ......(4)$$

Let the external forces be conservative and deriavable from a single valued potential function V. Then $\bar{F} = -\nabla V$ and hence (4) becomes

$$\frac{D\bar{q}}{Dt} = -\nabla V - \frac{1}{\rho}\nabla p \quad ......(5)$$

$$\therefore \quad \frac{D\bar{q}}{Dt}d\bar{r} = -\nabla V\cdot d\bar{r} - \frac{1}{\rho}\nabla p\cdot d\bar{r}$$

$$= -dV - \frac{dp}{\rho} \qquad \text{......(6)}$$

Also $\bar{q} \cdot d\bar{q} = \frac{1}{2} d(\bar{q} \cdot \bar{q}) = \frac{1}{2} dq^2$ ......(7)

Using (6) and (7), (3) reduces to

$$\frac{D}{Dt}(\bar{q} \cdot d\bar{r}) = -dV - \frac{1}{\rho} dp + \frac{1}{2} dq^2 \qquad \text{......(8)}$$

Using (8) and assuming that ρ is a single-valued function of p, (2) reduces to

$$\frac{D\Gamma}{Dt} = \left[\frac{1}{2} q^2 - V - \int_C \frac{dp}{\rho} \right]_C \qquad \text{......(9)}$$

where $[]_C$ denotes change in the quantity enclosed within brackets on moving once round C. Since q, V and p are single-valued functions of $\bar{r}$, so R.H.S. of (9) vanishes. (9) gives the rate of change of flow along any closed circuit moving with the fluid. Hence it follows that the circulation in any closed circuit moving with the fluid is constant for all time.

Permanance of Irrotational motion :

When the external forces are consecvative and derivable from a single-valued potential, and density is a function of pressure only, then the motion of an inviscid fluid, if once irrotational, remains irrotational even afterwards.

Proof :

From Stokes's theorem, the circulation is given by

$$\Gamma = \int_C \bar{q} \cdot d\bar{r} = \int_C \text{Curl}\, \bar{q} \cdot \hat{n} dS \qquad \text{......(1)}$$

At any time t, let the motion be irrotational so that Curl $\bar{q} = 0$. Then (1) shows that $\Gamma = 0$ at that instant. Hence it follows from Kelvin's circulation theorem that $\Gamma = 0$ for all time. Hence at any subsequent time, (1) shows that

$$\int_C \text{Curl}\, \bar{q} \cdot \hat{n} dS = 0 \qquad \text{......(2)}$$

Since S is arbitrary, (2) shows that Curl $\bar{q} = 0$ at all subsequent time i.e., the motion remains irrotational even afterwards.

Green's Theorem :

If ϕ, ϕ' are both single-valued and continuously differentiable scalar point functions such that $\nabla\phi$ and $\nabla\phi'$ are also continuously differentiable, then

$$\int_V (\nabla\phi \cdot \nabla\phi')dV = \int_s \phi \frac{\partial\phi}{\partial n} dS - \int_V \phi\nabla^2\phi' dV$$

$$= \int_s \phi' \frac{\partial\phi}{\partial n} dS - \int_V \phi'\nabla^2\phi dV,$$

where S is closed surface bounding any simply-connected region, δn is an element of inward normal at a point on S, and V is the volume enclosed by S.

Proof :

From vector calculus, we have

$$\nabla \cdot (\phi\bar{a}) = \bar{a} \cdot (\nabla\phi) + \phi(\nabla \cdot \bar{a}) \quad \text{......(1)}$$

where ϕ is a scalar point function and $\bar{a}$ is a vector point function.

Replacing $\bar{a}$ by $\nabla\phi'$ in (1), we get

$$\nabla \cdot (\phi\nabla\phi') = (\nabla\phi') \cdot (\nabla\phi) + \phi(\nabla \cdot \nabla\phi') \quad \text{......(2)}$$

Integrating both sides of (2) over volume V, we get

$$\int_V \nabla \cdot (\phi\nabla\phi')dV = \int_V (\nabla\phi') \cdot (\nabla\phi)dV + \int_V \phi(\nabla \cdot \nabla\phi')dV \quad \text{...(3)}$$

By Gauss divergence theorem, we have

$$\int_V \nabla \cdot (\phi\nabla\phi')dV = \int_V \hat{n} \cdot (\phi\nabla\phi')dS$$

where $\hat{n}$ is the unit vector drawn to the surface S.

or $$\int_V \nabla \cdot (\phi\nabla\phi')dV = -\int_S \phi(\hat{n} \cdot \nabla\phi')dS$$

or $$\int_V \nabla \cdot (\phi\nabla\phi')dV = -\int_S \phi \frac{\partial\phi'}{\partial n} dS \quad \text{......(4)}$$

Again, $\nabla \cdot \nabla\phi' = \nabla^2\phi'$ and $\nabla\phi' \cdot \nabla\phi = \nabla\phi \cdot \nabla\phi'$ ......(5)

Using (4) and (5), (3) reduces to

$$-\int_S \phi \frac{\partial\phi'}{\partial n} dS = \int_V (\nabla\phi \cdot \nabla\phi')dV + \int_V \nabla^2\phi' dV$$

$$\int_V (\nabla\phi \cdot \nabla\phi')dV = -\int_S \phi \frac{\partial\phi'}{\partial n} dS - \int_V \phi\nabla^2\phi' dV \quad \text{......(6)}$$

Interchanging ϕ and ϕ' in (6), we have

$$\int_V (\nabla\phi' \cdot \nabla\phi)dV = -\int_S \phi' \frac{\partial\phi}{\partial n} dS - \int_V \phi'\nabla^2\phi dV$$

or $$\int_V (\nabla\phi \cdot \nabla\phi')dV = -\int_S \phi' \frac{\partial\phi}{\partial n} dS - \int_V \phi' \nabla^2 \phi dV \qquad \text{......(7)}$$

(6) and (7) together prove the Green's theorem.

Deduction from Green's Theorem :

Deduction I.

Let ϕ, ϕ' be the velocity potentials of two liquid motions taking place within S. Then $\nabla^2\phi = 0 = \nabla^2\phi'$ and hence green's theorem yields

$$\int_S \phi \frac{\partial\phi'}{\partial n} dS = \int_S \phi' \frac{\partial\phi}{\partial n} dS$$

or $$\int_S \rho\phi\left(-\frac{\partial\phi'}{\partial n}\right) dS = \int_S \rho\phi'\left(-\frac{\partial\phi}{\partial n}\right) dS \qquad \text{......(1)}$$

But $-\frac{\partial\phi}{\partial n}$ is the normal velocity inwards and $\rho\phi$ is the impulsive pressure at any point on the surface which will produce velocity potential ϕ from rest. Hence (1) shows that if there be two possible motions inside S by means of two different impulsive pressures on the boundary, then the work done by the first in acting through the displacement produced by the second must be equal to the work done ny the second in acting through the displacement produced by the first.

Deduction II.

Let $\phi' = \text{constant}$ (=k, say). Then $\nabla^2\phi' = 0 = \frac{\partial\phi'}{\partial n}$ everywhere. If ϕ be the velocity potential of a liquid motion within S, then by Green's theorem, we have

$$\int_S k \frac{\partial\phi}{\partial n} dS = 0 \text{ or } \int_S \frac{\partial\phi}{\partial n} dS = 0 \qquad (2)$$

Since $\frac{\partial\phi}{\partial n}$ is the normal velocity outwards, $\frac{\partial\phi}{\partial n} dS$ represents the flow across dS per unit time. Then (2) shows that the total flow across S is zero i.e. the quantity of a liquid inside S remains constant.

Deduction II.

Let $\phi = \phi'$ and let ϕ be the velocity potential of a liquid motion within S. Then $\nabla^2\phi = 0$ and hence Green's theorem gives

$$\int_V (\nabla\phi.\nabla\phi)dv = -\int_S \phi\frac{\partial\phi}{\partial n}dS$$

or $$\int_V\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]dv = -\int_S \phi\frac{\partial\phi}{\partial n}dS \qquad (3)$$

Let q be the velocity and ρ be the density of the liquid, then (3) reduces to

$$\frac{1}{2}\rho\int_V q^2 dv = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}ds \qquad (4)$$

Clearly the L.H.S of (4) represents the kinetic energy T of the liquid within S. Hence (4) reduces to

$$T = -\frac{1}{2}\rho\int_S \frac{\partial\phi}{\partial n}ds \qquad (5)$$

Now $\rho\phi$ is the impulsive pressure that would set up the motion instantaneously from rest, and $-\frac{\partial\phi}{\partial n}$ is the inward normal velocity at the surface. Hence (5) shows that the kinetic energy set up by impulses, in a system starting from rest, is the sum of the products of each impulses and half the velocity of its point of application. From (5), we also find that the **kinetic energy of a given mass of liquid moving irrotationaly in a simply connected region depends only on the motion of its boundaries.**

Suppose on the boundary $\frac{\partial\phi}{\partial n}=0$. Then (4) reduces to $\int_V q^2 dv = 0$. Since q^2 is positive, (6) implies that q =0 everywhere. Hence the liquid is at rest. Thus, a cyclic irrotational motion is impossible in a liquid bounded by fixed rigid boundary.

Kinetic Energy of Infinite Liquid :

Consider an infinite mass of liquid moving irrotationally, at rest at infinity, and bounded internally by a solid surface S and externally by a large surface S′. Let ϕ be the single-valued velocity potential. Then from deduction III of above, the kinetic energy T of the liquid contained into the region bounded by S and S′ is given by

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS - \frac{1}{2}\rho\int_{S'} \phi\frac{\partial\phi}{\partial n}dS' \qquad ......(1)$$

Since there in no flow into the region across S, the equation of continuity takes the form

$$\int_S \frac{\partial\phi}{\partial n}dS + \int_{S'} \frac{\partial\phi}{\partial n}dS' = 0 \qquad ......(2)$$

Multiplying (2) by $\frac{1}{2}C$, a constant, and subtracting from (1), we get

$$T=-\frac{1}{2}\rho\int_S(\phi-C)\frac{\partial\phi}{\partial n}dS-\frac{1}{2}\rho\int_{S'}(\phi-C)\frac{\partial\phi}{\partial n}dS'=0 \quad ......(3)$$

Since for the solid boundary S, $\int_S\frac{\partial\phi}{\partial n}dS=0$, it follows from (2) that $\int_{S'}\rho\frac{\partial\phi}{\partial n}dS'=0$, i.e.,

$\int_{S'}\frac{\partial\phi}{\partial n}dS'$ is independent of S'. Let $\phi\to C$ at infinity and let the surface S' be enlarged indefinitely in all directions. Then the second integral in (3) vanishes and hence the required kinetic energy of infinite liquid is given by

$$T=-\frac{1}{2}\rho\int_S(\phi-C)\frac{\partial\phi}{\partial n}dS$$

i.e., $$T=-\frac{1}{2}\rho\int_S\phi\frac{\partial\phi}{\partial n}dS \quad \left[\because \quad \int_S\frac{\partial\phi}{\partial n}dS=0\right] \quad ......(4)$$

Remark :

For the motion of liquid to exist, T must not vanish. Hence all internall boundaries must not be at rest.

Kelvin's minimum energy theorem :

The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Proof :

Let T_1 be the kinetic energy, $\bar{q}_1$ the fluid velocity of the actual irrotational motion with a velocity potential ϕ. Then

$$\bar{q}_1=-\nabla\phi \quad ......(1)$$

Let T_2 be the kinetic energy, $\bar{q}_2$ the fluid velocity of any other possible state of motion consistent with the same normal velocity of the boundary S.

Continuity equations for the above two motions give

$$\nabla\cdot\bar{q}_1=0 \text{ and } \nabla\cdot\bar{q}_2=0 \quad ......(2)$$

Let $\hat{n}$ denote the unit normal at a point of S. then using the fact that the boundary has the same normal velocity in both motions, we have

$$\hat{n}\cdot\bar{q}_1=\hat{n}\cdot\bar{q}_2 \quad ......(3)$$

Now, $T=-\frac{1}{2}\rho\int_V q_1^2dV=\frac{1}{2}\rho\int_V\bar{q}_1^2dV$

and $T = \frac{1}{2}\rho\int_V q_2{}^2 dV = \frac{1}{2}\rho\int_V \bar{q}_2{}^2 dV$

$\therefore \quad T_2 - T_1 = \frac{1}{2}\rho\int_V \left(\bar{q}_2{}^2 - \bar{q}_1{}^2\right) dV$

$$= \frac{1}{2}\rho\int_V \left\{2\bar{q}_1 . (\bar{q}_2' - \bar{q}_1) + (\bar{q}_2 - \bar{q}_1)^2\right\} dV$$

$$= \rho\int_V \bar{q}_1 . (\bar{q}_2 - \bar{q}_1) dV + \frac{1}{2}\rho\int_V (\bar{q}_2 - \bar{q}_1)^2 dV$$

$$= -\rho\int_V (\nabla\phi)\cdot(\bar{q}_2 - \bar{q}_1) dV + \frac{1}{2}\rho\int_V (\bar{q}_2 - \bar{q}_1)^2 dV \quad ...(4)$$

[Using (1)]

But $\nabla\cdot[(\phi(\bar{q}_2 - \bar{q}_1))] = \phi[\nabla\cdot(\bar{q}_2 - \bar{q}_1)] + (\nabla\phi)\cdot(\bar{q}_2 - \bar{q}_1)$

$= (\nabla\phi)\cdot(\bar{q}_2 - \bar{q}_1)$ [Using (2)]

$\therefore \quad \int_V (\nabla\phi)\cdot(\bar{q}_2 - \bar{q}_1) dV = \int_V \nabla . [\phi(\bar{q}_2 - \bar{q}_1)] dV$

$= \int_S \phi\hat{n} . (\bar{q}_2 - \bar{q}_1) dS$ [by Divergence Theorem]

$= 0$ [Using (3)] ......(5)

Making use of (5), (4) reduces to

$$T_2 - T_1 = \frac{1}{2}\rho\int_V \left(\bar{q}_2{}^2 - \bar{q}_1{}^2\right) dV \quad ......(6)$$

Since R.H.S. of (6) is non-negative, we have $T_2 - T_1 \geq 0$, i.e., $T_1 \leq T_2$. Hence the result.

●●●

Motion of Cylinders

Motion of a Circular Cylinder :

To determine the motion of a circular cylinder moving in an infinite mass of the liquid at rest at infinity, with velocity U in the direction of x-axis.

To find the velocity potential ϕ that will satisfy the given boundary conditions, We have the following considerations :

(i) ϕ satisfies the Laplace's equation $\nabla^2\phi = 0$ at every point of the liquid. In polar coordinates in two dimensions $\nabla^2\phi = 0$ takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial^2\theta} = 0 \qquad \text{......(1)}$$

We know that (1) has solutions of the form

$r^n \cos n\theta$, $r^n \sin n\theta$,

where n is any integer, positive or negative. Hence, the sum of any number of terms of the form

$A_n r^n \cos n\theta$, $B_n r^n \sin n\theta$, is also a solution of (1),

(ii) Normal velocity at any point of the cylinder

= velocity of the liquid at that point in that direction

i.e. $$-\frac{\partial\phi}{\partial r} = U\cos\theta, \qquad \text{when } r = a \qquad \text{......(2)}$$

(iii) Since the liquid is at rest at infinity, velocity must be zero there.

Thus, $$-\frac{\partial\phi}{\partial r} = 0, \quad \text{and} \quad -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = 0 \qquad \text{at } r = \infty \qquad \text{......(3)}$$

The above considerations suggest that we must assume the following suitable form of ϕ.

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta \qquad \text{......(4)}$$

From (4), $$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{B}{r^2}\right)\cos\theta \qquad \text{..........(5)}$$

Putting $r = a$ in (5) and using (3), we get

$$U\cos\theta = -\left(A - \frac{B}{a^2}\right)\cos\theta \quad \text{or} \quad A - \frac{B}{a^2} = -U \qquad \text{.......(6)}$$

Putting $r = \infty$ in (5) and using (3), we get

$0 = -A\cos\theta$ so that $A = 0$.

Then (6) gives $B = Ua^2$. Hence, (4) reduces to

$$\phi = \frac{Ua^2}{r}\cos\theta \qquad ....(7)$$

It may be noted that (7) also satisfies the second condition given by (3). Hence (7) gives the required velocity potential.

But $$\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad ....(8)$$

$$\therefore \quad \frac{\partial \psi}{\partial r} = \frac{Ua^2}{r^2}\sin\theta, \qquad \text{by (7) and (8)}$$

Integrating, (neglecting const of integration) $\psi = -\frac{Ua^2}{r}\sin\theta,$ .... (9)

which gives the stream function of the motion. The complex potential $w(=\phi + i\psi)$ is given by

$$w = \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{r}e^{-i\theta} = \frac{Ua^2}{z} \qquad .... (10)$$

where $z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Note 1 : From (7) and (9), we find that the velocity potential and stream function are the same as for a two dimensional doublet of strength Ua^2 on the axis of the cylinder in an infinite mass of liquid.

Note 2 : The stream lines are given by $\psi = \text{const.}$ i.e by

$$-\frac{Ua^2}{r}\sin\theta = -\frac{Ua^2}{c} \quad \text{or} \quad cr\sin\theta = r^2$$

i.e. $$x^2 + y^2 - cy = 0 \quad \text{or} \quad x^2 + \left(y - \frac{c}{2}\right)^2 = \left(\frac{c}{2}\right)^2$$

which are circles all touching x-axis at the origin and having the center $\left(0, \frac{c}{2}\right)$

Liquid streaming past a fixed Circular Cylinder :

Let the cylinder be at rest and let the liquid flow past the cylinder with velocity U in the negative direction of x-axis. This motion may be deduced from that of the previous article by imposing a velocity -U parallel to the x-axis on both the cylinder and the liquid. The cylinder is then reduced to rest and we must add to the velocity potential a term Ux (i.e. $Ur\cos\theta$) to account for the additional velocity; consequently a term $Ur\sin\theta$ must be added to ψ. Thus, we have

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta, \quad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta \qquad ....(1)$$

and $\quad w = \phi + i\psi = U(r\cos\theta + ir\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$

$$= Ure^{i\theta} + \frac{Ua^2}{re^{i\theta}} = Uz + \frac{Ua^2}{z} \qquad ......(2)$$

Note 1 : Here, the equation $\left(r - \frac{a^2}{r}\right)\sin\theta = \text{const}$ represents the stream lines relative to the cylinder, and this is true whether the cylinder be moving or at rest.

Note 2 : The velocity distribution at any point $z = ae^{i\theta}$ on the cylinder is given by

$$q = \left|\frac{dw}{dz}\right| \quad \text{when} \quad z = ae^{i\theta}$$

$$= \left|U - \frac{Ua^2}{z^2}\right| \quad \text{when} \quad z = ae^{i\theta}, \quad \text{by (2)}$$

$$= \left|U - \frac{Ua^2}{a^2e^{2i\theta}}\right| = |U - Ue^{-2i\theta}| = |U||1 - e^{-2i\theta}| = |U||1 - \cos 2\theta + i\sin 2\theta|$$

$$= |U|\sqrt{\{(1-\cos 2\theta)^2 + \sin^2 2\theta\}} = |U|\sqrt{2 - 2\cos 2\theta} = 2|U|\sin\theta \quad .....(3)$$

The maximum value of q occurs where $\sin\theta = 1$ i.e., $\theta = \frac{\pi}{2}$. Thus, we have

$$q_{max} = 2|U| = \text{twice the velocity of free stream.}$$

Stagnation points (or critical points) occur where $q = 0$ i.e., $\sin\theta = 0$ i.e. $\theta = 0$ or $\theta = \pi$.

Note 3:

We now determine pressure on the boundary of the cylinder. Let Π be the pressure at infinity and U be free stream velocity at infinity. Then Bernoulli's equation gives

$$p + \frac{1}{2}\rho q^2 = \text{constant} = \Pi + \rho\frac{1}{2}U^2$$

or $\quad p - \Pi = \frac{1}{2}\rho U^2(1 - 4\sin^2\theta) \quad (\text{by } 3) \qquad ......(4)$

We know that a liquid is not able to sustain a negative pressure. Moreover it maintains contact with boundary so long as the pressure remains positive everywhere. When the pressure p given by (4) becomes negative, the theory breaks down and cavitation occurs. Hence, for p to be positive, we must have

$$\Pi+\frac{1}{2}\rho U^2(1-4\sin^2\theta)>0 \text{ at } \theta=\frac{\pi}{2}$$

i.e. $$\Pi-\frac{3}{2}\rho U^2>0 \text{ or } U^2<\frac{2\Pi}{3\rho}$$

Thus, if $U>\sqrt{\frac{2\Pi}{3\rho}}$, cavitation occurs.

Exercise :

A circular cylinder is placed in a uniform stream, find the forces acting on the cylinder.

Solution :

We know that the complex potential for the undisturbed motion is given by $w=(u-iv)z$. Using Milne-Thomson's circle theorem (for statement see next page), the complex potential for the present problem is

$$w=(u-iv)z+(u+iv)\left(\frac{a^2}{z}\right)$$

$$\therefore \quad \frac{dw}{dz}=u-iv-(u+iv)\left(\frac{a^2}{z^2}\right)$$

If the pressure thrusts on the contour of the fixed circular cylinder be represented by a force (X, Y) and a couple of moment N about the origin of co-ordinates, then by Blasius' theorem,(for statement see next page) we have

$$X-iY=\frac{1}{2}i\rho\int_c\left(\frac{dw}{dz}\right)^2dz=\frac{1}{2}i\rho\int\left\{(u-iv)-(u+iv)\left(\frac{a^2}{z^2}\right)\right\}^2dz=0$$

so that $X=0$ and $Y=0$

Also $$N=\text{Real part of } -\frac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2dz$$

$$=\text{real part of } -\frac{1}{2}\rho\int_C z\left\{u-iv-(u+iv)\frac{a^2}{z^2}\right\}^2dz$$

$$=\text{real part of } -\frac{1}{2}\rho\{-2(u^2+v^2)a^2\}2\pi i \;=0$$

$\therefore \quad X=Y=N=0$, showing that neither a force nor a couple acts on the cylinder.

Statement of Milne Thomson Circle Theorem

Let f(z) be the complex potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z|=a$. Then, introducing the solid circular cylinder $|z|=a$ in to the flow, the new complex potential is given by

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \quad \text{for } |z| \geq a$$

Statement of Blasius Theorem

In a steady two dimensional irrotational motion of an incompressible fluid under no external forces given by the complex potential $w = f(z)$, if the pressure thrust on the fixed cylinder of any shape are represented by a force (X,Y) and a couple of moment M about the origin of coordinates, then

$$X - iY = \frac{1}{2} i\rho \int_c \left(\frac{dw}{dz}\right)^2 dz, \qquad M = \text{Real part of} \left\{-\frac{1}{2}\rho \int_c z \left(\frac{dw}{dz}\right)^2 dz\right\},$$

where ρ is the fluid density and integrals are taken round the contour c of the cylinder.

Circulation about a Circular Cylinder :

Let k be the constant circulation about the cylinder. Then the suitable form of ϕ may be obtained by equating to k the circulation round a circle of radius r. Thus, we have

$$\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)(2\pi r) = k \quad \text{i.e.} \quad \frac{\partial\phi}{\partial\theta} = -\frac{k}{2\pi}$$

so that $\quad \phi = -\left(\frac{k\theta}{2\pi}\right)$

Since ϕ and ψ are conjugate functions, we have

$$\psi = \left(\frac{k}{2\pi}\right)\log r.$$

Thus the complex potential due to the circulation about a circular cylinder is given by

$$w = \phi + i\psi = -\frac{k\theta}{2\pi} + \frac{ik}{2\pi}\log r = \frac{ik}{2\pi}(\log r + i\theta)$$

Thus $\quad w = \left(\frac{ik}{2\pi}\right)\log z$, as $z = re^{i\theta}$.

Streaming and Circulation about a fixed Circular Cylinder :

We know that the complex potential w_1 due to the circulation of strength k about the cylinder is given by

$$w_1 = \left(\frac{ik}{2\pi}\right)\log z \qquad \text{......(1)}$$

Again, the complex potential w_2 for streaming past a fixed circular cylinder of radius a, with velocity U, in the negative direction of x-axis is given by

$$w_2 = Uz + \left(\frac{Ua^2}{z}\right) \qquad \text{......(2)}$$

Hence the complex potential $w(=\phi + i\psi)$ to the combined effects at any point $z = re^{i\theta}$ is given by

$$w = w_1 + w_2 = U\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi}\log z \qquad \text{......(3)}$$

or $\phi + i\psi = U\left[r\cos\theta + ir\sin\theta + \frac{a^2}{r}(\cos\theta - i\sin\theta)\right] + \frac{ik}{2\pi}(\log r + i\theta)$

so that $\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi}$

and $\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{k}{2\pi}\log r$.

Since the velocity will be only tangential at the boundary of the cylinder, $\left(-\frac{\partial\phi}{\partial r}\right) = 0$ and hence the magnitude of the velocity q is given by

$$q = \left|-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=a} = \left|2U\sin\theta + \frac{k}{2\pi a}\right| \qquad \text{......(4)}$$

If there were no circulation $(k = 0)$ there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$. However, in the presence of circulation, the stagnation (or critical) points are given by $q = 0$, i.e.

$$\sin\theta = -\frac{k}{4\pi Ua} \qquad \text{......(5)}$$

and such points exist when

$$|k| < 4\pi Ua \qquad \text{......(6)}$$

Remark :

From the above discussion, it follows that any point on the circumference might be made a critical point by a suitable choice of the ratio $\frac{k}{U}$. **This fact is employed in the theory of aerofoils.**

We now determine the pressure at points of the cylinder. The pressure p is given by Bernoulli's theorem

$$\frac{p}{\rho} = F(t) - \frac{1}{2}q^2 \qquad \text{......(7)}$$

Let Π be the pressure at infinity. Then $p = \Pi$ and $q = U$, so that

$$\frac{\Pi}{\rho} = F(t) - \frac{1}{2}U^2 \qquad \text{or} \qquad F(t) = \frac{\Pi}{\rho} + \frac{1}{2}U^2 \qquad \text{......(8)}$$

Using (8), (7) reduces to

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{2}U^2 - \frac{1}{2}q^2$$

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(2U\sin\theta + \frac{k}{2\pi a}\right)^2 \qquad \text{......(9)}$$

If X, Y be the components of thrust on the cylinder, we have

$$X = -\int_0^{2\pi} p\cos\theta\cdot(a\,d\theta), \qquad Y = -\int_0^{2\pi} p\sin\theta\cdot(a\,d\theta) \qquad \text{......(10)}$$

Using (9), (10) reduces to (after simplification)

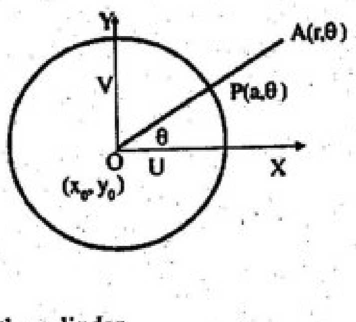
$$X = 0,\ Y = \rho k U \qquad \text{......(11)}$$

showing that the cylinder experiences an upward lift. **This effect may be attributed to circulation phenomenon.**

Equations of Motion of a Circular Cylinder :

A circular cylinder is moving in a liquid at rest at infinity. To calculate the forces acting on the cylinder owing to the pressure of the fluid.

Let U, V be the components of velocity of the cylinder when the centre of the cross-section O is (x_0, y_0). Then, we have



$$U = \dot{x}_0$$

and $\quad V = \dot{y}_0 \qquad \text{......(1)}$

Let $\quad z_0 = x_0 + iy_0$

and $\quad z - z_0 = re^{i\theta} \qquad \text{......(2)}$

where r denotes the distances from the axis of the cylinder.

On the surface of the cylinder r = a, we must have

Normal velocity of the liquid = Normal velocity of the cylinder

i.e. $\quad$ at r = a, $\quad -\frac{\partial\phi}{\partial r} = U\cos\theta + V\sin\theta \qquad \text{......(3)}$

Since liquid is at rest at infinity, so

$$-\frac{\partial \phi}{\partial r} = 0 \text{ at } r = \infty \qquad \text{......(4)}$$

Keeping (3) and (4) in mind, we take

$$\phi = \left(Ar + \frac{B}{r} \right) \cos\theta + \left(Cr + \frac{D}{r} \right) \sin\theta \qquad \text{......(5)}$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \left(A - \frac{B}{a^2} \right) \cos\theta + \left(C - \frac{D}{r^2} \right) \sin\theta \qquad \text{......(6)}$$

Using (6), (3) reduces to

$$-\left(A - \frac{B}{a^2} \right) \cos\theta - \left(C - \frac{D}{a^2} \right) \sin\theta = U\cos\theta + V\sin\theta \quad \text{...(7)}$$

so that $$\frac{B}{a^2} - A = U \text{ and } \frac{D}{a^2} - C = V \qquad \text{......(8)}$$

Again, using (6), (4) reduces to

$$-A\cos\theta - C\sin\theta = 0$$

so that $$A = 0 \text{ and } C = 0 \qquad \text{......(9)}$$

From (8) and (9),

$$B = = a^2U, \qquad D = a^2V \qquad \text{......(10)}$$

Using (9) and (10), (5) reduces to

$$\phi = \left(\frac{a^2}{r} \right) (U\cos\theta + V\sin\theta) \qquad \text{......(11)}$$

But $$\frac{\partial \psi}{\partial r} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

$$\therefore \quad \frac{\partial \psi}{\partial r} = -\frac{a^2}{r^2}(-U\sin\theta + V\cos\theta)$$

Integrating $$\psi = -\left(\frac{a^2}{r} \right) (-U\sin\theta + V\cos\theta) \qquad \text{......(12)}$$

Hence the complex potential is given by

$$w = \phi + i\psi = \frac{a^2}{r}[U(\cos\theta - i\sin\theta) + iV(\cos\theta - i\sin\theta)]$$

$$= \frac{a^2 e^{-i\theta}}{r}(U + iV)$$

Thus, $w = \dfrac{a^2(U+iV)}{z-z_0}$, by (2) ......(13)

$$\therefore \quad \frac{\partial w}{\partial t} = \frac{a^2(\dot{U}+i\dot{V})}{z-z_0} - \frac{a^2(U+iV)}{(z-z_0)^2}(-\dot{z}_0)$$

$$= \frac{a^2(\dot{U}+i\dot{V})}{z-z_0} + \frac{a^2(U+iV)^2}{(z-z_0)^2} \quad [\because \quad \dot{z}_0 = \dot{x}_0 + i\dot{y}_0 = U + iV]$$

or $$\frac{\partial \phi}{\partial t} + i\frac{\partial \psi}{\partial t} = \frac{a^2}{r}(\dot{U}+i\dot{V})(\cos\theta - i\sin\theta)$$

$$+\frac{a^2}{r^2}(U+iV)^2(\cos 2\theta - i\sin 2\theta), \text{ by (2)}$$

$$\therefore \quad \frac{\partial \phi}{\partial t} = \frac{a^2}{r}(\dot{U}\cos\theta + \dot{V}\sin\theta) + \frac{a^2}{r^2}[(U^2 - V^2)\cos 2\theta$$

$$+ 2UV\sin 2\theta] \qquad ......(14)$$

The velocity q is given by help of (13). Thus, we have

$$q^2 = \left|\frac{dw}{dz}\right|^2 = \left|-a^2\frac{U+iV}{(z-z_0)^2}\right| = \frac{a^4(U^2+V^2)}{r^4} \qquad ......(15)$$

Omitting the external forces, the pressure at any point is given by Bernoulli's equation, namely,

$$\frac{p}{\rho} = F(t) + \frac{\partial \phi}{\partial t} - \frac{1}{2}q^2 \qquad ......(16)$$

Using (14) and (15), (16) reduces to

$$\frac{p}{\rho} = F(t) + \frac{a^2}{r}(\dot{U}\cos\theta + \dot{V}\sin\theta) + \frac{a^2}{r^2}[(U^2 - V^2)\cos 2\theta$$

$$+ 2UV\sin 2\theta] - \frac{1}{2}\frac{a^4}{r^4}(U^2 + V^2) \qquad ......(17)$$

Let p_1 be the pressure at (a, θ) on the boundary of the cylinder. Then q_1 is given by putting $r = a$ in (17). Thus, we have

$$p_1 = \rho F(t) + \rho a(\dot{U}\cos\theta + \dot{V}\sin\theta) + \rho[(U^2 - v^2)\cos 2\theta$$

$$+ 2UV\sin 2\theta] - \frac{1}{2}(U^2 - V^2) \qquad ......(18)$$

Let x and Y be the compotents of forces on the cylinder due to fluid thrusts. Then, we have

$$X = -\int_0^{2\pi} p_1 \cos\theta \cdot a d\theta, \quad Y = -\int_0^{2\pi} p_1 \sin\theta \cdot a d\theta \quad \text{.......(19)}$$

Using (18), (19) gives

$$X = -\rho a^2 \int_0^{2\pi} \dot{U} \cos^2\theta d\theta, \text{ on simplification}$$

$$= -\pi a^2 \rho \dot{U} = -M'\dot{U} \quad \text{......(20)}$$

where $M' = \pi a^2 \rho$ = the mass of the liquid displaced by the cylinder of unit length.

Similarly,

$$Y = -\pi a^2 \rho \dot{V} = -M'\dot{V} \quad \text{......(21)}$$

Corollary : To show that the effect of the pressence of the liquid is to reduce the extraneous forces on th eratio $\sigma - \rho : \sigma + \rho$ where σ, ρ are the desities of the cylinder and liquid respectively.

Let M the mass of the cylinder per unit length and X', Y' be the components of the extraneous (external) forces on the cylinder if there were no liquid.

Let f_x be the acceleration of the extraneous forces in x-direction. Then, due to presence of liquid the resultant force in x-direction

$$= \pi a^2 \sigma f_x - \pi a^2 \rho f_x$$

$$= \frac{\sigma - \rho}{\sigma}(\pi a^2 \sigma f_x) = \frac{\sigma - \rho}{\sigma} X'$$

The the equation of motion in x-direction is of the form

$$M\dot{U} = -M'\dot{U} + \frac{\sigma - \rho}{\sigma} X'$$

or $$(M + M')\dot{U} = \frac{\sigma - \rho}{\sigma} X'$$

or $$M\dot{U} = \frac{M}{M + M'} \frac{\sigma - \rho}{\sigma} X'$$

$$= \frac{\pi a^2 \sigma}{\pi a^2 gs + \pi a^2 \rho} \frac{\sigma - \rho}{\sigma} X'$$

$\therefore$ $$M\dot{U} = \frac{\sigma - \rho}{\sigma + \rho} X' \quad \text{.......(22)}$$

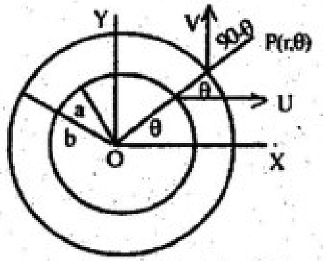
Similarly, $$M\dot{V} = \frac{\sigma - \rho}{\sigma + \rho} Y' \quad \text{.......(23)}$$

Hence the effect of the presence of the liquid is to reduce the extenal forces in the ratio $\sigma - \rho : \sigma + \rho$.

Two Coaxial Cylinders (Problems of initaial motion)

To determine the velocity potential and stream function at any point of a liquid contained between two coaxial cylinders of radii a and b (a < b) when the cylinders are moved suddenly parallel to themselves in directions at right angles with velocities U and V respectively.

Let ϕ be the velocity potential and ψ be the current function at any point (r, θ) in the liquid. Here the boundary conditions for the velocity potential ϕ are :



(i) when $r = a$, $-\frac{\partial\phi}{\partial r} = U\cos\theta$

(ii) when $r = b$, $-\frac{\partial\phi}{\partial r} = V\sin\theta$

Moreover, ϕ must satisfy he Laplac's equation $\nabla^2\phi = 0$ at every point of the liquid. In polar coordinate the Laplac's equation takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad (1)$$

Since, (1) has solutions of the form $r^n\cos n\theta$, $r^n\sin n\theta$, where n is any integer, (positive or negative) hence, the sum of any number of terms of the form

$A_n r^n\cos n\theta$, $B_n r^n\sin n\theta$, is also a solution of (1).

The above considerations suggest that we must assume the following suitable form of ϕ.

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta \qquad ......(2)$$

so that

$$\frac{\partial\phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta \qquad ....(3)$$

Using the boundary conditions (i) and (ii), (3) gives

$$-U\cos\theta = \left(A - \frac{B}{a^2}\right)\cos\theta + \left(C - \frac{D}{a^2}\right)\sin\theta \qquad ...(4)$$

$$-V\sin\theta = \left(A - \frac{B}{b^2}\right)\cos\theta + \left(C + \frac{D}{b^2}\right)\sin\theta \qquad ...(5)$$

Since (4) and (5) must hold for all values of θ, we have

$$A - \frac{B}{a^2} = -U, \qquad C - \frac{D}{a^2} = 0$$

$$A-\frac{B}{b^2}=0, \qquad C-\frac{D}{b^2}=-V$$

$$\therefore A=\frac{Ua^2}{b^2-a^2}, \quad B=\frac{Ua^2b^2}{b^2-a^2}, \quad C=-\frac{Vb^2}{b^2-a^2}, \quad D=-\frac{Va^2b^2}{b^2-a^2}$$

So,

$$\phi=\frac{a^2U}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta-\frac{b^2V}{b^2-a^2}\left(r+\frac{a^2}{r}\right)\sin\theta \quad ......(6)$$

Now, since $\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$, (6) gives

$$\frac{\partial\psi}{\partial\theta}=\frac{a^2U}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\cos\theta-\frac{b^2V}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\sin\theta$$

$$\therefore \quad \psi=\frac{a^2U}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\sin\theta+\frac{b^2V}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\cos\theta \quad ....(7)$$

Note : Equation (6) and (7) represents the motion at the instant when the cylinders are coaxial. Thus they give the initial motion.

Exercise 1. :

The space between two infinitely long coaxial cylinder of radii a and b respectively is filled with homogeneous liquid of density ρ and is suddenly moved with velocity U perpendiculer to the axis, the outer one being kept fixed. Show that the resultant impulsive pressure on a length ℓ of the inner cylinder is

$$\pi\rho a^2\ell\frac{b^2+a^2}{b^2-a^2}U.$$

Solution :

In reference to the above discussion of problem of initial motion, here we have V=0. Also, the boundary conditions become

(i) when $r=a$, $\quad -\frac{\partial\phi}{\partial r}=U\cos\theta$

(ii) when $r=b$, $\quad -\frac{\partial\phi}{\partial r}=0 \quad (\text{as } V=0)$

Moreover, ϕ must satisfy he Laplac's equation $\nabla^2\phi=0$ at every point of the liquid, which in polar coordinate takes the form

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0$$

We assume the following suitable form of ϕ.

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta \qquad \text{......(1)}$$

so that

$$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta \qquad \text{.......(2)}$$

Using the boundary conditions (i) and (ii), (2) gives

$$-U\cos\theta=\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta \qquad \text{...(3)}$$

$$0=\left(A-\frac{B}{b^2}\right)\cos\theta+\left(C+\frac{D}{b^2}\right)\sin\theta \qquad \text{...(4)}$$

Since (3) and (4) must hold for all values of θ, we have

$$A=\frac{Ua^2}{b^2-a^2},\quad B=\frac{Ua^2b^2}{b^2-a^2},\quad C=0,\quad D=0$$

(calculations exactly similar to the earlier case)

So, from (1)

$$\phi=\frac{a^2U}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta$$

But we know that the impulsive pressure at any point is $\rho\phi$. (see the equation of motion under impulsive forces). Hence impulsive pressure p_1 at any point $P(a,\theta)$ of the inner cylinder is given by

$$p_1=\rho\phi \quad \text{at } r=a$$

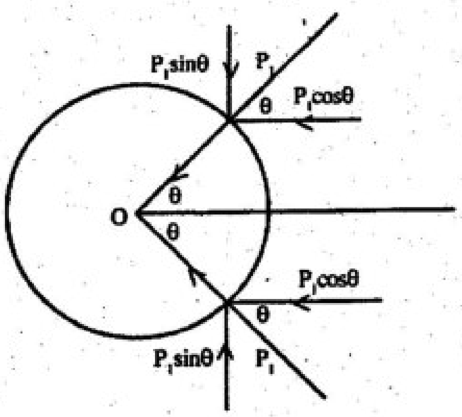
$$\therefore p_1=\rho\frac{Ua^2}{b^2-a^2}\left(a+\frac{b^2}{a}\right)\cos\theta$$

$$\therefore p_1=\rho Ua\frac{b^2+a^2}{b^2-a^2}\cos\theta$$

Hence, the total impulsive pressure on the cylinder of length ℓ is given by

$$\int_0^{2\pi}p_1\cos\theta.a\,d\theta\;\ell=\ell\rho Ua^2\frac{b^2+a^2}{b^2-a^2}\int_0^{2\pi}\cos^2\theta\,d\theta \;=\pi\rho a^2\ell\frac{b^2+a^2}{b^2-a^2}U$$

Note : In writing the expression for total impulsive pressure the following figure should be considered, where it is seen that the components $p_1\sin\theta$ of p_1 cancel each other.



Exercise 2. :

The space between two infinitely long coaxial cylinders of radii a and b respectively is filled with homogeneous liquid of density ρ and is suddenly moves with velocity V perpendicular to the a .. the outer one being kept fixed. Show that the resultant implusive pressure on **unit length** of the inner cylinder is

$$\pi\rho a^2 \frac{b^2+a^2}{b^2-a^2} V.$$

Solution : Exactly similar to the earlier problem. Note that here $\ell = 1$.

Exercise 3. :

An infinite cylinder of radius a and density σ is surrounded by a **fixed** concentric cylinder of radius b and the the intervening space is filled with liquid of density ρ. Prove that the impulse **per unit length** necessary to start the inner cylinder with velocity V is

$$\frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho)b^2-(\sigma-\rho)a^2\right\}V.$$

Solution : As in Exercise 2 above, the total impulsive pressure on the cylinder of unit length is

$$\pi\rho a^2 \frac{b^2+a^2}{b^2-a^2} V$$

Moreover, the impulse needed to move the inner cylinder with a velocity V

$$= (\text{mass of the cylinder}) \times V = \pi a^2 \sigma V$$

Hence the total impulse per unit length to move the inner cylinder with velocity V

$$\pi a^2 \sigma V + \pi\rho a^2 \frac{b^2+a^2}{b^2-a^2} V = \frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho)b^2-(\sigma-\rho)a^2\right\}V$$

Exercise 4. :

A circular cylinder is fixed across a stream of velocity U with a circulation k round the cylinder. Show that the maximum velocity in the liquid is $2U+\left(\frac{k}{2\pi a}\right)$, where a is the radius of the cylinder.

Unit-3

Irrotational Motion in Three-Dimensions (Motion of a Sphere)

Stokes's Stream Function

Introduction :

We propose to study irrotational motion in three-dimensions with a particular reference to the motion of a sphere. We shall consider certain special forms of solution of the equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \qquad \text{......(1) (Laplac's equation)}$$

which, in spherical polar co-ordinates (r, θ, ω) reduces to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\omega^2}=0 \quad \text{...(2)}$$

When there is symmetry about a line (say, z-axis), ϕ is independent of ω and hence (2) reduces to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0 \qquad \text{......(3)}$$

In the case of motion of a sphere the velocity potential is known to have the form $f(r)\cos\theta$. Substituting $\phi = f(r)\cos\theta$ in (3), we have

$$\left(\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{df}{dr}\right)\cos\theta-\frac{f(r)}{r^2}\cos\theta-\frac{\cos\theta}{r^2}f(r)=0$$

so that $$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}-\frac{2f}{r^2}=0$$

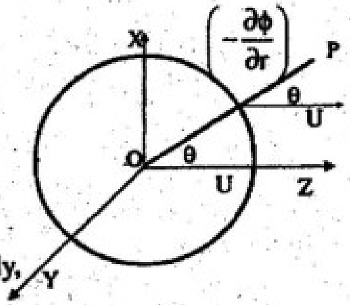
or $$r^2\frac{d^2f}{dr^2}+2r\frac{df}{dr}-2f=0$$

which is homogeneous differential equation. As usual, its solution is $f(r)=Ar+\frac{B}{r^2}$. Hence a solution of (3) of the form $f(r)\cos\theta$ may be taken as

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \qquad \text{......(4)}$$

Motion of a sphere through an infinite mass of a liquid at rest at infinity :

Take the origin at the centre of the sphere and the axis of z in the direction of motion. Let the sphere move with velocity U along the z-axis. To determine the velocity potential ϕ that will satisfy the given boundary conditions, we have the following considerations :



(i) ϕ satisfies the Laplace's equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}=0 \quad ......(1)$$

wherein we have used the fact that there is symmetry of flow about z-axis.

(ii) Boundary condition at the surface of the sphere r = a, namely,

Normal velocity at any point of the sphere

= velocity of the liquid at that point in that direction

i.e., $$-\frac{\partial\phi}{\partial r}=U\cos\theta,\ \text{when } r=a \quad ......(2)$$

(iii) Since the liquid is at rest at infinity, we must have

$$-\frac{\partial\phi}{\partial r}=0,\ \text{at } r=\infty \quad ......(3)$$

The above considerations (i) and (ii) suggest that ϕ must be of the form $f(r)\cos\theta$ and hence it may be assumed as

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \quad ......(4)$$

From (4),

$$-\frac{\partial\phi}{\partial r}=-\left(A-\frac{2B}{r^3}\right)\cos\theta \quad ......(5)$$

Putting $r=\infty$ in (5) and using (3), we get $0=A\cos\theta$ so that $A=0$ ......(6)

Putting $r=a$ in (5) and using (2) and (6), we get

$$U\cos\theta=\left(\frac{2B}{a^3}\right)\cos\theta$$

so that $$B=\frac{1}{2}Ua^3 \quad ......(7)$$

Thus $$\phi=\frac{1}{2}Ua^3\frac{\cos\theta}{r^2} \quad ......(8)$$

which determines the velocity potential for the flow.

We now determine the equations of lines (stream lines) of flow. The differential equation of the lines of flow at the instant the centre of sphere is passing through the origin is given by

$$\frac{dr}{\frac{\partial\phi}{\partial r}} = \frac{rd\theta}{\frac{\partial\phi}{r\partial\theta}}$$

or $$\frac{dr}{\frac{Ua^3}{r^3}\cos\theta} = \frac{rd\theta}{\frac{Ua^3}{2r^3}\sin\theta}, \qquad \text{using (8)}$$

or $$\frac{dr}{r} = 2\frac{\cos\theta}{\sin\theta}d\theta$$

Integrating,

$$\log r = 2\log\sin\theta + \log c \quad \text{or} \quad r = c\sin^2\theta,$$

which is the equation of the lines of flow.

Liquid streaming past a fixed sphere :

Let the sphere be at rest and let the liquid flow past the cylinder with velocity U in the negative direction of z-axis. This motion may be deduced from that of the previous article by imposing a velocity -U parallel to the z-axis on both the sphere and the liquid. The sphere is then reduced to rest and we must add to the velocity potential a term $Ur\cos\theta$ to account for the additional velocity. Thus

$$\phi = \frac{1}{2}Ua^3\frac{\cos\theta}{r^2} + Ur\cos\theta = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad \text{......(1)}$$

To determine the lines of flow relative to the sphere.

Now the stream lines are given by

$$\frac{dr}{\frac{\partial\phi}{\partial r}} = \frac{rd\theta}{\frac{\partial\phi}{r\partial\theta}}$$

or $$\frac{dr}{U\left(1 - \frac{a^3}{r^3}\right)\cos\theta} = \frac{rd\theta}{-U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta}$$

or $$-2\cot\theta d\theta = \frac{2r^3 + a^3}{r^3 - a^3}\frac{dr}{r} = \left(\frac{3r^2}{r^3 - a^3} - \frac{1}{r}\right)dr$$

Integrating,

$$-2\log\sin\theta = \log(r^3 - a^3) - \log r - \log c$$

i.e., $\sin^2\theta = \dfrac{cr}{r^3 - a^3}$ or $r^2 \sin^2\theta\left(1 - \dfrac{a^3}{r^3}\right) = c$ ......(2)

(2) gives the lines of flow relative to the sphere.

Exercise 1. :

Show that when a sphere of radius a moves with uniform velocity U through a perfect incompressible infinite fluid, **the acceleration of a particle of the fluid** at (r, 0) is

$$3U^2\left(\frac{a^3}{r^4} - \frac{a^6}{a^7}\right).$$

Solution :

Superimpose a velocity - U both to the sphere and the liquid. This reduces the sphere to rest and the velocity potential of the flow is gives by [see the case of liquid streaming past a fixed sphere]

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad ......(1)$$

$$\therefore \quad \dot{r} = -\frac{\partial\phi}{\partial r} = -U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \qquad ......(2)$$

$$\text{and} \quad r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \qquad ......(3)$$

Again, from (2), we have

$$\ddot{r} = U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} - U\frac{3a^3}{a^4}\dot{r}\cos\theta$$

$$= U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} + \frac{3a^3}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta, \text{ by (2)}$$

Clearly for a point (r, θ), the velocity is only along the direction of r and hence the acceleration will also be only along r so that $\dot{\theta} = 0$.

Thus the required acceleration

$= \ddot{r}$ only {at(r,0)}

$$= \frac{3a^3}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right), \text{ from (3) with } \theta = \dot{\theta} = 0$$

$$= 3U^2\left(\frac{a^3}{r^4} - \frac{a^6}{a^7}\right)$$

Exercise 2. :

An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is distributed by the sphere and the pressure in the liquid at a great distance from the obstacles is Π. Show that the thrust on that half of the sphere on whcih the liquid impinges is $\pi a^2\left(\Pi - \rho\frac{U^2}{16}\right)$.

Solution : The velocity potential of the motion of the liquid streaming past fixed sphere with velocity U in the negative direction of z-axis is given by

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad \text{......(1)}$$

$$\therefore \quad \left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[U\left(1 - \frac{a^3}{r^3}\right)\cos\theta\right]_{r=0} = 0$$

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \left[\frac{U}{r}\left(r + \frac{a^3}{2r^2}\right)(-\sin\theta)\right]_{r=a} = -\frac{3}{2}U\sin\theta$$

Let q be the velocity at any point of the boundary of the sphere r = a. Then, we have

$$q^2 = \left\{\left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right\}_{r=a} = \frac{9}{4}U^2\sin^2\theta \qquad \text{......(2)}$$

In steady motion in absence of external forces, the pressure at any point by Bernoulli's equation is given by

$$\frac{p}{\rho} + \frac{1}{2}q^2 = C \qquad \text{......(3)}$$

But $p = \Pi$, $q = U$ at infinity. So (3) gives

$$\frac{\Pi}{\rho} + \frac{1}{2}U^2 = C \qquad \text{......(4)}$$

Subtracting (4) from (3), we obtain

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2 \qquad \text{......(5)}$$

Using (2), the pressure p' at any point P on the surface of the sphere r = a is given by

$$p' = \Pi + \frac{1}{2}\rho U^2 - \left(\frac{9}{8}\right)\rho U^2 \sin^2\theta$$

Hence the required thrust on that half of the sphere on which the liquid impings

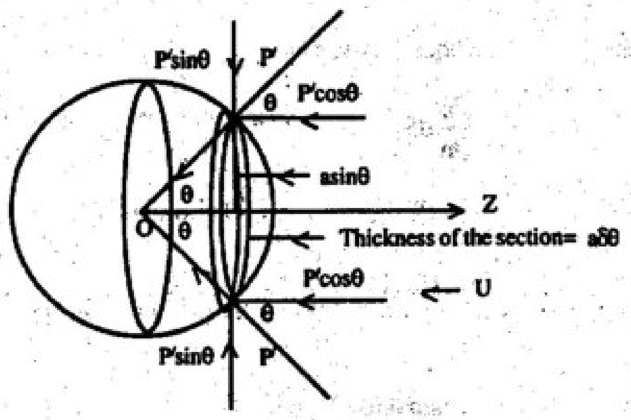
$$= \int_0^{\frac{\pi}{2}} (p'\cos\theta) 2\pi a \sin\theta \cdot a d\theta$$

$$= 2\pi a^2 \int_0^{\frac{\pi}{2}} \left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta \right] \sin\theta \cos\theta d\theta$$

$$= 2\pi a^2 \left[\left(\pi + \frac{1}{2}\rho U^2 \right) \cdot \frac{1}{2} - \frac{9}{8}\rho U^2 \cdot \frac{1}{4} \right]$$

$$= \pi a^2 \left(\Pi - \rho \frac{U^2}{16} \right)$$

Note : In writing the expression for the required thrust following figure should be considered, where it is seen that the components $p'\sin\theta$ of p' cancel each other.



Exercise 3 :

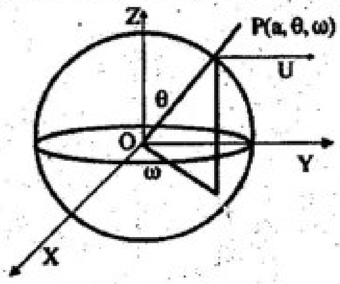
A sphere of radius a is moving with constant velocity U through an infinite liquid at rest at infinity. If p_0 be the pressure at infinity, show that the pressure at any point of the surface of the sphere, the radius to which point makes an angle θ with the direction of motion is given by

$$p = p_0 + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}\sin^2\theta \right).$$

Solution : Exactly similar to the previous problem.

Equations of Motion of a Sphere :

Take the origin at the ncentre of the spher and the axis of z in the direction of motion. Let the sphere move with velocity W along the z-axis in an infinite mass of liquid at rest at infinity. Then as discussed earlier in the case of **'Motion of a sphere through an infinite mass of a liquid at rest at infinity'**, velocity potential of the motion is given by



$$\phi = \frac{Wa^3}{2r^2}\cos\theta$$

so that $$\frac{\partial\phi}{\partial r} = -\frac{Wa^3}{r^3}\cos\theta$$

Let P(a, θ, ω) be the spherical polar co-ordinates of any point on the surface of the sphere. Then elementary surface area dS at P is adθ.asinθdω. Again the value of $\phi\left(\frac{\partial\phi}{\partial r}\right)$ at P is given by

$$\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[\frac{Wa^3}{2r^2}\cos\theta\left(-\frac{Wa^3}{r^3}\cos\theta\right)\right]_{r=a}$$

$$= -\frac{1}{2}W^2 a\cos^2\theta \qquad \text{......(1)}$$

As, the kinetic energy T_1 of the liquid is given by

$$T_1 = -\frac{1}{2}\rho\iint\phi\frac{\partial\phi}{\partial n}dS$$

integrated over the surface. Using (1), we obtain

$$T_1 = -\frac{1}{2}\rho\int_{\omega=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-\frac{1}{2}W^2 a\cos^2\theta\right)(a^2\sin\theta\, d\theta\, d\omega)$$

$$= \frac{1}{4}W^2\rho a^3\left[\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta \times\left[\int_0^{2\pi}d\omega\right]\right]$$

$$= \frac{1}{3}\pi\rho a^3 W^3 \quad = \frac{1}{4}\frac{4}{3}\pi\rho a^3 W^2$$

$$= \frac{1}{4}M'W^2 \qquad \text{......(2)}$$

where $$M' = \frac{4}{3}\pi\rho a^3 \qquad \text{......(3)}$$

is the of the liquid displaced by the sphere, σ being the density of the liquid. Let σ be the density of the sphere and M be the mass of the sphere so that

$$M = \frac{4}{3}\pi\sigma a^3 \quad \text{......(4)}$$

and K.E. of the sphere = $T_2 = \frac{1}{2}MW^2$. ......(5)

Let T be the total kinetic energy of the liquid and the sphere.

Then, $T = \frac{1}{2}\left(M + \frac{1}{2}M'\right)W^2$, by (2) and (5).

Let Z be the external force parallel to the z-axis (i.e. in the direction of motion of the sphere). Then from the principle of energy, we have

Rate of increase of total K.E. = rate at which work is being done

i.e., $$\frac{d}{dt}\left[\frac{1}{2}\left(M + \frac{1}{2}M'\right)W^2\right] = ZW$$

or $$\left(M + \frac{1}{2}M'\right)W\dot{W} = ZW \text{ where } \dot{W} = \frac{dW}{dt}$$

or $$M\dot{W} = Z - \frac{1}{2}M'\dot{W} \quad \text{.....(6)}$$

Let Z′ be the external force on the sphere when no liquid is present. Then from Hydrostaical considerations, there exists a relation between Z and Z′ of the form

$$Z = \left[\frac{(\sigma - \rho)}{\sigma}\right]Z' \quad \text{......(7)}$$

From (6) and (7), we have

$$M\dot{W} + \frac{1}{2}M'\dot{W} = \left[\frac{(\sigma - \rho)}{\sigma}\right]Z' \quad \text{or} \quad \left(M + \frac{1}{2}M'\right)\dot{W} = \left[\frac{(\sigma - \rho)}{\sigma}\right]Z'$$

or $$M\dot{W} = \frac{M}{M + \left(\frac{1}{2}\right)M'}\frac{\sigma - \rho}{\sigma}Z'$$

or $$M\dot{W} = \frac{\frac{4}{3}\pi\sigma a^3}{\frac{4}{3}\pi\sigma a^3 + \frac{1}{2}\frac{4}{3}\pi\rho a^3}\frac{\sigma - \rho}{\sigma}Z' \quad \text{[by (3) and (4)]}$$

or $$M\dot{W} = \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} Z' \qquad \text{......(8)}$$

(8) shows that the whole effect of the presence of the liquid is to reduce the external force in the ratio $\sigma - \rho : \sigma + \frac{1}{2}\rho$.

Remark 1. :

When liquid is absent (so that $M' = 0$), (6) reduces to

$$M\dot{W} = Z \qquad \text{......(9)}$$

Comparing (9) with (6), we find that the presence of liquid offers resistance of amount $\frac{1}{2}M'\dot{W}$ to the motion of the sphere.

Remark 2. :

When U, V, W are the components of velocity of the centre of the sphere and X', Y', Z' are the components of the external force on the sphere in absence of liquid, then equations of motion of the sphere are of the form

$$\left.\begin{aligned} M\dot{U} &= \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} X' \\ M\dot{V} &= \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} Y' \\ MW &= \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} Z' \end{aligned}\right] \qquad \text{......(10)}$$

Pressure Distribution on a Sphere :

To show that at a point on a sphere moving through an infinite liquid the pressure is given by the formula

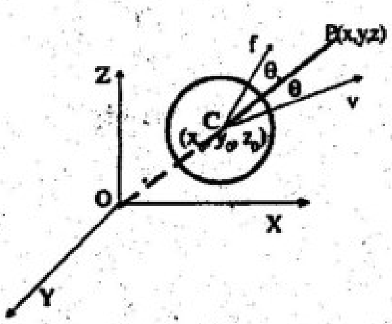
$$\frac{p - p_0}{\rho} = \frac{1}{2} af \cos\theta_1 + \frac{1}{8} v^2 (9\cos^2\theta - 5)$$

where v is the velocity, f the accelleration of the sphere, and θ, θ_1 are the angles between the radii and the direction of v, f respectively, and p_0 is the pressure at infinity.

Proof :

Let the co-ordinates of the centre C of the moving sphere reffered to fixed axes be (x_0, y_0, z_0) and let

$$\dot{x}_0 = U, \dot{y}_0 = V, \dot{z}_0 = W \quad ....(1)$$



Let (x, y, z) be the co-ordinates of any point P in the liquid.

Let θ, θ_1 be the angels between CP and the direction of v, f respectively.

Let CP = r. Then, we have

$$r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - y_0)^2 \quad ......(2)$$

Let ℓ, m, n be the direction cosines of CP, then

$$\ell = \frac{x - x_0}{r}, \quad m = \frac{y - y_0}{r}, \quad n = \frac{z - z_0}{r} \quad ....(3)$$

Also $\quad v^2 = U^2 + V^2 + W^2 \quad ....(4)$

$v\cos\theta$ = resolved part of v along CP

$= U\ell + Vm + Wn$

$$= U\frac{x - x_0}{r} + V\frac{y - y_0}{r} + W\frac{z - z_0}{r} \quad ......(5)$$

and $\quad f\cos\theta_1$ = reslovcd part of f along CP $= \dot{U}\ell + \dot{V}m + \dot{W}n$

$$= \dot{U}\frac{x - x_0}{r} + \dot{V}\frac{y - y_0}{r} + \dot{W}\frac{z - z_0}{r} \quad ......(6)$$

Then from the previous discussion on **'motion of a sphere through an infinite mass of liquid at rest at infinity'** the velocity potential at a fixed point of space (x, y, z) is given by

$$\phi = \frac{a^3}{2r^2} v\cos\theta$$

or $$\phi = \frac{a^3}{2r^3}[U(x - x_0) + V(y - y_0) + W(z - z_0)] \quad ......(7)$$

From (2),

$$2r\frac{\partial r}{\partial x} = 2(x - x_0)$$

or $$\frac{\partial r}{\partial x} = \frac{x - x_0}{r} \quad ......(8)$$

Differentiating (7) partially w.r.t. 'x' we get

$$\frac{\partial\phi}{\partial x}=\frac{1}{2}\frac{a^3U}{r^3}-\frac{3a^3}{2r^4}\frac{\partial r}{\partial x}[U(x-x_0)+V(y-y_0)+W(z-z_0)]$$

$$=\frac{1}{2}\frac{a^3U}{r^3}-\frac{3a^3}{2r^4}(x-x_0)v\cos\theta, \quad \text{by (5) and (8)}$$

Similarly, differentiating (7) partially w.r.t. y and z, we get

$$\frac{\partial\phi}{\partial y}=\frac{1}{2}\frac{a^3V}{r^3}-\frac{3a^3}{2r^4}(y-y_0)v\cos\theta$$

$$\frac{\partial\phi}{\partial z}=\frac{1}{2}\frac{a^3W}{r^3}-\frac{3a^3}{2r^4}(z-z_0)v\cos\theta$$

$$\therefore \quad q^2=\left(-\frac{\partial\phi}{\partial x}\right)^2+\left(-\frac{\partial\phi}{\partial y}\right)^2+\left(-\frac{\partial\phi}{\partial z}\right)^2$$

$$=\frac{1}{4}\frac{a^6}{r^6}(U^2+V^2+W^2)-\frac{3a^6}{2r^7}v\cos\theta[U(x-x_0)+V(y-y_0)$$

$$+W(z-z_0)]+\frac{9a^6}{4r^8}v^2\cos^2\theta[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]$$

$$=\frac{1}{4}\frac{a^6v^2}{r^6}-\frac{3a^6}{2r^6}v^2\cos^2\theta+\frac{9a^6}{4r^6}v^2\cos^2\theta, \quad \text{by (2) and (5)}$$

$$=\frac{1}{4}\frac{a^6v^2}{r^6}(1+3\cos^2\theta) \quad \text{......(9)}$$

From (2),

$$r\frac{\partial r}{\partial t}=-(x-x_0)\dot{x}_0-(y-y_0)\dot{y}_0(z-z_0)\dot{z}_0$$

$$=-U(x-x_0)-V(y-y_0)-W(z-z_0) \quad \text{.......(10)}$$

Differentiating (7) partially w.r.t. 't', we get

$$\frac{\partial\phi}{\partial t}=\frac{a^3}{2r^3}[\dot{U}(x-x_0)+\dot{V}(y-y_0)+\dot{W}(z-z_0)-(U\dot{x}_0+U\dot{y}_0+U\dot{z}_0)$$

$$-\frac{3a^3}{2r^4}\frac{\partial r}{\partial t}[U(x-x_0)+V(y-y_0)+W(z-z_0)]$$

$$= \frac{a^3}{2r^3}[fr\cos\theta_1 - (U^2 + V^2 + W^2)] + \frac{3a^3}{2r^5}[U(x - x_0)$$

$$+ V(y - y_0) + W(z - z_0)]^2, \qquad \text{by (1), (6) and (10)}$$

$$= \frac{a^3}{2r^3}(fr\cos\theta_1 - v^2) + \frac{3a^3}{2r^5}(r^2 v^2 \cos^2\theta) \qquad \text{by (5)}$$

$$= \frac{a^3}{2r^3}(fr\cos\theta_1 - v^2 + 3v^2\cos^2\theta) \qquad \text{......(11)}$$

Let P be the potential function due to external forces. Then the pressure at any point in the liquid is given by Bernoulli's equation, namely,

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + \frac{1}{2}q^2 + P = F(t) \qquad \text{......(12)}$$

At infinity $r = \infty$, $p = p_0$ and so $\frac{\partial\phi}{\partial t} = 0$ and $q = 0$ from (11). Hence (12) gives $F(t) = \frac{p_0}{\rho} + P$. So (12) reduces to

$$\frac{p - p_0}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2$$

$$= \frac{a^3}{2r^3}(fr\cos\theta_1 - v^2 + 3v^2\cos^2\theta) - \frac{1}{8}\frac{a^6 v^6}{r^6}(1 + 3\cos^2\theta)$$

$$\therefore \quad \frac{p - p_0}{\rho} = \frac{a^3 f}{2r^2}\cos\theta_1 - \frac{a^3 v^2}{8r^6}(4r^3 + a^3) + \frac{3a^3}{8r^6}v^2\cos^2\theta(4r^3 - a^3) \text{....(13)}$$

Putting $r = a$ in (13), pressure at any point on the surface of the sphere is given by

$$\frac{p - p_0}{\rho} = \frac{1}{2}af\cos\theta_1 + \frac{1}{8}v^2(9\cos^2\theta - 5) \qquad \text{......(14)}$$

Corollary 1. :

When sphere moves uniformly, i.e., when $f = 0$, pressure at point on the surface of the sphere $r = a$ is given by [putting $f = 0$ in (14)]

$$\frac{p - p_0}{\rho} = \frac{1}{8}v^2(9\cos^2\theta - 5) \qquad \text{......(15)}$$

or $$\frac{p - p_0}{\rho} = \frac{1}{8}v^2\left[9\frac{(1 + \cos 2\theta)}{2} - 5\right]$$

or $$\frac{p-p_0}{\rho}=\frac{1}{16}v^2(9\cos 2\theta-1) \qquad ......(16)$$

Corollary 2. : resultant thrust when there is no acceleration :

In this case pressure p is given by (15), i.e.,

$$p=p_0+\frac{1}{8}\rho v^2(9\cos^2\theta-1) \qquad ......(17)$$

So, the resultant thrust on the sphere

$$=-\int p\cos\theta ds$$

$$=-\int_0^\pi p\cos\theta(ad\theta)\ (2\pi a\sin\theta)$$

$$=-2\pi a^2\int_0^\pi\left[p_0+\frac{1}{8}v^2\rho(9\cos^2\theta-5)\right]\sin\theta\cos\theta d\theta=0$$

which is in conformity with D' Alembertt's paradox.

D'Alembert's Paradox :

A body moving with uniform velocity through an infinite liquid, otherwise at rest, will experience no resistance at all. This result is known as D'Alembert's Paradox.

Corollary 3. Resultant thrust when there is acceleration :

When f is not zero, the resultant thrust due to that part will be

$$=-\int_0^\pi\rho\frac{1}{2}af\cos\theta_1\,2\pi a\sin\theta_1 ad\theta_1$$

$$=-\pi a^3\rho f\int_0^\pi\cos^2\theta_1\sin\theta_1 d\theta_1$$

$$=-\frac{2}{3}\pi a^3 f\rho=-\frac{1}{2}M'f$$

where $M'=\left(\frac{4}{3}\right)\pi a^3\rho$ = mass of the liquid displaced.

Exrecise 1:

Prove that at a point on the sphere moving through an infinite liquid the pressure is given by the formula

$$\frac{p-p_0}{\rho}=\frac{1}{2}af\cos\theta_1+\frac{1}{8}v^2(9\cos^2\theta-5)$$

where v is the velocity, f the acceleration of the sphere, and θ, θ_1 are the angles between the radius and the direction of v, f respectively, and p_0 is the hydrostatic pressure.

Solution :

Proceed as in the previous discussion under the heading **Pressure Distribution on a Sphere** upto equation (12). If p_0 is hydrostatic pressure, i.e., when there is no motion so that $q = 0, \frac{\partial \phi}{\partial t} = 0$ and $p = p_0$. Then (12) gives $F(t) = \frac{p_0}{\rho} + P$ as before.

Hence, we get the required formula from equation (14).

Exercise 2:

A solid sphere is moving through a frictionless liquid. Prove that when the sphere is in motion with uniform velocity v, the pressure at the part of its surface where the radius makes and angle θ with the direction of motion is increased an account of the motion by the amount,

$$\frac{1}{16}\rho v^2(9\cos 2\theta - 1) \quad \text{where } \rho \text{ is the density of the liquid.}$$

Solution :

Proceed like the previous discussion under the heading **Pressure Distribution on a Sphere** upto equation (16).

Exercise 3:

A solid sphere is moving through a frictionless liquid. Compare the velocities of slip of the liquid past it at different parts of its surface.

Solution :

Proceed like the previous discussion under the heading **Pressure Distribution on a Sphere** upto equation (7). Now, the velocity of slip at any point (a, θ) on the surface of the sphere

$$= \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)_{r=a} = \left(\frac{1}{r}\frac{a^3}{2r^2} v \sin\theta\right)_{r=a} = \frac{1}{2} v \sin\theta.$$

Concentric Sphere (Problem of Initial Motion) :

A sphere of radius a is surrounded by a concentric sphere of radius b, the space between them being filled with liquid at rest. The inner sphere is given a velocity U and outer sphere a velocity V in the same direction. To determine the initial motion of the liquid.

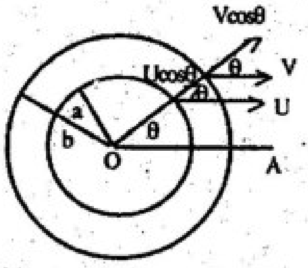
Let O be the common centre and φ be the velocity potential of the initial motion. Let U and V be in the direction of initial line OA as shown in the figure. Then to determine φ we have the following considerations :

(i) φ satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r}\frac{\partial^2 \phi}{\partial^2 \theta} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0 \quad \text{......(1)}$$

wherein we have used the fact that there is symmetry about the initial line.

(ii) ϕ satisfies the followin boundary conditions :



$$-\frac{\partial\phi}{\partial r} = U\cos\theta, \text{ when } r = a \qquad \text{......(2)}$$

and $$-\frac{\partial\phi}{\partial r} = V\cos\theta, \text{ when } r = b \qquad \text{......(3)}$$

The above considerations (1) and (2) suggest that ϕ must be of the from $f(r)\cos\theta$ and hence it may be assumed as

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta \qquad \text{......(4)}$$

so that $$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{2B}{r^3}\right)\cos\theta \qquad \text{......(5)}$$

Using boundary conditions (2) and (3), (5) gives

$$U\cos\theta = -\left(A - \frac{2B}{a^3}\right)\cos\theta \text{ or } -A + \frac{2B}{a^3} = U \qquad \text{.......(6)}$$

and $$V\cos\theta = -\left(A - \frac{2B}{b^3}\right)\cos\theta \text{ or } -A + \frac{2B}{b^3} = V \qquad \text{......(7)}$$

Solving (6) and (7) for A and B, we get

$$A = \frac{Ua^3 - Vb^3}{b^3 - a^3} \qquad \text{and} \qquad B = \frac{(U - V)a^3b^3}{2(b^3 - a^3)}$$

Therefore, at the instant of starting the motion, the velocity potential is given by

$$\phi = \frac{Ua^3 - Vb^3}{b^3 - a^3} r\cos\theta + \frac{(U - V)a^3b^3}{2(b^3 - a^3)}\frac{\cos\theta}{r^2} \qquad \text{.......(8)}$$

Corollary :

A sphere of radius a is surrounded by a concentric spherical shell of radius b, the space between is filled with liquid. If the sphere be moving with velocity U, to show that

$$\phi = \frac{Ua^3}{b^3 - a^3}\left(r + \frac{b^3}{2r^2}\right)\cos\theta.$$

Also, to discuss the motion so produced.

Since the outer sphere is at rest, V = 0 and hence (8) reduces to the desired result.

Let I be the impulse necessary to produce the velocity U in the inner sphere. Then by principle of momentum, we have

$$I = MU + \iint \overline{\omega} \cos\theta \, dS \qquad \text{......(A)}$$

where $\overline{\omega} = (\rho\phi)_{r=a}$ is the impulsive pressure of the liquid at a point on the sphere. Hence, we have

$$\overline{\omega} = \frac{Ua^3\rho}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right)\cos\theta$$

$$\therefore \quad \iint \overline{\omega}\cos\theta \, dS = \int_0^{\pi} \frac{Ua^3\rho}{b^3 - a^3}\left(a + \frac{b^3}{2a^2}\right)\cos\theta \; \cos\theta \;\; 2\pi a \sin\theta \;\; a d\theta$$

$$= \frac{2\pi\rho U a^3 (2a^3 + b^3)}{3(b^3 - a^3)}$$

$$= \frac{1}{2}\frac{M'U(2a^3 + b^3)}{b^3 - a^3} \qquad \text{......(B)}$$

where $\quad M'$ = mass of the liquid displaced by the sphere

$$= \left(\frac{4}{3}\right)\pi a^3 \rho \qquad \text{.......(C)}$$

So from (A) and (B), we have

$$I = MU + \frac{1}{2}\frac{M'U(2a^3 + b^3)}{b^3 - a^3} \qquad \text{......(D)}$$

Let $b \to \infty$, then we have

$$\lim_{b\to\infty} \frac{2a^3 + b^3}{b^3 - a^3} = \lim_{b\to\infty} \frac{2\left(\frac{a^3}{b^3}\right) + 1}{1 - \left(\frac{a^3}{b^3}\right)} = 1$$

Thus, if the outer sphere becomes infinitely large (i.e., $b \to \infty$), the impulse required to give a sphere in unbounded liquid, a velocity U is

$$I = MU + \frac{1}{2}M'U = \left(M + \frac{1}{2}M'\right)U,$$

showing that it effectively increases the mass of the sphere by an amount $\frac{1}{2}M'$.

We also notice that the impulse required to impart a velocity U is the same when the sphere is in a mass of liquid at rest at infinity or is surrounded by a fixed spherical envelope of a very large radius.

Exercise 1:

Liquid of density ρ fills the space between a solid sphere of radius a and density σ and a fixed concentric spherical envelope of radius b. Prove that the work done by an impulse which starts the solid sphere with velocity U is

$$\frac{1}{3}\pi a^3 U^2\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right)$$

Solution :

As in corollary of discussion of **'Concentric Sphere (Problem of Initial Motion)'**, the total impulse I is given by

$$I = MU + \iint \overline{\varpi}\cos\theta\, dS$$

But $\iint \overline{\varpi}\cos\theta\, dS = \frac{2}{3}\pi\rho a^3\frac{2a^3+b^3}{b^3-a^3}$, by (B) of the previous discussion

and $\quad M = \text{mass of inner solid} = \left(\frac{4}{3}\right)\pi a^3\sigma$

$$\therefore \quad I = \frac{2\pi a^3 U^2}{3}\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right).$$

Hence the work done by impulse I

= I × (mean of the initial and final velocities)

$$= I\times\frac{0+U}{2} = \frac{1}{2}UI = \frac{\pi a^3 U^2}{3}\left(2\sigma+\frac{2a^3+b^3}{b^3-a^3}\rho\right).$$

Motion symmetrical About an Axis, the Lines of Motion being in planes Passing Through the Axis : Stokes's Stream Function :

When the motion is the same in every plane through a given line, called the axis, the motion is called **axi-symmetrical**. Such a motion occurs, for example, in uniform flow past a stationary sphere, the sphere moving with uniform velocity in a fluid at rest, the motion of a solid of revolution moving in the direction of the axis of revolution etc. Such motions give rise to some analogies with the two-dimensional case; for example a stream function can be defined for such motions as illustrated below.

The equation of continuity in cylindrical co-ordinates for the case of incompressible fluid is,

$$\frac{1}{r}\frac{\partial}{\partial r}(rq_r)+\frac{1}{r}\frac{\partial q_\theta}{\partial \theta}+\frac{\partial q_z}{\partial z}=0 \qquad \text{......(1)}$$

When the motion is symmetrical about z-axis, $q_\theta = 0$ and hence (1) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{\partial q_z}{\partial z} = 0 \qquad \text{.......(2)}$$

Now, let x-axis be taken as the axis of symmetry in place of z-axis and let $\overline{\omega} = \left[\sqrt{(y^2+z^2)}\right]$ denote the distance from the x-axis. Let u, v denote components of velocity in the directions of x and $\overline{\omega}$ respectively. Then replacing r and z by $\overline{\omega}$ and x respectively and replacing q_r and q_z by v and u respectively in (2), we have

$$\frac{1}{\overline{\omega}}\frac{\partial}{\partial \overline{\omega}}(\overline{\omega}v) + \frac{\partial u}{\partial x} = 0$$

or $$\frac{\partial}{\partial \overline{\omega}}(\overline{\omega}v) = \frac{\partial}{\partial x}(-\overline{\omega}u) \qquad \text{......(3)}$$

But (3) is the condition that

$$\overline{\omega}v dx - \overline{\omega}u d\overline{\omega},$$

may be an exact differential, $d\psi$, say.

Thus $\overline{\omega}v dx - \overline{\omega}u d\overline{\omega} = d\psi$

$$\therefore \quad \overline{\omega}v dx - \overline{\omega}u d\overline{\omega} = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial \overline{\omega}}d\overline{\omega} \qquad \text{......(4)}$$

so that $$u = -\frac{1}{\overline{\omega}}\frac{\partial \psi}{\partial \overline{\omega}} \text{ and } v = \frac{1}{\overline{\omega}}\frac{\partial \psi}{\partial x} \qquad \text{......(5)}$$

The function ψ defined by (5) is known as **Stokes's stream function.**

The strem lines are given by

$$\frac{dx}{u} = \frac{d\overline{\omega}}{v} \quad \text{or} \quad \overline{\omega}v dx - u\overline{\omega}d\overline{\omega} = 0$$

or $d\psi = 0$, using (4)

Integrating,

ψ = constant, which represents the stream lines.

Remark :

Stokes's stream function ψ represents the strem lines ψ = constant in an analogous way to the stream function in two-dimesional flow as defined earlier. But the existence of Stokes's stream function does not depend upon the existence of the velocity potential ϕ, i.e., **the Stokes's stream function exists even if the motion is not irrotational which is not true in the case of two-dimensional stream function defined earlier.**

A Property of Stokes's Function :

2π times the difference of the values of Stokes's stream function at two points in the meridian

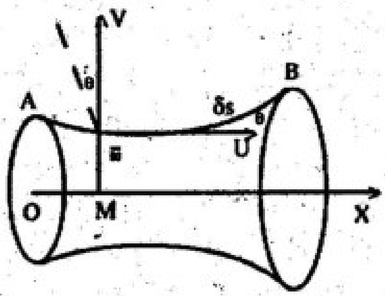
plane is equal to the flow across the annular surface obtained by the revolution round the axis of curve joining the points.

Proof :

Let AB be an arc of a curve which when rotated about the axis (x-axis) will describe an annular surface. Let P be a point in AB, ds an elementary arc at P. Let θ be the inclination of ds to the axis. Then velocity across the element ds

$= v\cos\theta - u\sin\theta$

$$= \frac{1}{\varpi}\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\frac{d\varpi}{ds} = \frac{1}{\varpi}\frac{\partial\psi}{ds} \quad \ldots\ldots (1)$$



∴ Flow across the annular surface

$$= \int_A^B (v\cos\theta - u\sin\theta) 2\pi\varpi ds$$

$$= 2\pi\int_A^B \frac{1}{\varpi}\frac{\partial\psi}{\partial s}\varpi ds \quad \text{(using (1))}$$

$$= 2\pi\int_A^B d\psi = 2\pi(\psi_B - \psi_A)$$ which proves the required result.

Unit 4

Vortex Motion (Rectangular Vortices)

Introduction :

It is known that all possible motions of an ideal liquid can be subdivided into two classes; vortex free irrotational or potential flows, whose characteristics can be derived from a velocity potential $\phi(x, y, t)$, and vortex or rotational motions for which this is not the case. Rotational motions differ from potential flows in that, as the name applies, all particles of the fluid or at least part of them rotate about an axis which moves with the fluid. Potential flow, on the other hand, is irrotational by definition. So far we paid attention almost entirely to cases involving irrotational motion only. In the present chapter we wish to discuss the theory of rotational or vortex motion.

Helmholtz's Vorticity Theorems, Properties of Vortex Tube :

(1) The product of the cross section and vorticity (or angular velocity) at any point on a vortex filament is constant along the filament and for all time when the body forces are conservative and the pressure is a single-valued function of density only.

Let Ω be the vorticity **vector** and let ω be the angular velocity **vector**. Then we have

$$\Omega = \text{curl}\,\bar{q} \quad \text{and also} \quad \Omega = 2\omega \qquad ......(1)$$

Let $\delta S_1, \delta S_2$ be two sections of a vortex tube and let n_1, n_2 be the unit normals to these sections drawn outwards from the fluid between them. Again, suppose δS be the curved surface of the vortex tube and

$$\Delta S = \text{total surface area element} = \delta S_1 + \delta S + \delta S_2$$

$$\Delta V = \text{total volume which } \Delta S \text{ contains.}$$

Then $$\int_{\Delta S} \bar{n}\cdot\Omega dS = \int_{\Delta S} \bar{n}\cdot\Omega dV = 0 \qquad ......(2)$$

Since $\nabla\cdot\Omega = \nabla\cdot\text{curl}\bar{q} = 0$. Hence (2) gives

$$\int_{\delta S_1} \bar{n}\cdot\Omega dS + \int_{\delta S} \bar{n}\cdot\Omega dS + \int_{\delta S_2} \bar{n}\cdot\Omega dS = 0 \qquad ......(3)$$

Since Ω is tangential to the curved surface of the vortex tube, $\bar{n}\cdot\Omega = 0$ at each point δS. Hence (3) reduces to

$$\int_{\delta S_1} \bar{n}\cdot\Omega dS = -\int_{\delta S_2} \bar{n}\cdot\Omega dS$$

or $$\int_{\delta S_1} \Omega\cdot dS = \int_{\delta S_2} \Omega\cdot dS \qquad ......(4)$$

or $$\int_{\delta S_1} 2\omega\cdot dS = \int_{\delta S_2} 2\omega\cdot dS \qquad ......(5)$$

Thus, to first order of approximation, (4) and (5) give

$$\Omega_1\delta S_1 = \Omega_2\delta S_2 \quad \text{and} \quad \omega_1\delta S_1 = \omega_2\delta S_2 \qquad ......(6)$$

Equation (6) shows that $\Omega\delta S$ or $2\omega\delta S$ is constant over every section δS of the vortex tube. Its value is known as the **strength of the vortex tube**. A vortex tube whose strength is unity is called a **unit vortex tube.**

(2) Vortex lines and tubes cannot originate or terminate at internal points in a fluid.

Let S be any closed surface containing a volume V. Then we have

$$\int_S \Omega \cdot dS = \int_S \bar{n} \cdot \Omega dS = \int_V \nabla \cdot \Omega dV = 0$$

which shows that the total strength of vortex tubes emerging from S must be equal to that entering S. Hence, vortex lines and tubes cannot begin or and at any point within the liquid. They must either form closed curves or have their extremities on the boundary of the liquid.

(3) Vortex lines move along with the liquid (i.e., they are composed of the same elements of the liquid) provided that body forces are conservative and the pressure is a single-valued function of density.

Let C be a closed circuit of liquid particles and let S be an open surface with C as rim. Then the circulation Γ is constant in the moving circuit C by **Kelvin's circulation theorem.** Thus we have

$$\Gamma = \int_C \bar{q} \cdot d\bar{r} = \int_S \text{curl}\bar{q} \cdot dS = \int_S \Omega \cdot dS \quad \text{(by Stoke's Theorem)}$$

so that

$$\int_S \Omega \cdot dS = \text{constant} \qquad ....(7)$$

since Γ is constant. Thus for a surface S moving with the fluid (7) holds. The L.H.S. of (7) represents the total strength of vortex tubes passing through S. This shows that the vortex tubes move with the fluid. By taking $S \to 0$, it follows that the vortex lines move with the liquid.

Exercise :

Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are

$$u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

where μ, ϕ are function of x, y, z, t.

OR

Find the necessary and sufficient condition that vortex lines may be at right angles to the stream lines.

Solution :

Stream line are $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ ......(1)

and vortex lines are $\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$ ......(2)

(1) and (2) will be at right angles, if

$$u\Omega_x + v\Omega_y + w\Omega_z = 0 \qquad ......(3)$$

But $\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ ......(4)

Using (4), (3) may be re-written as

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

which is the necessary and sufficient condition in order that udx + vdy + wdz may be a perfect differential. So we may write

$$udx + vdy + wdz = \mu d\phi$$

$$= \mu\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right)$$

$$\therefore \quad u = \mu\frac{\partial \phi}{\partial x}, \quad v = \mu\frac{\partial \phi}{\partial y}, \quad w = \mu\frac{\partial \phi}{\partial z}.$$

Rectilinear Vortices :

Vortex lines being straight and parallel, all vortex tubes are cylindrical, with generators perpendicular to the plane of motion. Such vortices are known as rectilinear vortices.

Derivation of velocity potential, stream function, velocity components and complex potential due to a rectilinear vortex filament :

Consider a retilinear vortex with its axis parallel to the axis of z. The motion being similar in all planes parallel to xy-plane, we have no velocity along that axis i.e., w = 0. Moreover u and v are independent of z, i.e.,

$$\frac{\partial u}{\partial z} = 0 \quad \text{and} \quad \frac{\partial v}{\partial z} = 0 \qquad ......(1)$$

If $\Omega_x, \Omega_y, \Omega_z$ be the vorticity components, then

$$\Omega_x = 0, \Omega_y = 0 \text{ and } \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad ......(2)$$

Now the equations of lines of flow are

$$\frac{dx}{u} = \frac{dy}{v}, \text{ i.e., } vdx - udy = 0 \qquad ....(3)$$

The equation of continuity is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

so that $$\frac{\partial v}{\partial y}=\frac{\partial(-u)}{\partial x} \qquad ......(4)$$

Equation (4) shows that vdx - udy must be perfect differential, $d\psi$ (say). Thus

$$vdx-udy=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

so that $$u=-\frac{\partial \psi}{\partial y} \quad \text{and} \quad v=\frac{\partial \psi}{\partial x} \qquad ......(5)$$

Then the lines of flow are given by $d\psi=0$, i.e, $\psi=$ constant. Hence ψ is the strem function. Using (5), (2) gives

$$\Omega_z=\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2} \qquad ......(6)$$

Thus the strem function ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\Omega_z, \text{ on the vortex filament} \qquad ......(7A)$$

$$=0, \text{ outside the filament} \qquad ......(7B)$$

Let $P(r, \theta)$ be any point outside the vortex filament. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists such that

$$\frac{\partial \psi}{\partial r}=-\frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad ......(8)$$

Moreover, outside vortex filament, ψ satisfies the equation (re-writing (7B) in polar co-ordinates)

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}=0 \qquad ......(9)$$

There being symmetry about the origin. ψ must be independent of θ and so (9) reduces to

$$\frac{\partial^2 \psi}{\partial r^2}+\frac{1}{r}\frac{\partial \psi}{\partial r}=0$$

or $$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=0$$

Integrating

$$r\frac{\partial \psi}{\partial r}=c \qquad ......(10).$$ Integrating (10),

$$\psi = c \log r. \qquad \text{......(11)}$$

Now (8) and (10) give

$$\frac{c}{r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \quad \text{so that} \quad \phi = -c\theta \qquad \text{......(12)}$$

If $w(=\phi + i\psi)$ be the complex potential outside the filament, the we have

$$w = -c\theta + ic\log r = ic(\log r + i\theta) = ic\log(re^{i\theta}) = ic\log z.$$

Let k be the circulation in the circut embracing the vortex.

$$\therefore \quad k = \int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) r\,d\theta = c\int_0^{2\pi} d\theta = 2\pi c, \quad \text{by (12)}$$

$$\therefore \quad c = \frac{k}{2\pi}.$$

Hence, we have

$$\phi = -\frac{k}{2\pi}\theta, \quad \psi = \frac{k}{2\pi}\log r \text{ and } w = \frac{ik}{2\pi}\log z \qquad \text{......(13)}$$

Here k is called the strength of the vortex.

If there be a rectilinear vortex of strength k at $z_0(= x_0 + iy_0)$, then

$$w = \frac{ik}{2\pi}\log(z - z_0) \qquad \text{......(14)}$$

We now determine velocity components due to a rectilinear vortex of strength k at $A_0(z_0)$. Let P(x, y) be any point in the fluid. Then, if r_0 be the distance between $A_0(z_0)$ and P(z), then we have

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2$$

and $$\psi = \frac{k}{2\pi}\log r_0$$

$$\therefore \quad u = -\frac{\partial\psi}{\partial y} = -\frac{\partial\psi}{\partial r_0}\frac{\partial r_0}{\partial y} = -\frac{k}{2\pi r_0}\frac{y - y_0}{r_0}$$

$$= -\frac{k}{2\pi}\frac{y - y_0}{r_0^2} \qquad \text{......(15)}$$

and $$v = \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r_0}\frac{\partial r_0}{\partial x} = \frac{k}{2\pi r_0}\frac{x - x_0}{r_0} = \frac{k}{2\pi}\frac{x - x_0}{r_0^2}$$

$$\therefore \quad q = \sqrt{u^2 + v^2} = \frac{k}{2\pi r_0^2}\sqrt{\{(x - x_0)^2 + (y - y_0)^2\}} = \frac{k}{2\pi r_0} \quad \text{...(17)}$$

which gives velocity at P(x, y).

Remark 1. :

Some writes define $K = \frac{k}{2\pi}$ as the strength of the vortex. Accordingly, they take

$$\left.\begin{array}{llll} \phi = -K\theta, & \psi = K\log r, & w = iK\log z, & w = iK\log(z - z_0) \\ u = -K\dfrac{y - y_0}{r_0^2}, & v = K\dfrac{x - x_0}{r_0^2}, & q = \dfrac{K}{r_0} & \end{array}\right\} \quad \text{......(18)}$$

However, we shall not use these results in the present discussion unless otherwise stated.

Remark 2. : The case of several rectilinear vortices :

Let there be a number of vortices of strength $k_1, k_2, k_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$ Then the complex potential is given by

$$w = \frac{ik_1}{2\pi}\log(z - z_1) + \frac{ik_2}{2\pi}\log(z - z_2) + \frac{ik_3}{2\pi}\log(z - z_3) + \ldots$$

i.e., $$w = \frac{1}{2\pi}\Sigma k_n \log(z - z_n) \quad \text{......(19)}$$

Here vortices of strength $k_1, k_2, k_3, \ldots$ are situted at $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ Hence, using (15) and (16), the velocity components u and v due to these vortices are given by

$$u = -\frac{1}{2\pi}\Sigma\frac{y - y_n}{r_n^2} \quad \text{and} \quad v = \frac{1}{2\pi}\Sigma\frac{x - x_n}{r_n^2} \quad \text{......(20)}$$

where $r_n^2 = (x - x_n)^2 + (y - y_n)^2, \quad n = 1, 2, 3, \ldots$ ......(21)

Let k_m be the strength of a vortex situated at (x_m, y_m). **Then we omit the term containing k_m while finding the velocity of that vortex.** Thus the motion of the mth situated at (x_m, y_m) is given by

$$\dot{x}_m = -\frac{1}{2\pi}\Sigma k_n \frac{y_m - y_n}{r_{mn}^2}, \qquad \dot{y}_m = -\frac{1}{2\pi}\Sigma k_n \frac{x_m - x_n}{r_{mn}^2} \quad \text{......(22)}$$

where $m \neq n$ and $r_{mn}^2 = (x_m - x_n)^2 + (y_m - y_n)^2$ ......(23)

Using (22), we have

$$\sum_m k_m \dot{x}_m = -\frac{1}{2\pi}\sum_m \sum_n k_m k_n \frac{y_m - y_n}{r_{mn}^2} = 0 \quad \text{......(24)}$$

since m, n can be interchanged and the denominator is positive.

Similarly, $$\sum_m k_m \dot{y}_m = 0 \quad \text{......(25)}$$

Since k_m is independent of t, integration of (24) and (25) yield

$$\Sigma k_m x_m = \text{constant} \quad \text{and} \quad \Sigma k_m y_m = \text{constant} \quad \text{......(26)}$$

Also, $\bar{x} = \dfrac{\Sigma k_m x_m}{\Sigma k_m}$ and $\bar{y} = \dfrac{\Sigma k_m y_m}{\Sigma k_m}$ ......(27)

Using (26), (27) show that $\bar{x}, \bar{y}$ are constants. Hence if $k_1, k_2, k_3, \ldots$ be supposed to be the masses situated at $z_1, z_2, z_3, \ldots$, then their centre of gravity is fixed throughout the motion. This point is known as the **centre of vortices. Thus if there be several vortices, they move in such a manner that their centre is stationary.**

Remark 3. Single vortex in the field of several vortices :

"To show that a single rectilinear vortex in an unlimited mass of liquid remains stationary, and when such a vortex is in the presence of other vortices it has tendency to move of itself but its motion through the liquid is entirely due to the vortices caused by the other vortices."

Proof :

The value of stream function ψ at any point inside of a circular vortex tube is given by

$$2\zeta = \frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr}$$

or $\quad \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{d\psi}{dr}\right) = 2\zeta \quad$ or $\quad \dfrac{d}{dr}\left(r\dfrac{d\psi}{dr}\right) = 2\zeta dr$

Integrating twice,

$$\psi = \frac{1}{2}\zeta r^2 + c_1 \log r + c_2 \qquad ......(1)$$

$\therefore$ **velocity at right angles to the radius vector**

$$= \frac{d\psi}{dt} = \zeta r + \frac{c}{r} \qquad ......(2)$$

Since the velocity at the origin is finite, c must be zero. Then (2) gives

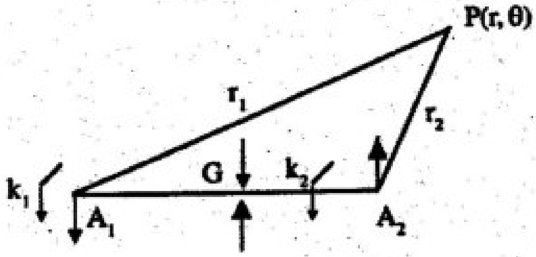
$$\left(\frac{d\psi}{dr}\right)_{r=0} = 0,$$

showing that the velocity at the origin due to a single vortex must vanish. **It follows that a vortex filament (vortex) induces no velocity at its centre. Thus, if a vortex is in the presence of other vortices it has no tendency to move of itself but its motion through the liquid will be caused by the other vortices.**

Two Vortex filaments :

Case I : When the filaments are in the same sense :

Let us consider two rectilinear vortices of strength k_1 and k_2 at A_1 $(z = z_1)$ and $A_2 (z = z_2)$. Then complex potential due to stationary system is



$$w = \frac{ik_1}{2\pi}\log(z - z_1) + \frac{ik_2}{2\pi}\log(z - z_2) \qquad \text{......(1)}$$

However, vortices situated at A_1 and A_2 would start moving due to the presence of each other. Let u_1, v_1 be the components of the velocity q_1 of A_1 which due to A_2 alone. Then, we have

$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik_1}{z - z_1} + \left(-\frac{dw}{dz}\right)\right]_{z=z_1} = -\frac{ik_2}{2\pi}\frac{1}{z_1 - z_2} \qquad \text{...(2)}$$

$$\therefore \qquad q_1 = |u_1 - iv_1| = \frac{k_2}{2\pi|z_1 - z_2|} = \frac{k_2}{2\pi(A_1A_2)} \qquad \text{......(3)}$$

similarly,

$$u_2 - iv_2 = -\frac{ik_1}{2\pi}\frac{1}{z_2 - z_1} \qquad \text{......(4)}$$

and $$q_2 = \frac{k_1}{2\pi(A_1A_2)} \qquad \text{.......(5)}$$

From (2) and (4),

$$\frac{u_1 - iv_1}{k_2} = -\frac{u_2 - iv_2}{k_1}$$

i.e., $k_1(u_1 - iv_1) - k_2(u_2 - iv_2) = 0$

i.e., $(k_1u_1 + k_2u_2) - i(k_1v_1 + k_2v_2) = 0$

so that $k_1u_1 + k_2u_2 = 0$ and $k_1v_1 + k_2v_2 = 0$ ......(6)

Since $k_1 + k_2 \neq 0$, (6) shows that a point G, the centroid of masses k_1, k_2 at z_1 and z_2, moving with velocities (u_1, v_1), (u_2, v_2) is at rest. Hence the line A_1A_2 rotates about G. Since G is C. G. of k_1 and k_2, we have

$$k_1 . A_1G = k_2 . A_2G$$

$$\text{or} \qquad \frac{A_1G}{k_2} = \frac{A_2G}{k_1} = \frac{A_2G + A_1G}{k_1 + k_2} = \frac{A_1A_2}{k_1 + k_2}$$

$$\text{so that} \qquad A_1G = \frac{k_2}{k_1 + k_2} A_1A_2, \qquad A_2G = \frac{k_1}{k_1 + k_2} A_1A_2 \quad \text{......(7)}$$

Re-writing (3), we have

$$q_1 = \frac{k_2 A_1 A_2}{k_1 + k_2} \cdot \frac{k_1 + k_2}{2\pi (A_1A_2)^2} = A_1G \cdot \omega \quad \text{......(8)}$$

$$\text{where} \qquad \omega = \frac{k_1 k_2}{2\pi (A_1A_2)^2} \qquad \text{and} \qquad A_1G = \frac{k_2 A_1 A_2}{k_1 + k_2} \quad \text{.......(9)}$$

The angular velocity of A_1 is ω about G. Similarly, we may show that the angular velocity of A_2 is ω about G. Hence the line A_1A_2 revolves about G with uniform angular velocity ω.

Remark :

As a particular case, let $k_1 = k_2 = k$ and $A_1A_2 = 2a$. Then, we have

$$q_1 = \frac{k}{4\pi a}, \qquad q_2 = \frac{k}{4\pi a} \quad \text{and} \quad \omega = \frac{k}{4\pi a^2}$$

and the stream function is given by

$$\psi = \frac{k}{2\pi} \log r_1 + \frac{k}{2\pi} \log r_2 = \frac{k}{2\pi} \log(r_1 r_2)$$

where $r_1 = A_1P$, $r_2 = r_2P$ and P is any point in the fluid. The stream lines are given by

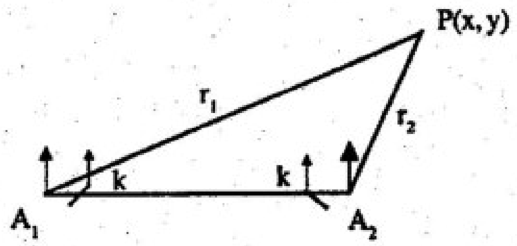
$\psi = \text{constant}$, i.e., $r_1 r_2 = \text{constant}$.

Case II. : when the filaments are in the opposite sense :

Let k_1 and k_2 be of opposite signs. Then G will not lie in between A_1 and A_2. However, if $k_1 > k_2$, then G will lie on A_1A_2 produced and if $k_2 > k_1$, it will lie on A_1A_2 produced. As before, it can be shown that the line A_1A_2 revolves about G with uniform velocity ω.

Vortex Pair :

Two vortex filaments of strengths k and - k form a vortex pair.



Let us consider two rectilinear vortices of strengths k and - k at $A_1(z = z_1)$ and $A_2(z = z_2)$.

Then complex potential at any point P(x, y) due to stationary system is

$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2) \qquad \text{......(1)}$$

However, the vortices situated at A_1 and A_2 would start moving due to the presence of each other. Let u_1, v_1 be the components of the velocity q_1 of A_1 which is due to A_2 alone. Then, we have

$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik}{z - z_1} + \left(-\frac{dw}{dz}\right)\right]_{z=z_1} = \frac{ik}{2\pi}\frac{1}{z_1 - z_2} \qquad \text{...(2)}$$

$$\therefore \quad q_1 = |u_1 - iv_1| = \frac{k}{2\pi|z_1 - z_2|} = \frac{k}{2\pi(A_1A_2)} \qquad \text{......(3)}$$

Similarly,

$$u_2 - iv_2 = -\frac{ik}{2\pi}\frac{1}{z_2 - z_1} \qquad \text{......(4)}$$

and $$q_2 = \frac{k}{2\pi(A_1A_2)} \qquad \text{......(5)}$$

Let $$q_1 = q_2 = q \text{ (say)} \qquad \text{......(6)}$$

Thus, the velocity q_1 of A_1 due to A_2 is q and perpendicular to A_1A_2. Similarly, the velocity q_2 of A_2 due to A_1 is q and perpendicular to A_2A_1 in the same sense as that of A_1. Hence the vortices situated at A_1 and A_2 move in the same direction perpendicular to A_1A_2 with uniform velocity q. However, the line may move forward or backward according to the directions of rotation.

Let $w = \phi + i\psi$, $z = (x, y)$, $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.
Then (1) gives

$$\phi + i\psi = \frac{ik}{2\pi}\log[(x - x_1) + i(y - y_1)] - \frac{ik}{2\pi}\log[(x - x_2) + i(y - y_2)]$$

Equating imaginary parts of both sides, we get

$$\psi = \frac{ik}{2\pi}\frac{1}{2}\log[(x - x_1)^2 + (y - y_1)^2] - \frac{ik}{2\pi}.\frac{1}{2}\log[(x - x_2)^2 + (y - y_2)^2]$$

or $$\psi = \frac{ik}{4\pi}(\log r_1^2 - \log r_2^2) = \frac{ik}{2\pi}\log\frac{r_1}{r_2}$$

where $r_1 = A_1P$ and $r_2 = A_2P$.

The statements are given by ψ = constant, i.e., $\frac{r_1}{r_2}$ = constant which clearly form a system of coaxial circles having A_1 and A_2 as their limiting points.

Motion of any Vortex :

When there are any number of vortices in an infinite liquid, we can find the motion of any one of them. It depends not on itself but on others, hence to find the motion we have to subtract from the stream function of the system the term that corresponds to it.

Let there be a number of vortices of strengths $k_1, k_2, k_3, \ldots$ situated at $z_1, z_2, z_3, \ldots$ respectively, where $z_n = x_n + iy_n$. Then the complex potential of the system at any outside point is

$$w = \frac{i}{2\pi}\Sigma k_n \log(z - z_n)$$

or $$\phi + i\psi = \frac{i}{2\pi}\Sigma k_n \log[(x - x_n) + i(y - y_n)]$$

or $$\phi + i\psi = \frac{i}{2\pi}\Sigma k_n \left[\frac{1}{2}\log\left\{(x - x_n)^2 + (y - y_n)^2\right\} + i\tan^{-1}\frac{y - y_n}{x - x_n}\right]$$

$$\therefore \quad \psi = \Sigma\frac{k_n}{4\pi}\left\{(x - x_n)^2 + (y - y_n)^2\right\}$$

$\therefore$ The stream function ψ' at the vortex (x_n, y_n) is given by

$$\psi' = \Sigma\frac{k_n}{4\pi}\left\{(x - x_n)^2 + (y - y_n)^2\right\} - \frac{k_m}{4\pi}\left\{(x - x_m)^2 + (y - y_m)^2\right\}$$

If χ be the stream function for the motion of vortex (x_m, y_m), we have

$$-\frac{\partial\chi}{\partial y_m} = \left(-\frac{\partial\psi'}{\partial y}\right)_m, \qquad \frac{\partial\chi}{\partial x_m} = \left(\frac{\partial\psi'}{\partial x}\right)_m$$

by equating the components of velocity of the vortex (x_m, y_m).

Suppose there is a single vortex k at (x_1, y_1) in front of a fixed wall taken as $y = 0$.

We have to introduce the image - k at $(x_1, -y_1)$ and the stream function of the system is

$$\psi = \frac{k}{4\pi}\log\left\{(x - x_1)^2 + (y - y_1)^2\right\} - \frac{k}{4\pi}\log\left\{(x - x_1)^2 + (y + y_1)^2\right\}$$

$$\therefore \quad \psi' = -\frac{k}{4\pi}\log\left\{(x - x_1)^2 + (y + y_1)^2\right\}$$

$$\therefore \quad \frac{\partial\chi}{\partial y_1} = \left(-\frac{\partial\psi'}{\partial y}\right)_{x=x_1, y=y_1} = \left\{\frac{k}{4\pi}\frac{2(y + y_1)}{(x - x_1)^2 + (y + y_1)^2}\right\}_{\substack{x=x_1\\y=y_1}}$$

$$= \frac{k}{4\pi}\frac{4y_1}{4y_1^2} = \frac{k}{4\pi y_1}$$

and $$\frac{\partial \chi}{\partial x_1} = \left(\frac{\partial \Psi'}{\partial x}\right)_{x=x_1, y=y_1} = \left\{-\frac{k}{4\pi}\frac{2(x-x_1)}{(x-x_1)^2+(y+y_1)^2}\right\}_{\substack{x=x_1\\ y=y_1}} = 0$$

$$\therefore \quad \chi = -\frac{k}{4\pi}\log y_1$$

Hence the path of the cortex is the streamline for the vortex, i.e., $y_1 =$ constant.

Kirchhoff Vortex Theorem : General System of Vortex Filament :

If $(r_1, \theta_1), (r_2, \theta_2), ..., (r_n, \theta_n)$ be the polar co-ordinates at any time t of a system of rectilinear vortices of strength $K_1, K_2, ..., K_n$ then

$$\sum_{p=1}^{n} K_p x_p = A, \quad \sum_{p=1}^{n} K_p y_p = B, \quad \sum_{p=1}^{n} K_p r_p^2 = C, \quad \sum_{p=1}^{n} K_p r_p^2 \dot{\theta}_p = D$$

where A, B, C, D are constants and $\dot{\theta}_p = \frac{d\theta_p}{dt}$.

Proof :

The complex potential w due to n vortex filaments of strengths K_p at the points $z_p = x_p + iy_p = r_p(\cos\theta_p + i\sin\theta_p)$ is given by

$$w = \sum_{p=1}^{n} \frac{iK_p}{2\pi}\log(z - z_p).$$

Hence, the velocity at any point of the fluid, not occupied by any vortex is given by

$$u - iv = -\frac{dw}{dz} = -\sum_{p=1}^{n} \frac{iK_p}{2\pi(z - z_p)}.$$

Since the velocity (u_p, v_p) of vortex K_p is produced by the remaining other vortices (because any particular vortex cannot move solely on its own account),

$$\therefore \quad u_p - iv_p = \left(-\frac{dw_p}{dz}\right)_{z=z_p} = \left[-\frac{d}{dz}\sum_{q\neq p}\frac{iK_q}{2\pi}\log(z - z_q)\right]_{z=z_p}$$

or $$u_p - iv_p = -\sum_{q\neq p} \frac{iK_q}{2\pi(z_p - z_q)} \qquad ......(1)$$

Multi plying (1) by K_p and summing up from p = 1 to p = n, we obtain

$$\sum_{p=1}^{n} K_p(u_p - iv_p) = -\sum_{p=1} \sum_{q\neq p} \frac{iK_pK_q}{2\pi(z_p - z_q)} = 0 \qquad \text{......(2)}$$

the double summation on R.H.S is zero because the terms cancel in pairs, for example $\frac{iK_pK_q}{(z_p - z_q)}$ cancels

$\frac{iK_pK_q}{(z_q - z_p)}$ and there are no terms in $K^2{}_p$ etc.

Equating real and imaginary parts, (2) gives

$$\Sigma K_p u_p = 0 \quad \text{and} \quad \Sigma K_p v_p = 0$$

$$\text{or} \quad \Sigma K_p \frac{dx_p}{dt} = 0 \quad \text{and} \quad \Sigma K_p \frac{dy_p}{dt} = 0 \qquad \text{.......(3)}$$

Integrating (3),

$$\Sigma K_p x_p = A \quad \text{and} \quad \Sigma K_p y_p = B.$$

where A and B are constants of integration.

Again, multiplying (1) by K_p, z_p and summing from p = 1 to p = n, we obtain

$$\sum_{p=1}^{n} K_p z_p (u_p - iv_p) = -\sum_{p=1}^{n} \sum_{q\neq p} \frac{iK_pK_qz_p}{2\pi(z_p - z_q)}$$

$$\text{or} \quad \sum_{p=1}^{n} K_p(x_p + iy_p)(u_p - iv_p) = -\frac{i}{2\pi}\sum_{p=1}^{n} \sum_{q\neq p} \frac{K_pK_qz_p}{z_p - z_q}$$

$$\text{or} \quad \sum_{p=1}^{n} K_p[(x_pu_p + y_pv_p) - i(x_pv_p - y_pu_p)] = -\frac{i}{2\pi}\Sigma K_pK_q \quad \text{...(4)}$$

because in the double summation on R.H.S., the sum of pairs of terms such as $\frac{K_pK_qz_p}{(z_p - z_q)}$ and $\frac{K_pK_qz_q}{(z_q - z_p)}$

reduces to K_pK_q and there are no terms in $K^2{}_p$.

Equating real and imaginary parts, (4) gives

$$\Sigma K_p(x_pu_p + y_pv_p) = 0 \qquad \text{.......(5)}$$

$$\text{and} \quad \Sigma K_p(x_pv_p - y_pu_p) = \frac{1}{2\pi}\Sigma K_pK_q = \text{cons tan t} = D\text{, say} \quad \text{..(6)}$$

Re-writing (5), we have $\Sigma K_p\left[2x_p\frac{dx_p}{dt} + 2y_p\frac{dy_p}{dt}\right] = 0$

or $\quad \Sigma K_p \frac{d}{dt}(x_p^2 + y_p^2) = 0$

or $\quad \Sigma K_p \frac{dr_p^2}{dt} = 0 \qquad [\because \quad r_p^2 = x_p^2 + y_p^2]$

Integrating,

$$\Sigma K_p r_p^2 = C, \quad \text{where C is constant of integration.}$$

From (6),

$$\Sigma K_p \left(x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt} \right) = D \qquad \text{......(7)}$$

But, $\frac{y_p}{x_p} = \tan\theta_p \qquad [\because \quad x_p = r_p\cos\theta_p, \quad y_p = r_p\sin\theta_p]$

Differentiating both sides w.r.t. 't', we get

$$\frac{x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt}}{x_p^2} = \sec^2\theta_p \frac{d\theta_p}{dt}$$

or $\quad x_p \frac{dy_p}{dt} - y_p \frac{dx_p}{dt} = x_p^2 \sec^2\theta_p \dot{\theta}_p = r_p^2 \dot{\theta}_p \qquad [\because \quad x_p = r_p\cos\theta_p]$

$\therefore \quad$ (7) can be re-written as $\quad \Sigma K_p r_p^2 \dot{\theta}_p = D.$

Exercise :

(a) When an infinite liquid contains two parallel, equal and opposite rectililinear vortices at a distance 2a, prove that the streamlines relative to the vortex are given by the equation,

$$\log \frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} + \frac{y}{a} = c,$$

the origin being the middle point of the join, which is taken for axis of y.

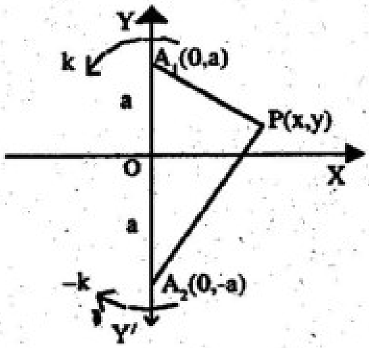
(b) Show that for a vortex pair the relative streamlines are given by

$$k\left\{\left(\frac{y}{2a}\right) + \log\left(\frac{r_1}{r_2}\right)\right\} = \text{cons tan t},$$

where 2a is the distance between the vortices and r_1, r_2 are the distances of any point from them.

Solution : Part (a) :

Let there be two rectilinear vortices of strength k and $-$ k at $A_1(z = 0 + ia)$ and $A_2(z = 0 - ia)$ respectively. Thus $A_1A_2 = 2a$, origin being the middle point of A_1A_2 and y-axis being taken along A_1A_2 as shown in figure.



Here we have a vortex pair and hence the pair will move with a uniform velocity $\frac{k}{2\pi A_1A_2}$ or $\frac{k}{4\pi a}$ perpendicular to the line A_1A_2 (i.e., along the x-axis). To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $\frac{k}{4\pi a}$ of motion of the vortex pair. Accordingly, we add a term $\frac{kz}{4\pi a}$ to the complex potential of the vortex pair. Note that

$$-\frac{d}{dz}\left(\frac{kz}{4\pi a}\right) = -\frac{kz}{4\pi a},$$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = \frac{ik}{2\pi}\log(z - ia) - \frac{ik}{2\pi}\log(z + ia) + \frac{kz}{4\pi a}$$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log[x^2 + (y - a^2)] - \frac{k}{4\pi}\log[x^2 + (y + a^2)] + \frac{ky}{4\pi a}$$

or $$\psi = \frac{k}{2\pi}\left[\log\frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} + \frac{y}{a}\right] \qquad ......(1)$$

Hence the required relative streamlines are given by

ψ = constant

i.e., $$\log\frac{x^2 + (y-a)^2}{x^2 + (y+a)^2} + \frac{y}{a} = c$$

Part (b) :

As in part (a), do upto (1). Let $r_1 = A_1P$ and $r_2 = A_2P$ so that $r_1^2 = x^2 + (y - a)^2$ and $r_2^2 = x^2 + (y + a)^2$.

Putting these in (1) of part (a), we obtain

$$\psi = \frac{k}{4\pi}\left[\log\frac{r_1^2}{r_2^2} + \frac{y}{a}\right] = \frac{k}{2\pi}\left[\log\frac{r_1}{r_2} + \frac{y}{2a}\right] \qquad ......(2)$$

Hence the relative streamlines are given by

$$\psi = \text{constant}$$

i.e.,
$$k\left\{\left(\frac{y}{2a}\right) + \log\left(\frac{r_1}{r_2}\right)\right\} = \text{constant}$$

Exercise 2 :

An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance 2a. Show that the paths of the fluid particles relative to the vortices can be represented by the equation

$$\log\frac{r^2 + a^2 - 2ar\cos\theta}{r^2 + a^2 + 2ar\cos\theta} + \frac{r\cos\theta}{a} = \text{constant}.$$

Solution :

To obtain the desired result, modify solution of Exercise 1(a) as follows :

Let the vortex pair lie along x-axis in place of y-axis. Then interchanging x and y, we obtain

$$\psi = \frac{k}{2\pi}\left[\log\frac{y^2 + (x-a)^2}{y^2 + (x+a)^2} + \frac{x}{a}\right]$$

or
$$\psi = \frac{k}{2\pi}\left[\log\frac{x^2 + y^2 + a^2 - 2ax}{x^2 + y^2 + a^2 + 2ax} + \frac{x}{a}\right] \qquad (1)$$

Let $x = r\cos\theta$, $y = r\sin\theta$. Then (1), in polar coordinates, takes the form

$$\psi = \frac{k}{2\pi}\left[\log\frac{r^2 + a^2 - 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta} + \frac{r\cos\theta}{a}\right] \qquad (2)$$

Hence, the relative streamlines are given by

$$\psi = \text{const}$$

or
$$\log\frac{r^2 + a^2 - 2ar\cos\theta}{r^2 + a^2 - 2ar\cos\theta} + \frac{r\cos\theta}{a} = \text{const}$$

Exercise 3:

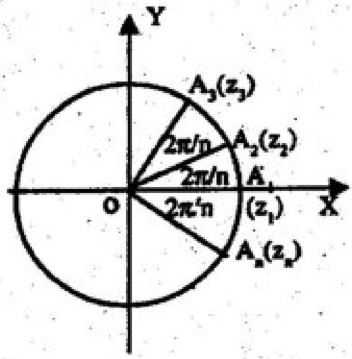
If n recilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)k}$, and find the velocity of any part of the liquid.

Solution :

Let $A_1A_2A_3$ be the circle of radius a. Suppose that n rectilinear vortices each of strength k be situated at points

$$z_m = ae^{\frac{2\pi i m}{n}}, \quad m = 0, 1, 2, ..., n-1$$

of the circle. Then the complex potential due to these n vortices is given by



$$w = \frac{ik}{2\pi}\sum_{m=0}^{n-1}\log\left(z - ae^{\frac{2\pi i m}{n}}\right)$$

$$= \frac{ik}{2\pi}\log\prod_{m=0}^{n-1}\left(z - ae^{\frac{2\pi i m}{n}}\right) = \frac{ik}{2\pi}\log\left(z^n - a^n\right)$$

Now, the fluid velocity q at any point out of all the n vortices is given by

$$q = \left|-\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{nz^{n-1}}{z^n - a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n - a^n}\right|$$

Again the velocity induced at $A_1(z = a)$, by others is given by the complex potential

$$w' = \frac{ik}{2\pi}\log\left(z^n - a^n\right) - \frac{ik}{2\pi}\log(z - a)$$

$$= \frac{ik}{2\pi}\log\frac{\left(z^n - a^n\right)}{(z - a)}$$

or $\quad w' = \frac{ik}{2\pi}\log\left(z^{n-1} + z^{n-2}a + ... + za^{n-2} + a^{n-1}\right)$

so that $\quad \frac{dw'}{dz} = \frac{ik}{2\pi}\frac{(n-1)z^{n-2} + (n-2)z^{n-3} + ... + a^{n-2}}{z^{n-1} + z^{n-2}a + ... + za^{n-2} + a^{n-1}}$

$$\therefore \left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi}\frac{(n-1) + (n-2) + ... + 2 + 1}{na} = -\frac{ik(n-1)}{4\pi a}$$

$$\therefore \ u_1 - iv_1 = \left(-\frac{dw'}{dz}\right)_{z=a} = -\frac{ik(n-1)}{4\pi a}$$

so that $u_1 = 0$ and $v_1 = \dfrac{k(n-1)}{4\pi a}$

If q_r and q_θ be the radial and transverse velocity components of the velocity at z = a, then we have

$$q_r = 0 \quad \text{and} \quad q_\theta = \frac{k(n-1)}{4\pi a}$$

Due to symmetry of the problem, it follows that each vortex move with the same transverse velocity

$q_\theta = \dfrac{k(n-1)}{4\pi a}$. Hence the required time T is given by

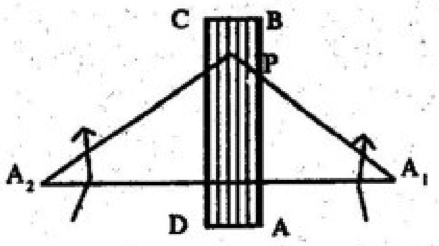
$$T = \frac{2\pi a}{\frac{k(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{(n-1)k}$$

Image of a Vortex Filament in a Plane :

To show that the image of a vortex filament in a plane to which it is parallel is an equal and opposite vortex filament at its optical image in the plane.

Proof :

Let the vortex filaments of strength k and - k be situated at $A_1(z = z_1)$ and $A_2(z = z_2)$ respectively. The complex potential due to the vortices at any point P(z) is given by



$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2)$$

$$\therefore \quad \phi + i\psi = \frac{ik}{2\pi}\left[\log(r_1 e^{i\theta_1}) - \log(r_2 e^{i\theta_2})\right] \qquad \text{......(1)}$$

where $r_1 = |z - z_1|$, $\theta_1 = \arg(z - z_1)$, $r_2 = |z - z_2|$, $\theta_2 = \arg(z - z_2)$.

Equating imaginary parts, (1) gives

$$\psi = \frac{ik}{2\pi}\log\frac{r_1}{r_2} \qquad \text{.......(2)}$$

Let ABCD be a plane bisecting A_1A_2 at right angles and let P be any point on it. Then $r_1 = r_2$ on P so that $\psi = 0$ from (2). Thus there would be no flow acrosss the plane AB. Hence, the motion would remain unchanged if the plane were made a rigid barrier. This proves the required result.

Remark :

$A_1A_2 = 2a$. Then the uniform velocity of vortex filament A_1 parallel to the plane AB (induced by B) is given by

$$\left|\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{1}{z_1 - z_2}\right| = \frac{k}{2\pi A_1A_2} = \frac{k}{4\pi a}$$ ....(3) Moreover the velocity

midway A_1 and A_2 due to both the vortices is $\frac{k}{\pi a}$. **Thus the vortex moves parallel to the plene with one-fourth of the velocity of the liquid at the boundary.**

Image of Vortex in a Quadrant :

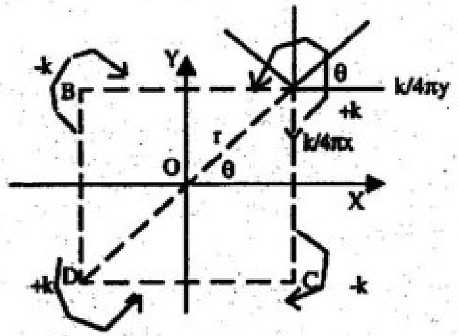
The image system of vortex of strength k, at the point A(x, y) in xy-plane with respect to quadrant XOY consists of

(i) a vortex of strength - k at B(- x, y)
(ii) a vortex of strength - k at C(x, - y)
(iii) a vortex of strength k at D(- x, - y)

The velocity at A is only on account of its images and hence its components (making use of remark of '**Image of a vortex filament in a plane**') are as indicated in the figure. Thus the radial and transverse components of velocity at A are (fig. next page) given by

$$\frac{dr}{dt} = \frac{k\cos\theta}{4\pi y} - \frac{k\sin\theta}{4\pi x} = \frac{k\cos\theta}{4\pi r\sin\theta} - \frac{k\sin\theta}{4\pi r\cos\theta}$$

$$= \frac{k(\cos^2\theta - \sin^2\theta)}{2\pi r\sin 2\theta} \quad (1)$$



or $$\frac{dr}{dt} = \frac{k\cos 2\theta}{2\pi r\sin 2\theta}$$ ....(1)

$$r\frac{d\theta}{dt} = \frac{k}{4\pi r} - \frac{k\sin\theta}{4\pi y} - \frac{k\cos\theta}{4\pi x} = \frac{k}{4\pi r} - \frac{k\sin\theta}{4\pi r\cos\theta} - \frac{k\cos\theta}{4\pi r\cos\theta}$$

or $$r\frac{d\theta}{dt} = -\frac{k}{4\pi r}$$ ....(2)

On dividing (1) by (2), $\frac{1}{r}\frac{dr}{d\theta} = -2\frac{\cos 2\theta}{\sin 2\theta}$.

Integrating it, $\log r = -\log\sin 2\theta + \log c$ i.e., $r\sin 2\theta = c$

Transforming into cartesian, it becomes (using $x = r\cos\theta$, $y = r\sin\theta$)

$$2r\sin\theta\cos\theta = c \quad \text{or} \quad 4r^4\cos^2\theta\sin^2\theta = c^2r^2$$

i.e., $4(r\cos\theta)^2(r\sin\theta)^2 = c^2r^2$

i.e., $4x^2y^2 = c^2(x^2 + y^2)$ or $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{c^2}$.

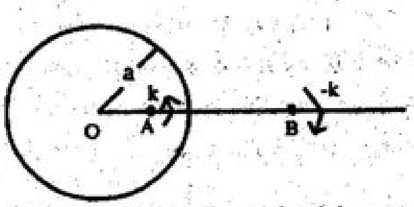
Vortex inside an infinite circular cylinder :

Let the vortex of strength K be situated at A(OA = f) inside the circular cylinder of radius a with axis parallel to the axis of the cylinder. Let a vortex of strength - K be placed at B, where B is the inverse point of A with respect to the circular section of the cylinder so that

$$OB \,.\, OA = a^2$$

$$\Rightarrow OB \,.\, f = a^2$$

$$\Rightarrow OB = \frac{a^2}{f}.$$



The circle is one of the co-axial system having A and B as limiting points and so it is a stream line. The velocity of A

$$= \frac{K}{2\pi AB} = \frac{K}{2\pi(OB - OA)} = \frac{K}{2\pi\left(\frac{a^2}{f} - f\right)} = \frac{Kf}{2\pi(a^2 - f^2)}$$

which is perpendicular to OA. B also has the above mentioned velocity so that OAB will not remain a straight line at the next instant. But if A describes a circle about O with the above velocity, then at every instant the circle will be a stream line, the positions of B, of course, changing from instant to instant.

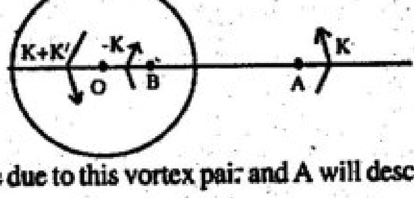
Vortex outside a circular cylineder :

Let the vortex of strength K be situated at A(OA = f) outside the circular cylinder of radius a with axis parallel to the axis of the cylinder. Let a vortex of strength - K be placed at B, where B is the inverse point of A with respect to the circular section of the cylinder so that

$$OB \,.\, OA = a^2$$

$$\Rightarrow OB \,.\, f = a^2$$

$$\Rightarrow OB = \frac{a^2}{f}.$$



Then the circle will be an instantaneous streamline due to this vortex pair and A will describe a circle with velocity

$$= \frac{K}{2\pi AB} = \frac{K}{2\pi(OA - OB)} = \frac{K}{2\pi\left(f - \frac{a^2}{f}\right)} = \frac{Kf}{2\pi(f^2 - a^2)}$$

But the introduction of a vortex - K at B gives a circulation - K about the cylinder and let the circulation about the cylinder be K'. The circulation - K about the cylinder due to the vortex B can be annuled by putting a vortex K to O and therefore to get the final circulation K' about the cylinder, we must put an additional vortex K' at O.

Thus we have a vortex K at A, - K at B, K + K' at O. Hence, the velocity of A due to the above system

$$= \frac{K+K'}{2\pi .OA} - \frac{K}{2\pi .AB} = \frac{K+K'}{2\pi f} - \frac{K}{2\pi(AB - OB)}$$

$$= \frac{K+K'}{2\pi f} - \frac{K}{2\pi\left(f - \frac{a^2}{f}\right)}$$

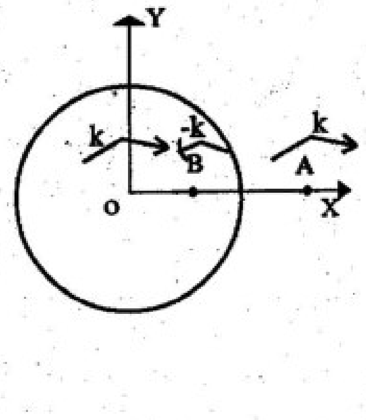
$$= \frac{K+K'}{2\pi f} - \frac{Kf}{2\pi(f^2 - a^2)}$$ and A describes a circle with this velocity.

Image of a vortex outside a circular cylinder :

To show that the image system of a vortex k outside the circular cylinder consists of a vortex of strength = k at the inverse point and a vortex of strength k at the centre.

Let us determine the image of a vortex filament of strength k placed at $A(z = c > a)$ with respect to a circular cylinder $|z| = a$ with O as centre. Let B be the inverse point of A with respect to $|z| = a$ so that

$$OB = \frac{a^2}{OA} = \frac{a^2}{c}.$$



In absence of $|z| = a$, the complex potential at any point due to vortex at A is given by

$$\frac{ik}{2\pi}\log(z - c)$$

When the circular cylinder $|z| = a$ is inserted in the fluid, the modified complex potential by Milne-Thomson's circle theorem is given by

$$w = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log\left(\frac{a^2}{z} - c\right)$$

$$= \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log\left[-\frac{c}{z}\left(z - \frac{a^2}{c}\right)\right]$$

$$= \frac{ik}{2\pi}\left[\log(z-c) - \log\left(z - \frac{a^2}{c}\right) + \log z - \log(-c)\right]$$

On adding the constant term $\left(\frac{ik}{2\pi}\right)\log(-c)$ to the above value, the complex potential takes the

form

$$w = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\left(z - \frac{a^2}{c}\right) + \frac{ik}{2\pi}\log z \qquad ....(1)$$

Putting $w = \phi + i\psi$, $z = ae^{i\theta}$ for any point on $|z| = a$ and equating imaginary parts, (1) gives $\psi = 0$. Thus there would be no flow across the boundary $|z| = a$. Hence the motion would remain unchanged if the cylindrical boundary $|z| = a$ were made a rigid barrier. From (1) the required image system follows :-

Remark 1. :

Complex potential w' induced at A, by a vortex - k at B and a vortex k at O is given by

$$w' = w - \frac{ik}{2\pi}\log(z-c) = -\frac{ik}{2\pi}\log\left(z - \frac{a^2}{c}\right) + \frac{ik}{2\pi}\log z$$

$$\therefore \qquad -\frac{dw'}{dz} = -\frac{ik}{2\pi}\frac{1}{z - \frac{a^2}{c}} + \frac{ik}{2\pi}\frac{1}{z} = -\frac{ik}{2\pi}\left[\frac{c}{cz - a^2} - \frac{1}{z}\right]$$

$$\therefore \qquad \left|-\frac{dw'}{dz}\right|_{z=c} = \frac{k}{2\pi}\left|\frac{c}{cz-a^2} - \frac{1}{z}\right|_{z=c} = \frac{k}{2\pi c}\frac{a^2}{c^2 - a^2}$$

which gives the velocity of the vortex A with which it moves round the cylinder.

Remark 2. :

Since the term iklogz denotes the circulation round the cylinder, the result of the above image system may be restated as under :

The image system of a vortex k outside the circular cylinder consists of a vortex of strength - k at the inverse point and a circulation of strength k round the cylinder.

Remark 3. :

Proceeding as above, we can also show that *the image system of a vortex - k outside the circular cylinder consists of a vortex of strength k at the inverse paint and a vortex of strength - k at the centre.*

Image of a vortex inside a circular cylinder :

To show that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

Let there be a vortex pair consisting of two vortices of strength k at $A(z = z_1)$ and - k at $B(z = z_2)$. Then the complex potentiality any point is given by

$$w = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - z_2)$$

or $$\phi + i\psi = \frac{ik}{2\pi}\log(r_1 e^{i\theta_1}) - \frac{ik}{2\pi}\log(r_2 e^{i\theta_2})$$

$$\therefore \quad \psi = \frac{ik}{2\pi}\log\frac{r_1}{r_2}, \text{ where } r_1 = |z - z_1|, r_2 = |z - z_2|.$$

Hence the streamlines are given by ψ = constant, i.e., $\frac{r_1}{r_2} = c$, which represents a family of co-axial circles with A and B as limiting points. Moreover, the motion is unsteady and hence stremlines go on changing and following the vortices which move through the liquid. However, if a particular circle of the family of co-axial circle be repleced by a similar rigid boundary and held fixed, then it follows that the image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

Remark :

Let O be the centre of the cylinder. Let OA = c. Then, if B is the inverse point A, $OB = \frac{a^2}{c}$, where a is the radius of the circular cylinder. The vortex at A will move round the circular cylinder with velocity q given by

$$q = \frac{K}{2\pi AB} = \frac{k}{2\pi(OB - c)} = \frac{k}{2\pi\left(\frac{a^2}{c} - c\right)} = \frac{kc}{2\pi(a^2 - c^2)}$$

Let w be the angular velocity of vortex at A. Then

$$w = \frac{q}{OA} = \frac{q}{c} = \frac{kc}{2\pi(a^2 - c^2)}.$$

Exercise 1 :

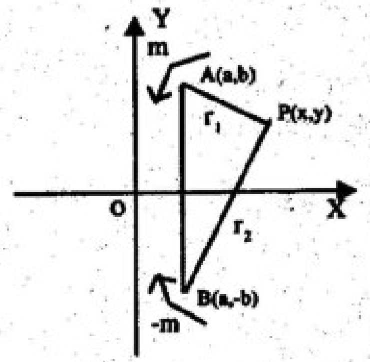
An infinitely long line vortex of strength m, parallel to the axis of z, is situated in an infinite liquid bounded by a rigid wall in the plane y = 0. Prove that, if there be no field of force, the surfaces of equal pressure are given by

$$\{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}=c\{(y^2+b^2)-(x-a)^2\},$$

where (a, b) are the co-ordinates of the vortex, and c is a parametric constant.

Solution :

The image of the vortex of strength m at A(a, b) is a vortex of strength - m at B(a, - b). The two vortices at A and B form a vortex pair with line joining them perpendicular to x-axis and AB = 2b. Hence these vortices move parallel to x-axis with velocity $\frac{m}{4\pi b}$.



The above system of vortices can be brought to rest by superimposing a velocity $-\frac{m}{4\pi b}$.

Hence the components of velocity q at a point P(x, y) are given by

$$u=-\frac{m}{2\pi}\left[\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right]-\frac{m}{4\pi b}$$

$$v=\frac{m}{2\pi}\left[\frac{x-a}{r_1^2}-\frac{x-a}{r_2^2}\right].$$

$\therefore \quad q^2=u^2+v^2$

$$=\frac{m^2}{4\pi^2}\left[\frac{(x-a)^2+(y-b)^2}{r_1^4}+\frac{(x-a)^2+(y+b)^2}{r_2^4}\right.$$

$$\left.-\frac{2\{(x-a)^2+y^2-b^2\}}{r_1^2r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}\right]$$

$$=\frac{m^2}{4\pi^2}\left[\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\{(x-a)^2+y^2-b^2\}}{r_1^2r_2^2}\right.$$

$$\left.+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}\right] \qquad \text{......(1)}$$

Since the system of vortices has been reduced to rest, the motion may be regarded as steady and hence in the absence of external field of force the pressure at any point (by Bernouilli's equation) is given by

$$\frac{p}{\rho}+\frac{1}{2}q^2=\text{constant} \qquad ......(2)$$

Hence the surface of equal pressure are given by p = constant. Using (2), the surfaces of equal pressure are given by q^2 = constant, i.e., by

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}-2\frac{\left\{(x-a)^2+y^2-b^2\right\}}{r_1^2 r_2^2}$$

$$+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)=\text{constant}=\frac{1}{c}, \text{(say)}$$

or

$$c\Big[r_1^2+r_2^2-2(x-a)^2-2(y^2-b^2)$$

$$+\frac{y}{b}(r_2^2-r_1^2)-(r_1^2+r_2^2)\Big]=r_1^2 r_2^2 \qquad ......(3)$$

But $\quad r_1^2=(x-a)^2+(y-b)^2, \qquad r_2^2=(x-a)^2+(y+b)^2$ ......(4)

so that $\quad r_2^2-r_1^2=4yb.$ ......(5)

Using (4) and (5), (3) becomes

$$c[(y^2+b^2)-(x-a)^2]=\{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}.$$

Exercise 2:

A vortex pair is situated within a cylinder. Show that it will remain at rest if the distance of either from the centre is given by $\left(\sqrt{5}-2\right)^{\frac{1}{2}}a$, where a is the radius of the cylinder.

Solution :

Let a vortex pair be situated at A, B where OA = OB = r. Let A′ and B′ be the inverse points of A and B respectively with regard to the circular cylinder so that

$OA'=\frac{a^2}{r}=OB'$. The vortex will remain at rest if its velocity due to other three vortices be zero i.e.,

$$\frac{k}{2\pi}\left[\frac{1}{AA'}-\frac{1}{BA}+\frac{1}{B'A}\right]=0$$

or $$\frac{1}{\frac{a^2}{r}-r}-\frac{1}{2r}+\frac{1}{\frac{a^2}{r}+r}=0$$

or $$\frac{r}{a^2-r^2}+\frac{r}{a^2+r^2}-\frac{1}{2r}=0$$

or $$r^4+4a^2r^2-a^4=0$$

or $$\left(\frac{r^2}{a^2}\right)^2+4\frac{r^2}{a^2}-1=0 \quad \text{or} \quad \frac{r^2}{a^2}=\sqrt{5}-2 \quad \text{or} \quad r=\left(\sqrt{5}-2\right)^{\frac{1}{2}}.$$

Vortex Rows :

When a body maoves slowly through a liquid, rows of vortices are often generated in its wake. When these vortices are stable, then they can be photographed. In the next articles we wish to consider infinite systems of parallel rectilinear vortices in two dimensional flow.

Infinite Number of Parallel Vortices of the Same Strength in one Row :

To show that the motion due to a set of line vortices of strength k at points $z = na(n = 0, 1, 2, 3, ...)$ is given by the relation

$$\omega=\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}.$$

Also to get velocity components and streamlines.

Proof :

Let there be $(2n + 1)$ vortices of strength k each situated at the point $(0, 0)$, $(\pm a, 0)(\pm 2a, 0)$, $(\pm 3a, 0)$, ..., $(\pm na, 0)$. The complex potential of these $(2n + 1)$ vortices at any point z is given by

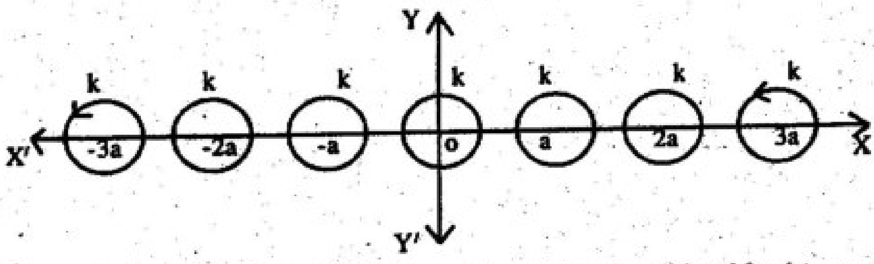
$$w_{2n+1}=\frac{ik}{2\pi}[\log z+\log(z-a)+\log(z+a)+\log(z-2a)$$

$$+\log(z+2a)+...+...+...+\log(z-na)+\log(z+na)]$$

$$=\frac{ik}{2\pi}\log[z(z^2-a^2)(z^2-2^2a^2)(z^2-3^2a^2)\ ...\ (z^2-n^2a^2)]$$

$$=\frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)...\left(1-\frac{z^2}{n^2a^2}\right)\right]$$

$$+\frac{ik}{2\pi}\log\left[(-1)^n\frac{a}{\pi}a^2 2^2a^2...n^2a^2\right] \quad ......(1)$$



The second term on R.H.S. of (1) being constant, it may be neglated for the purpose of complex potential. Hence the complex potential given by (1) may be also written as

$$\omega_{2n+1} = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right] \quad ..(2)$$

Making $n \to \infty$ in (2), the complex potential w of the entire system of vortices at points $z = na(n = 0, 1, 2, 3, \ldots, \infty)$ is given by

$$w = \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\left(1-\frac{z^2}{n^2a^2}\right)\cdots\infty\right] \quad ..(3)$$

But $$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots\infty \quad ......(4)$$

Putting $\theta = \frac{\pi z}{a}$ i.e., $\frac{z}{a} = \frac{\theta}{\pi}$ in (4), we get

$$\sin\left(\frac{\pi z}{a}\right) = \left(\frac{\pi z}{a}\right)\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\infty \quad ......(5)$$

Using (5), (3) becomes $$w = \left(\frac{ik}{2\pi}\right)\log\sin\left(\frac{\pi z}{a}\right) \quad ......(6)$$

Let u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then, we have

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cos\frac{\pi z}{a} \qquad \text{[using (6)]}$$

$$= -\frac{ik}{2\pi}\cot\frac{\pi(x+iy)}{a} = -\frac{ik}{2\pi}\frac{\cos\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}{\sin\frac{\pi}{a}(x+iy)\sin\frac{\pi}{a}(x-iy)}$$

$$= -\frac{ik}{2a}\frac{\sin\left(\frac{2\pi x}{a}\right)-\sin\left(\frac{2\pi iy}{a}\right)}{\cos\left(\frac{2\pi iy}{a}\right)-\sin\left(\frac{2\pi x}{a}\right)}$$

$$= -\frac{ik}{2a}\frac{\sin\left(\frac{2\pi x}{a}\right)-i\sinh\left(\frac{2\pi y}{a}\right)}{\cosh\left(\frac{2\pi y}{a}\right)-\cos\left(\frac{2\pi x}{a}\right)}$$

Equating real and imaginary parts, we have

$$u = -\frac{k}{2a}\frac{\sinh\left(\frac{2\pi y}{a}\right)}{\cosh\left(\frac{2\pi y}{a}\right)-\cos\left(\frac{2\pi x}{a}\right)} \qquad \text{......(7)}$$

$$v = -\frac{ik}{2a}\frac{\sin\left(\frac{2\pi x}{a}\right)}{\cosh\left(\frac{2\pi y}{a}\right)-\cos\left(\frac{2\pi x}{a}\right)} \qquad \text{......(8)}$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity q_0 of vortex at the origin is given by

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}-\frac{ik}{2\pi}\log z\right]\right\}_{z=0}$$

$$= -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a}-\frac{1}{z}\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\lim_{z\to 0}\left[\frac{\pi\cos\left(\frac{\pi z}{a}\right)}{a\sin\left(\frac{\pi z}{a}\right)}-\frac{1}{z}\right] \qquad [\infty - \infty]$$

$$= -\frac{ik}{2\pi a}\lim_{z\to 0}\frac{\pi z\cos\left(\frac{\pi z}{a}\right) - a\sin\left(\frac{\pi z}{a}\right)}{z\sin\left(\frac{\pi z}{a}\right)} \quad \left[\frac{0}{0}\right]$$

[By L Hospital's rule]

$$= \left(-\frac{ik}{2\pi a}\right)\times 0$$ [on evaluating the above indeterminate from us usual]

$$= 0.$$

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. **Thus, we find that the vortex row induces no velocity on itself.**

We now determine streamlines. From (6), we get

$$\phi + i\psi = \frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x+iy)\right\} \qquad \text{......(9)}$$

$$\therefore \quad \phi - i\psi = -\frac{ik}{2\pi}\log\sin\left\{\frac{\pi}{a}(x-iy)\right\} \qquad \text{......(10)}$$

Subtracting (10) from (9), we have

$$2i\psi = \frac{ik}{2\pi}\left[\log\sin\left\{\frac{\pi}{a}(x+iy)\right\} + \log\sin\left\{\frac{\pi}{a}(x-iy)\right\}\right]$$

or $$\psi = \frac{k}{4\pi}\log\left[\sin\left\{\frac{\pi}{a}(x+iy)\right\}\sin\left\{\frac{\pi}{a}(x-iy)\right\}\right]$$

or $$\psi = \frac{k}{4\pi}\log\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right]$$

or $$\psi = \frac{k}{4\pi}\log\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right) \qquad \text{......(11)}$$

on omiting the irrelevent constant. The required streamlines are given by ψ = constant,

i.e., $$\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a} = \text{cons tan t} \qquad \text{......(12)}$$

When y is very large, the second term on L.H.S. of (12) may be omitted. Then the resulting streamlines are given by

$$\cosh\frac{2\pi y}{a} = \text{cons tan t}, \text{ so that } y = \text{constant},$$

showing that at a great diatance from the row of vortices the streamlines are parallel to the row.

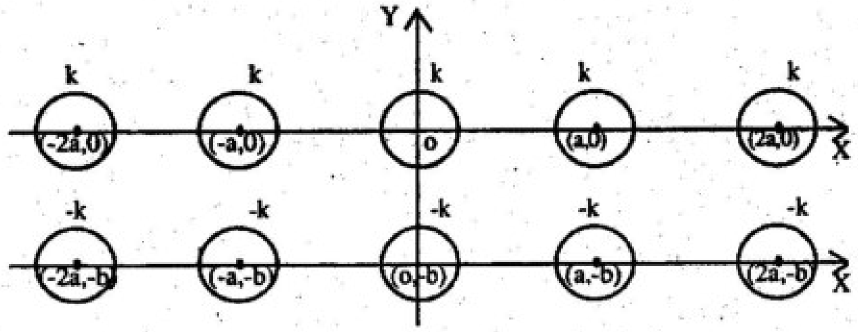
Two Infinite Rows of Parallel Rectilinear Vortices :

Let there be two infinite rows of vortices, one above the other at a distance b, the upper one having vortices each of strength k and lower one each of strength - k, one vortex of the upper row being exactly above each of the lower row. Taking the upper row as x-axis and y-axis passing through the centre of one of the vortices of strength k each are at the points (0, 0), (±a, 0), (±2, 0), ... and those of strength - k each are at the points

(0, -b), (±a, -b), (±2, - b), ...

The complex potential of the entire system (using the result of just concluded discussion) is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib) \qquad \text{......(1)}$$



Let u and v be the velocity components at any point of the fluid not occupied by any vortex filament. Then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2a}\cot\frac{\pi z}{a} + \frac{ik}{2\pi}\cot\frac{\pi}{a}(z+ib) \qquad \text{......(2)}$$

The velocity of the vortex at the origin is given by

$$u_0 - iv_0 = \left\{-\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib)\right]\right\}_{z=0}$$

$$u_0 - iv_0 = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\coth\frac{\pi}{a}(z+ib)\right]_{z=0}$$

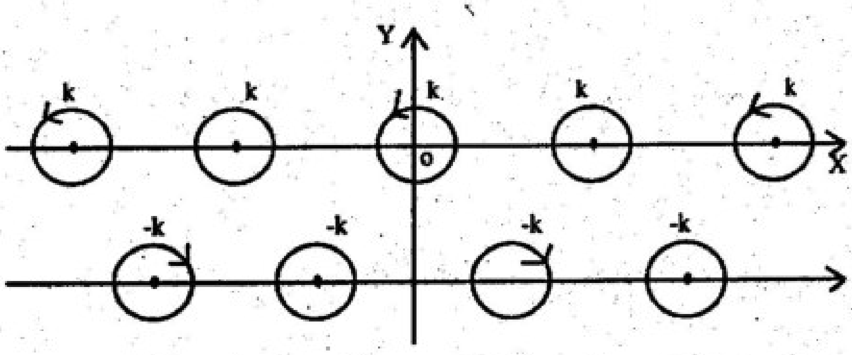
$$= -\frac{ik}{2a}\cot\frac{i\pi b}{a} = \frac{k}{2a}\cot\frac{\pi b}{a}$$

so that $\quad u_0 = \frac{k}{2a}\coth\frac{\pi b}{a}, \qquad v_0 = a,$

showing that the vortex system moves parallel to itself with velocity $\left(\frac{k}{2a}\right)\coth\left(\frac{\pi b}{a}\right)$.

Karman Vortex Street :

Let there be two parallel rows of vortices of equal but opposite strength placed in such a way that each vortex in one row is opposite to the point midway between two vortices of the other row. Accordingly. let vortices of strength k each be situated at the points $(0, 0), (\pm a, 0), (\pm 2a, 0)$



and the vortices of strength - k each be situated at the points

$$\left(\pm\frac{1}{2}a, -b\right), \left(\pm\frac{3}{2}a, -b\right), \ldots$$

The complex potential of the entire system is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right) \qquad \ldots\ldots(1)$$

If u and v be the velocity components at any point of the fluid not occupied by any vortex filamentt, then

$$u - iv = -\frac{dw}{dz} = -\frac{ik}{2\pi}\cot\frac{\pi z}{a} + \frac{ik}{2\pi}\cot\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right) \ldots(2)$$

Since the motion of the vortex at the origin is due to other vortices only, the velocity of vortex at the origin is given by

$$u_0 - iv_0 = -\left[\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z - \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right)\right\}\right]_{z=0}$$

$$= -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z} - \frac{\pi}{a}\coth\frac{\pi}{a}\left(z + \frac{1}{2}a + ib\right)\right]_{z=0}$$

$$= \frac{ik}{2a}\cot\left(\frac{\pi}{2} + \frac{i\pi b}{a}\right)$$

$$= -\frac{ik}{2a}\tan\frac{i\pi b}{a} = \frac{k}{2a}\tan h\frac{\pi b}{a}$$

so that $u_0 = \left(\frac{k}{2a}\right)\tan h\left(\frac{\pi b}{a}\right)$ and $v_0 = 0$, showing that the entire system would move parallel to itself with a uniform velocity $\left(\frac{k}{2a}\right)\tan h\left(\frac{\pi b}{a}\right)$.

Remark :

A Karman vortex street is often realized when a flat plate moves broadside through a liquid.

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Appendix:

Throughout the material we dealt with cartesian, cylindrical and spherical polar coordinate systems which are part of general orthogonal coordinate system. Once we find out a mathematical formula in orthogonal system from its vector form, it is very easy to deduce the corresponding formulas in the cartesian, cylindrical and spherical polar coordinate systems simply by changing the scale factors and the corresponding coordinates and the unit vectors. So, it is advantageous to remember the expressions for some of the formulas in the orthogonal system. Below, we mention some of them. Generally we take (λ, μ, ν) to be the coordinates, $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ the unit unit vectors and (h_1, h_2, h_3) the scale factors in the orthogonal system.

Coordinate system	coordinates	unit vectors	scale factors
cartesian	(x,y,z)	$(\hat{i},\hat{j},\hat{k})$	$h_1 = 1, h_2 = 1, h_3 = 1$
cylindrical	(r,θ,z)	$(\hat{r},\hat{\theta},\hat{k})$	$h_1 = 1, h_2 = r, h_3 = 1$
spherical	(r,θ,ϕ)	$(\hat{r},\hat{\theta},\hat{\phi})$	$h_1 = 1, h_2 = r, h_3 = r\sin\theta$

Expression for $d\bar{r}$ and ds in orthogonal system:

$$d\bar{r} = h_1\hat{e}_1 d\lambda + h_2\hat{e}_2 d\mu + h_3\hat{e}_3 d\nu, \qquad ds^2 = h_1^2(d\lambda)^2 + h_2^2(d\mu)^2 + h_3^2(d\nu)^2$$

Gradient in Orthogonal system:

$$\nabla\phi = \frac{\hat{e}_1}{h_1}\frac{\partial\phi}{\partial\lambda} + \frac{\hat{e}_2}{h_2}\frac{\partial\phi}{\partial\mu} + \frac{\hat{e}_3}{h_3}\frac{\partial\phi}{\partial\nu}$$

Divergence in Orthogonal system:

$$\nabla.\bar{F} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\lambda}(h_2h_3F_1) + \frac{\partial}{\partial\mu}(h_3h_1F_2) + \frac{\partial}{\partial\nu}(h_1h_2F_3)\right\} \quad \text{where } \bar{F} = (F_1, F_2, F_3)$$

Curl in Orthogonal system:

$$\nabla\times F = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial\lambda} & \frac{\partial}{\partial\mu} & \frac{\partial}{\partial\nu} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix} \quad \text{where } \bar{F} = (F_1, F_2, F_3)$$

Laplacian in Orthogonal system:

$$\nabla^2\phi = \nabla.\nabla\phi = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\lambda}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\lambda}\right)+\frac{\partial}{\partial\mu}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial\mu}\right)+\frac{\partial}{\partial\nu}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\nu}\right)\right\}$$

Exercise:

From Kinematics of Fluids in motion:

1. The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2u)=0$$ where u is the velocity at distance r.

2. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(\rho\omega)=0$$ where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

3. If ω is the area of cross-section of a stream filament prove that the equation of continuity is

$$\frac{\partial}{\partial\theta}(\rho\omega)+\frac{\partial}{\partial s}(\rho\omega q)=0,$$ where δs is an element of arc of the filament in the direction of flow and q is the speed.

4. A mass of fluid is in motion so that the lines of motion lie on the surface of co-exial cylinders. Show that the equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho u)+\frac{\partial}{\partial z}(\rho v)=0,$$ where u,v are the velocity perpendicular and parallel to z.

5. Air, obeying Boyle's Law, is in motion in a uniform tube of small cross section, prove that if r be the density and v the velocity at a distance x from a fixed point at time t, then

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$$

6. If the lines of motion are curves on the surface of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is

$$\frac{\partial\rho}{\partial r}+\frac{\partial}{\partial r}(\rho u)+\frac{2\rho u}{r}+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\theta}(\rho\omega)=0,$$

where u and ω are the velocity components in the direction in which r and ϕ increases.

7. If every particle moves on the surface of a sphere prove that the equation of continuity is

$$\frac{\partial\rho}{\partial r}\cos\theta+\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0,$$

ρ being the density, θ, ϕ the latitude and longitude of any element, and ω, ω' the angular velocities of the element in latitude and longitude respectively.

8. Show that $u=-\frac{2xyz}{(x^2+y^2)^2}$, $v=\frac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w=\frac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrotational.

9. Show that a fluid of constant density can have a velocity $\bar{q}$ given by

$$\bar{q}=\left[-\frac{2xyz}{(x^2+y^2)^2},\ \frac{(x^2-y^2)z}{(x^2+y^2)^2},\ \frac{y}{x^2+y^2}\right]$$

Find the vorticity vector.

10. If the velocity of an incompressible fluid at the point (x,y,z) is given by

$$\left(\frac{3xz}{r^5},\frac{3yz}{r^5},\frac{3z^2-r^2}{r^5}\right)$$

prove that the liquid motion is possible and that the velocity potential is $\frac{\cos\theta}{r^2}$. Also determine the stream lines.

11. Show that if the velocity potential of an irrotational fluid motion is equal to

$A(x^2+y^2+z^2)^{\frac{3}{2}}z\tan^{-1}\frac{y}{x}$, the lines of flow will be on the series of surfaces

$$x^2+y^2+z^2=c^{\frac{2}{3}}(x^2+y^2)^{\frac{2}{3}}$$

From Equation of Motion:

1. An infinite mass of homogeneous incompressible fluids is at rest subject to a uniform pressure Π and contains a spherical cavity of radius a, filled with a gas at pressure $m\Pi$; prove that if the inertia of the gas be neglected, and Boyle's law be supposed to to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na, where n is determined by the equation 1+3m log

$n-n^3=0$. If m be nearly equal to 1, the time of an oscillation will be $2\Pi\sqrt{\frac{a^2\rho}{3\Pi}}$, ρ being the density of the fluid.

2. A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at an infinite distance is Π and is such that the work done by this pressure on a unit area through a unit of length is one-half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time filling up the cavity will be

$$\Pi a\left[\left(\frac{\rho}{\Pi}\right)\left\{2-\left(\frac{3}{2}\right)^{\frac{3}{2}}\right\}\right]^{\frac{1}{2}}$$

a being the initial radius of the cavity, and ρ the density of the fluid.

Motions in two dimensions and Sources and Sinks:

1. λ denoting a variable parameter, and f a given function, find the condition that $f(x,y,\lambda)=0$ should be a possible system of stream lines for the steady irrotational motion in two dimensions.

2. In two dimensional motion show that, if the stream lines are confocal ellipses

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \quad \text{then} \quad \psi = A\log\left(\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\right)+B$$

3. Two sources each of strength m are placed at the points (-a,0),(a,0) and a sink of strength 2m at the origin. Show that the stream lines are the curves

$(x^2+y^2)^2=a^2(x^2-y^2+\lambda xy)$, where λ is a variable parameter. Show also that the fluid speed at any point is $\frac{2ma^2}{r_1r_2r_3}$, where r_1, r_2, r_3 are the distances of the points from the sources and sink.

4. In the case of the two dimensional fluid motion produced by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS meet the circle, prove that its magnitude at these points is

$$\frac{2m.OS}{(OS^2-a^2)}.$$

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2009

MATHEMATICS

Paper : 304

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

(New Syllabus)

(**CONTINUUM MECHANICS AND HYDRODYNAMICS**)

GROUP—B

(**Hydrodynamics**)

(*Marks* : 40)

5. Answer any *two* parts : 5×2=10

(a) Show that the variable ellipsoid

$$\frac{x^2}{a^2k^2t^4}+kt^2\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=1$$

is a possible form for the boundary surface of a liquid at any time t.

(b) In the steady motion of homogeneous liquid, if the surfaces $f_1=a_1$ and $f_2=a_2$ define streamlines, prove that the most general values of the velocity components u, v, w are

$$F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)},\ F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)},\ F(f_1, f_2)\frac{\partial(f_1, f_2)}{\partial(x, y)}$$

where F is any arbitrary function.

(c) Prove that if

$$\lambda=\frac{\partial u}{\partial t}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

and μ, ν are two similar expressions, then $\lambda dx+\mu dy+\nu dz$ is a perfect differential, if the forces are conservative and the density is constant.

6. *(a)* A two-dimensional motion of a liquid given by the complex potential w is given by $w = \log\left(z - \frac{a^2}{z}\right)$. Determine the velocity potential and the stream function of the motion. 4

Or

Find the complex potential for a two-dimensional source of strength m placed at the origin.

(b) Answer any *two* parts : 8×2=16

(i) In incompressible motion of two-dimension, prove that

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = q\nabla^2 q$$

where $q = |\vec{q}|$, $\vec{q}$ being the fluid velocity.

(ii) When a rigid boundary is in the form of the circle $(x+a)^2 + (y-4a)^2 = 8a^2$, there is a liquid motion due to a doublet of strength μ at the point $(0, 3a)$, with its axis along the axis of y. Show that the velocity potential is given by

$$\mu\left\{\frac{4(x-3a)}{(x-3a)^2 + y^2} + \frac{y-3a}{x^2 + (y-3a)^2}\right\}$$

(iii) The space between two infinitely long coaxial cylinders of radii a and b respectively is filled with homogeneous liquid of density ρ and the inner cylinder is suddenly moved with velocity U perpendicular to the axis, the outer one being kept fixed. Show that the resultant impulsive pressure on a length l of the inner cylinder is $\pi\rho a^2 l\left(\frac{b^2 + a^2}{b^2 - a^2}\right)U$.

7. Answer any *two* parts : 5×2=10

(a) If the external forces are conservative and the density is a function of pressure only, prove that

$$\frac{d}{dt}\left(\frac{\vec{\Omega}}{\rho}\right)=\left(\frac{\vec{\Omega}}{\rho}\cdot\vec{\nabla}\right)\vec{q}$$

where $\vec{\Omega}=\vec{\nabla}\times\vec{q}$.

(b) Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the streamlines is

$$(u, v, w)=\mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

where μ is a constant and ϕ is a function of x, y, z and t

(c) Show that the motion due to a set of vortices of strength k at the points $z=na\ (n=0, 1, 2, \cdots)$ is given by the relation

$$w=\frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right)$$

(d) Find the vorticity in the spherical polar coordinates (r, θ, ϕ) for the velocity components

$$q_r=\left(1-\frac{A}{r^3}\right)\cos\theta$$

$$q_\theta=-\left(1+\frac{A}{2r^3}\right)\sin\theta$$

$$q_\phi=0$$

2010

MATHEMATICS

Paper : 304

***Full Marks* : 80**

***Time* : 3 hours**

The figures in the margin indicate full marks for the questions

(New Syllabus)

(CONTINUUM MECHANICS AND HYDRODYNAMICS)

GROUP—B

(Hydrodynamics)

(*Marks* : 40)

5. Answer any *two* parts : 5×2=10

(a) A mass of fluid is in motion so that the lines of motion lie on the surface of coaxial cylinders, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u)+\frac{\partial}{\partial z}(\rho v)=0$$

where u, v are the velocity components perpendicular and parallel to z-axis.

(b) Prove that, if

$$\lambda=\frac{\partial u}{\partial t}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

and μ, ν are two similar expressions, then $\lambda dx+\mu dy+\nu dz$ is a perfect differential, if the forces are conservative and the density is constant.

(c) Prove that the equation of motion of a homogeneous inviscid fluid moving under conservative forces may be expressed in the form

$$\frac{\partial \vec{q}}{\partial t}-\vec{q}\times(\vec{\nabla}\times\vec{q})=-\vec{\nabla}\left(\frac{p}{\rho}+\frac{1}{2}q^2+\Omega\right)$$

with usual meanings of the symbols.

6. *(a)* Show that the velocity potential ϕ and the stream function ψ satisfy Laplace's equation. 4

Or,

What arrangement of sources and sinks will give rise to the function $w=\log\left(z-\frac{a^2}{z}\right)$? Also, obtain the stream function for the flow.

(b) Answer any *two* parts : 8×2=16

(i) In two-dimensional irrotational fluid motion, show that if the streamlines are confocal ellipses

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

then $\psi=A\log\left|\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right|+B$ and the velocity at any point is inversely proportional to the square root of the rectangle formed by the focal radii of the point.

(ii) A circular cylinder is moving in an infinite mass of a liquid at rest at infinity with velocity U in the direction of X-axis, show that the complex potential of the fluid motion is given by

$$W = \frac{Ua^2}{Z}$$

a being the radius of the cylinder.

(iii) An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is distributed by the sphere and the pressure in the liquid at a great distance is p_0. Show that the thrust on that half of the sphere on which the liquid impinges is

$$\pi a^2\left(p_0 - \frac{\rho U^2}{16}\right)$$

7. Answer any *two* parts : 5×2=10

(a) In an incompressible fluid, the vorticity at every point is constant in magnitude and direction, show that the components of velocity u, v, w are solutions of Laplace's equation.

(b) An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance $2a$. Show that the paths of the fluid particles relative to the vortices can be represented by the equation

$$\log \frac{r^2 + a^2 - 2ar\cos\theta}{r^2 + a^2 + 2ar\cos\theta} + \frac{r\cos\theta}{a} = \text{constant}$$

O is the middle point of the join which is taken as x-axis.

(c) If (r_1, θ_1), (r_2, θ_2), ⋯ be polar coordinates at time t of a system of rectilinear vortices of strengths $k_1, k_2, \cdots$, prove that

$$\Sigma kr^2 = \text{constant} \quad \text{and} \quad \Sigma kr^2\dot{\theta} = \frac{1}{2\pi}\Sigma k_1 k_2$$