M303

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Gauhati University

M.A./M.Sc. in Mathematics
Semester 3

Paper III
Continuum Mechanics

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Continuum Mechanics

Introduction:

Solid and fluid mechanics are two major subjects studied by all students of applied mathematics, physics and engineering. Traditionally these two subjects are taught separately by two different specialists whose approach, orientation and notation are in general different. The modern trend is to make a unified presentation of the ideas and general principles common to all branches of solid and fluid mechanics under the general heading of continuum Mechanics. A good knowledge of vectors and tensors is essential for a full appreciation of continuum mechanics. Since the Cartesion tensor formulation is sufficient for the development of continuum mechanics at an elementary level, we have limited our discussion of tensors to Cartesion tensors only.

Unit I is concerned with the topic analysis of stress. Deformation and motion of continuum are generally caused by external forces that give rise to interaction between neighbouring portions in the interior parts of a continuum. Such interaction are studied through the concept of stress.Unit II is devoted with the analysis of the geometrical changes that take place in a material body during its motion from one configuration to the other. The tensors which serve to measure these changes will be introduced and the related aspects will be considered in some detail in this unit. Unit III deals with the instantaneous motion of a continuum. The field equations of cotinuum mechanics are also presented in this unit. The last two units are devoted to the development of the governing equations of two basic areas of continuum mechanics: linear elasticity and mechanics of nonviscous and Newtonian viscous fluids. Unit IV deals with a class of continua called linear elastic solid . The classical elasticity theory is a essential part of solid mechanics and its scope is vast.We restrict ourselves to the derivation of the governing equations of the theory and some of its immediate consequences. Some simple and standard applications are also presented. Unit V is concerned with the equations of Fluid Mechanics. Here,also we restrict ourselves to the derivations of the governing equations for nonviscous and Newtonian viscous flows and their immediate consequences. Some simple and standard applications are also discussed.

Unit I.
Analysis of stress.

1.1. The continuum concept

The molecular nature of the structure of matter is well established.In numerous investigations of material behaviour, however, the individual molecule is of no concern and only the behaviour of the material as a whole is deemed important. For these cases the observed macroscopic behaviour is usually explained by disregarding molecular considerations, and, instead, by assuming the material to be continuously distributed throughout its volume and to completely fill the space it occupies. The continuum concept of matter is the fundamental postulate of continuum Mechanics. Within the limitations for which the continuum assumption is valid, the concept provides a framework studying the behaviour of solids, liquids and gases alike.

Adoption of the continuum view point as the basis for the mathematical description of material behaviour means the field quantities such as stress and displacement are expressed an piecewise continuous functions of the space coordinates and time.

1.2. Homogeneity, Isotropy, Mass-density

A homogeneous material is one having identical properties at all points. With respect to some property, a material is isotropic if that property is the same in all directions at a point. A material is called anisotropic with respect to those properties which are directional at a point.

The concept of density is developed from the mass-volume ratio in the neighbourhood of a point in the continuum. In fig 1.1. the mass in the small element of volume ΔV is denoted by ΔM. The average density of the material within ΔV is there fore

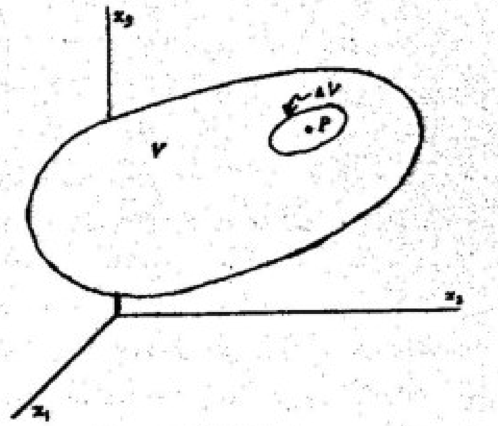

Fig. 1.1

$$\rho_{(av)} = \frac{\Delta M}{\Delta V} \qquad \cdots(1.1)$$

The density at some interior point P of the volume element ΔV is given mathematically in accordance with the continuum concept by the limit,

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V} = \frac{dM}{dV} \qquad \cdots(1.2)$$

Mass-density ρ is a scalar quantity.

1.3. Body and surface forces

Those forces which act on all elements of volume of continuum are known as body forces. Examples are gravity and inertia forces. These forces are represented by the symbol b_i (force per unit mass), or as p_i (force per unit volume). They are related through the density by the equation.

$$\rho b_i = p_i \quad \cdots(1.3)$$

Those forces which act on a surface element whether it is a portion of the bounding surface of the continuum or perhaps an arbitrary internal surface, are known as surface forces. These are denoted by f_i (force per unit area). Contact forces between bodies are types of surface force.

1.4. Cauchy's stress principle, The stress vector

A material continuum occupying the region R of space, and subjected to surface forces f_i and body forces b_i, is shown in fig 1.2.

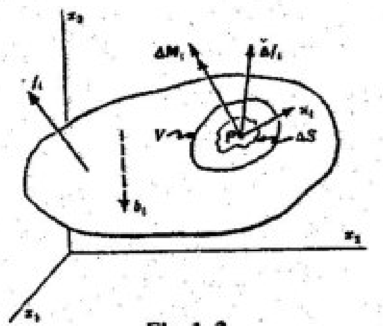

Fig. 1.2

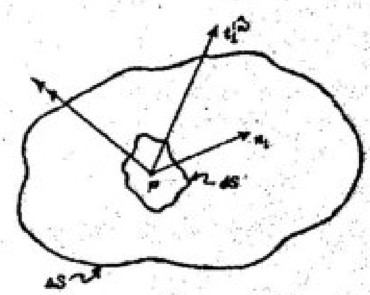

Fig. 1.3

Since the forces are transmitted from one portion of the continuum to another, the material within an arbitrary volume V enclosed by the surface S interacts with the material outside of this volume. Taking n_i as the outward unit normal at point P of a small element of surface ΔS of S, let Δf_i be the resultant force exerted across ΔS upon the material within V by the material outside of V. Clearly, the force element Δf_i will depend upon the choice of ΔS and upon n_i. The distribution of force on ΔS is not necessarily uniform.

The average force per unit area on ΔS is given by $\frac{\Delta f_i}{\Delta S}$. The Cauchy stress principle asserts that this ratio $\frac{\Delta f_i}{\Delta S}$ tends to a definite limit $\frac{df_i}{dS}$ as ΔS approaches zero at the point P while at the same time the moment of Δf_i about the point P vanishes in the limiting process. The resulting vector $\frac{df_i}{dS}$ (force per unit area) is called the stress vector $t_i^{(\hat{n})}$ and is shown in fig 1.3. If the moment at P were not to vanish in the limiting process, a couple - stress vector, shown by the double - headed arrow in fig 1.3 would also be defined at the point. Mathematically, the stress vector is defined by

$$t_i^{(\hat{n})} = \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S} = \frac{df_i}{dS} \quad ......(1.4)$$

The stress vector arising from the action across ΔS at P of the material within V upon the material outside is the vector $-t_i^{(\hat{n})}$. Thus by Newton's law of action and reaction,

$$-t_i^{(\hat{n})} = t_i^{(-\hat{n})} \quad ....(1.5)$$

The stress vector is also called **traction vector.**

1.5. State of stress at a point, stress tensor .

At an arbitrary point P in a continuum, Cauchy's stress principle associates a stress vector $t_i^{(\hat{n})}$ with each unit normal vector n_i, representing the orientation of an infinitesimal surface element having P as an interior point. This is illustrated in fig 1.3. The totality of all possible pairs of such vectors $t_i^{(\hat{n})}$ and n_i at P defines the state of stress at that point. Fortunately it is not necessary to specify every pair of stress and normal vectors to completely describe the state of stress at a given point. This may be accomplished by giving the stress vector on each of three mutually perpendicular planes at P. Coordinate transformation equations then serve to relate the stress vector on any other plane at the point to the given three.

Adopting planes perpendicular to the coordinate axes for the purpose of specifying the state of stress at a point, the appropriate stress and normal vectors are shown in fig 1.4.

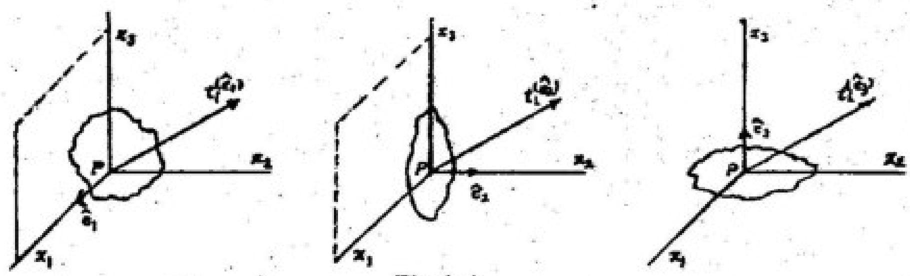

Fig. 1.4

For convenience, the three seperate diagrams in fig 1.4 are often combined into a single schematic representation as shown in fig 1.5

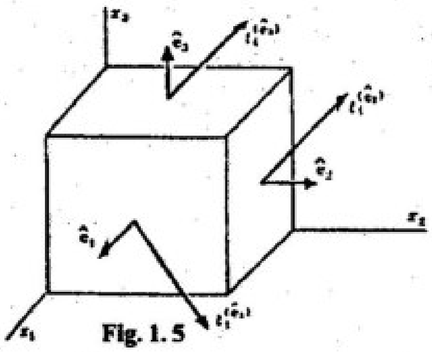

Fig. 1.5

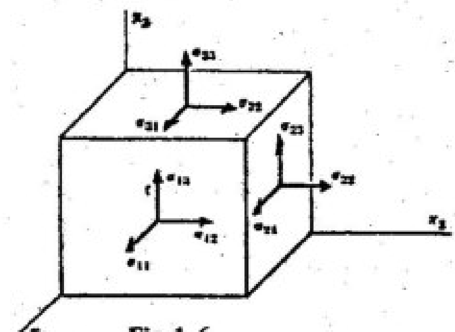

Fig. 1.6

Each of the three coordinate plane stress vectors may be written in terms of its cartesian components as

$$t_j^{(\hat{e}_1)}\hat{e}_j = t_1^{(\hat{e}_1)}\hat{e}_1 + t_2^{(\hat{e}_1)}\hat{e}_2 + t_3^{(\hat{e}_1)}\hat{e}_3$$
$$t_j^{(\hat{e}_2)}\hat{e}_j = t_1^{(\hat{e}_2)}\hat{e}_1 + t_2^{(\hat{e}_2)}\hat{e}_2 + t_3^{(\hat{e}_2)}\hat{e}_3 \qquad \cdots\cdots(1.6)$$
$$t_j^{(\hat{e}_3)}\hat{e}_j = t_1^{(\hat{e}_3)}\hat{e}_1 + t_2^{(\hat{e}_3)}\hat{e}_2 + t_3^{(\hat{e}_3)}\hat{e}_3$$

The nine stress vector components ,

$$t_j^{(\hat{e}_i)} \equiv \sigma_{ij} \quad \ldots \quad (1.7)$$

are the components of a second-order Cartesion tensor known as the sterss tensor . The matrix representation of the stress tensor is given by

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \text{or} \quad [\sigma_{ij}] \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \quad \ldots(1.8)$$

The stress tensor components may be displayed with reference to the coordinate planes as shown in Fig 1.6. The components perpendicular to the planes ($\sigma_{11}, \sigma_{22}, \sigma_{33}$) are called normal stresses. Those acting in (tangent to) the planes($\sigma_{12}, \sigma_{23}, \sigma_{31}, \sigma_{21}, \sigma_{32}, \sigma_{13}$) are called shear stresses. A stress component is positive when it acts in the positive direction of the coordinate exes, and on a plane whose outer normal points in one of the positive coordinate directions. The stress components shown in Fig 1.6 are all positive.

1.6. The stress vector -stress tensor relationship.

The relationship between the stress tensor σ_{ij} at a point P and the stress vector $t_i^{(\hat{n})}$ on a plane of arbitrary orientation at that point may be established through the force equilibrium or momentum balance of a small tetrahedran of the continuum, having its vertex at P. The base of the tetrahedron is taken perpendicular to n_i, and the three faces are taken perpendicular to the coordinate axes as shown on Fig 1.7. Designating the area of the base ABC as ds, the areas of the faces are the projected areas,

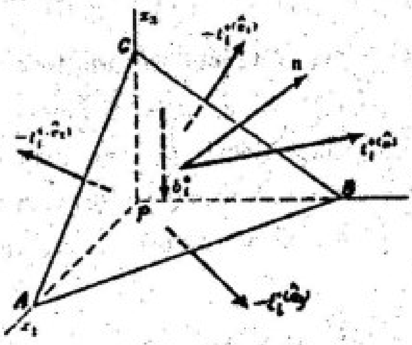

Fig. 1.7

$ds_1 = ds\, n_1$ for face CPB

$ds_2 = ds\, n_2$ for face APC

$ds_3 = ds\, n_3$ for face BPA

or, $ds_i = ds\, n_i$ . .....(1.9)

The average stress vectors $t_i^{*(\hat{e}_j)}$ on the faces and $t_i^{*(\hat{n})}$ on the base, together with the average body forces (including inertia forces, if present), acting on the tetrahedran are shown in the figure. Equilibrium of forces on the tetrahedran requires that

$$t_i^{*(\hat{n})}ds - t_i^{*(\hat{e}_1)}ds_1 - t_i^{*(\hat{e}_2)}ds_2 - t_i^{*(\hat{e}_3)}ds_3 + \rho b_i^* dv = 0 \qquad ...(1.10)$$

If now the linear dimensions of the tetrahedran are reduced in a constant ratio to one another, the body forces, being an higher order in the small dimensions, tend to zero more rapidly than the surface forces. At the same time, the average stress vectors approach the specifie values appropriate to the designated directions at P. Therefore by this limiting process and substitutions at (1.9), equation (1.10) reduces to

$$t_i^{(\hat{n})}ds = t_i^{(\hat{e}_1)}n_1 ds + t_i^{(\hat{e}_2)}n_2 ds + t_i^{(\hat{e}_3)}n_3 ds$$

$$= t_i^{(\hat{e}_j)}n_j ds \qquad ..(1.11)$$

Thus, we have

$$t_i^{(\hat{n})} = \sigma_{ji}n_j \left[\because t_i^{(\hat{e}_j)} \equiv \sigma ji\right] \qquad ..(1.12)$$

In matrix form, this equation can be written explicitly

$$\left[t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}\right] = \left[n_1, n_2, n_3\right] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad .....(1.13)$$

In matrix form (2.13) is equivalent to the component equations

$$t_1^{(\hat{n})} = n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}$$

$$t_2^{(\hat{n})} = n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32}$$
$$t_3^{(\hat{n})} = n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} \qquad ..(1.14)$$

1.7 Force and moment equilibrium, stress tensor symmetry.

Equilibrium of an arbitrary volume V of a continuum, subjected to a system of surface forces $t_i^{(\hat{n})}$ and body forecs b_i (including inertia forces, if present) as shown in fig 1.8.requires that the resultant forces and moment acting on the volume be zero.

Summation of surface and body forces results in the integral relation

$$\int_s t_i^{(\hat{n})}ds + \int_v \rho b_i dv = 0 \qquad \cdots(1.16)$$

or, $$\int_s \sigma_{ji}n_j ds + \int_v \rho b_i dv = 0$$

or, $$\int_v \left(\sigma_{ji,j} + \rho b_i\right)dv = 0 \qquad \cdots(1.17)$$

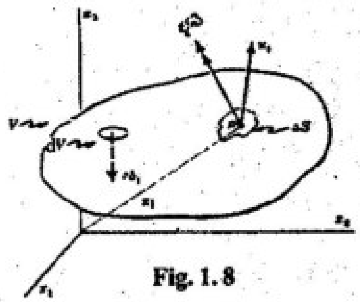

Fig. 1.8

(Replacing $t_i^{(\hat{n})}$ by $\sigma_{ji}n_j$ and then using Gauss divergence theorem)

Since the volume V is arbitrary, the integrand in (1.17) must vanish,so that

$$\sigma_{ji}n_j + pb_i = 0 \qquad ..(1.18)$$

In the absence of distributed moments or couple-stresses, the equilibrium of moments about the origin requires that

$$\int_s e_{ijk} x_j t_k^{(\hat{n})} ds + \int_v e_{ijk} x_j \rho b_k dv = 0 \qquad (1.19)$$

in which x_i is the position vector of the elements of surface and volume. Again, making the substitution $t_i^{(\hat{n})} = \sigma_{ji} n_j$ and applying the theorem of Gauss we have from (1.19)

$$\int_V \left(e_{ijk} x_j \sigma_{pk} \right)_{,p} + \int_V e_{ijk} x_j \rho b_k dv = 0$$

$$\text{or,} \quad \int_v e_{ijk} \left\{ x_{j,p} \sigma_{pk} + x_j (\sigma_{pk,p} + \rho b_k) \right\} dv = 0$$

But from equlibrium equation $\sigma_{pk,p} + \rho b_k = 0$ and since $x_{j,p} = \delta_{jp}$; the above volume integral reduces to

$$\int_v e_{ijk} \sigma_{jk} dv = 0 \qquad (1.20)$$

Since V is arbitrary therefore

$$e_{ijk} \sigma_{jk} = 0 \qquad (1.21)$$

Equation (1.21) represents the equations

$$\sigma_{12} = \sigma_{21}, \sigma_{23} = \sigma_{32}, \sigma_{13} = \sigma_{31} \quad \text{or in all}$$

$$\sigma_{ij} = \sigma_{ji} \qquad \dots (1.22)$$

which shows that the stress tensor is symmetric.

In view of (1.22), the equlibrium equation (1.18) reduces to

$$\sigma_{ij,j} + \rho b_i = 0 \qquad \dots (1.23)$$

which appear in expanded form as

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = 0 \qquad \dots (1.24)$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = 0$$

1.8 Stress quadric of Cauchy

At the point P in a continuum, let the stress tensor have the values σ_{ji} when referred to directions paralled to the local Cartesion axes $p\zeta_1\zeta_2\zeta_3$ shown in fig 1.9. The equation

$$\sigma_{ij} \zeta_i \zeta_j = \pm k^2 \text{ (a constant)} \qquad \dots (1.25)$$

represents geometrically similar quadric surfaces having a common centre at P. The plus or minus choice assures the surfaces are real.

The position vector $\bar{r}$ of an arbitrary point lying on the quadric surface has components $\zeta_i = rn_i$, where n_i is the unit normal in the direction of $\bar{r}$. At the point P the normal component $\sigma_N n_i$ of the stress vector $t_i^{(\hat{n})}$ has a magnitude

$$\sigma_N = t_i^{(\hat{n})} n_i = \sigma_{ij} n_i n_j \qquad ....(1.26)$$

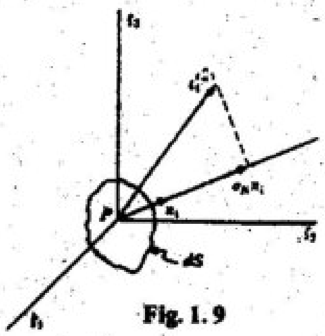

Fig. 1.9

Accordingly if the constant k^2 of (1.25) is set equal to $\sigma_N r^2$, the resulting quadric

$$\sigma_{ij}\zeta_i\zeta_j = \pm\sigma_N r^2 \qquad ......(1.27)$$

is called the stress quadric of Cauchy. From this definition it follows that the magnitude σ_N of the normal stress component on the surface element ds perpendicular to the position vector $\bar{r}$ of a point on Cauchy's stress quadric, is inversely proportional to r^2 i.e. $\sigma_N = \pm k^2/r^2$.

1.9. Principal stresses, stress invariants, stress ellipsoid.

At the point P for which the stress tensor components are σ_{ij}, the equation $t_i^{(\hat{n})} = \sigma_{ij} n_j$, associates with each direction n_i a stress vector $t_i^{(\hat{n})}$. Those directions for which $t_i^{(\hat{n})}$ and n_i are collinear as shown in Fig 1.10 are called principal stress directions. For a principal stress direction

$$t_i^{(\hat{n})} = \sigma n_i \quad ....(1.28)$$

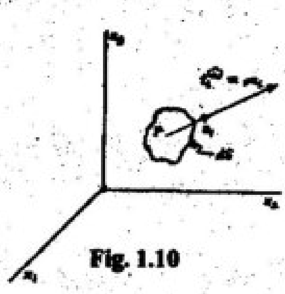

Fig. 1.10

in which σ, the magnitude of the stress vector, is called a principal stress value.

Now, we have $\quad \sigma_{ij} n_j = \sigma n_i$

$$\text{or,} \quad \sigma_{ij} n_j = \sigma\delta_{ij} n_j$$

$$(\sigma_{ij} - \sigma\delta_{ij}) n_j = 0 \qquad ...(1.29)$$

In the three equations in (1.29), there are four unknowns, namely the three direction consines n_i and the principal stress value σ.

For solutions of (1.29) other than the trivial one n_j=0, the determinant of coefficients, $|\sigma_{ij} - \sigma\delta_{ij}|$ must vanish. Explicitly,

$$|\sigma_{ij} - \sigma\delta_{ij}| = 0 \quad \text{or,} \quad \begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0 \quad ...(1.30)$$

which upon expansion yields the cubic polynomial in σ,

$$\sigma^3 - I_\Sigma \sigma^2 + II_\Sigma \sigma - III_\Sigma = 0 \qquad ...(1.31)$$

where $I_\Sigma = \sigma_{ii}$ ...(1.32)

$$II_\Sigma = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right) \qquad ...(1.33)$$

$$III_\Sigma = \left| \sigma_{ij} \right| \qquad ...(1.34)$$

are known respectively as the first, second and third invariants.

The three roots of (1.31), $\sigma_{(1)}, \sigma_{(2)}, \sigma_{(3)}$ are the three principal stress values. Associated with each principal stress $\sigma_{(k)}$, there is a principal stress direction for which the direction cosines $n_i^{(k)}$ are solutions of the equations

$$(\sigma_{ij} - \sigma_{(k)}\delta_{ij})\, n_j^{(k)} = 0 \qquad ..(1.35)$$

The expanded form of (1.35) for the first principal direction, therefore

$$\left(\sigma_{11} - \sigma_{(1)}\right) n_1^{(1)} + \sigma_{12} n_2^{(1)} + \sigma_{13} n_3^{(1)} = 0$$

$$\sigma_{21}\, n_1^{(1)} + (\sigma_{22} - \sigma_{(1)}) n_2^{(1)} + \sigma_{23} n_3^{(1)} = 0 \qquad ...(1.36)$$

$$\sigma_{31}\, n_1^{(n)} + \sigma_{32} n_2^{(1)} + (\sigma_{33} - \sigma_{(1)}) n_3^{(1)} = 0$$

Because the stress tensor is real and symmetric, the principal stress values are real and symmetric.

When referred to principal stress directions, the stress matrix $[\sigma_{ij}]$ is diagonal,

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{(1)} & 0 & 0 \\ 0 & \sigma_{(2)} & 0 \\ 0 & 0 & \sigma_{(3)} \end{bmatrix} \text{or, } [\sigma_{ij}] = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_{II} & 0 \\ 0 & 0 & \sigma_{III} \end{bmatrix} \qquad ...(1.37)$$

in which the principal stresses are ordered i.e $\sigma_I > \sigma_{II} > \sigma_{III}$.

Since the principal stress directions are coincident with the principal axes of Cauchy's stress quadric, the principal stress values include both maximum and minimum normal stress components at a point. In a principal stress space, i.e. a space whose axes are in the principal stress directions and whose coordinate unit of measure is stress $\left(t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}\right)$ as shown in Fig 1.11, the arbitrary stress vector $t_i^{(\hat{n})}$ has components

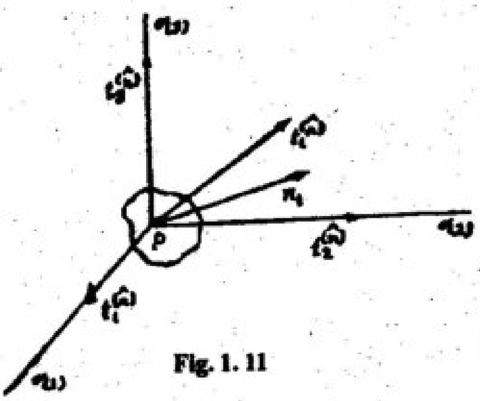

Fig. 1. 11

$$t_1^{(\hat{n})} = \sigma_{(1)} n_1, t_2^{(\hat{n})} = \sigma_{(2)} n_2, t_3^{(\hat{n})} = \sigma_{(3)} n_3 \qquad ...(1.38)$$

according to (1.12). But for the unit vector $n_i,(n_1)^2+(n_2)^2+(n_3)^2=1$, which requires that the stress vector $t_i^{(\hat{n})}$ satisfy the equation

$$\frac{\left(t_1^{(\hat{n})}\right)^2}{\left(\sigma_{(1)}\right)^2}+\frac{\left(t_2^{(\hat{n})}\right)^2}{\left(\sigma_{(2)}\right)^2}+\frac{\left(t_3^{(\hat{n})}\right)^2}{\left(\sigma_{(3)}\right)^2}=1 \quad ....(1.39)$$

in stress space. This equation is an ellipsoid known as the Lame' stress ellipsoid.

1.10. Deviator and spherical stress tensors

If is very often useful to split stress tensor σ_{ij} into two component tensors, one of which (the spherical or hydrostatic stress tensor) has the form

$$\Sigma_M=\sigma_M I=\begin{pmatrix}\sigma_M & 0 & 0\\ 0 & \sigma_M & 0\\ 0 & 0 & \sigma_M\end{pmatrix} \quad ...(1.40)$$

where $\sigma_m=-p=\sigma_{kk}/3$ is the mean normal stress, and the second (the deviator stress tensor) has the form

$$\Sigma_p=\begin{pmatrix}\sigma_{11}-\sigma_M & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma_M & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_M\end{pmatrix}\equiv\begin{pmatrix}s_{11} & s_{12} & s_{13}\\ s_{21} & s_{22} & s_{23}\\ s_{31} & s_{32} & s_{33}\end{pmatrix} \quad ...(1.41)$$

The decomposition is expressed by the equations

$$\sigma_{ij}=\delta_{ij}\,\sigma_{RR}/3+S_{ij} \quad ...(1.42)$$

The principal directions of the deviator stress tensor S_{ij} are the same as those of the stress tensor σ_{ij}. Thus principal deviator stress values are

$$S_{(k)}=\sigma_{(k)}-\sigma_M \quad ...(1.43)$$

The characteristic equation for the deviator stress tensor is the cubic

$$S^3+II_{\Sigma_D}S-III_D=0$$

$$\text{or,}\quad S^3+\left(S_I S_{II}+S_{II}S_{III}+S_{III}S_I\right)S-S_I S_{II} S_{III}=0 \quad ...(1.44)$$

It is easily shown that the first invariant of the deviator stress tensor is identically zero, which accounts for the absence in (1.44).

Solved Problems

Exp 1 The stress tensor values at a point P are given by the array

$$\Sigma = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Determine the stress (traction) vector an the plane at P whose unit normal is

$$\hat{n} = \left(\frac{2}{3}\right)\hat{e}_1 - \left(\frac{2}{3}\right)\hat{e}_2 + \left(\frac{1}{3}\right)\hat{e}_3.$$

Determine also (a) the component perpendicular to the plane, (b) the magnitude of $t_i^{(\hat{n})}$, (c) the angle between $t_i^{(\hat{n})}$ and $\hat{n}$.

Solution We have the relations

$$t_i^{(\hat{n})} = \sigma_{ij} n_j$$

Here,

$$\left[t_1^{(\hat{n})}, t_2^{(\hat{n})}, t_3^{(\hat{n})}\right] = \left[\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right] \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix} = \left[4, \ \frac{-10}{3}, \ 0\right]$$

Thus $\quad \vec{t}_i^{(n)} = 4\hat{e}_1 - \dfrac{10}{3}\hat{e}_2.$

(a) The component perpendicular to the plane is given by

$$t_i^{(\hat{n})}.\hat{n} = \left[4\hat{e}_1 - \frac{10}{3}\hat{e}_2\right]\left[\frac{2}{3}\hat{e}_1, -\frac{2}{3}\hat{e}_2, \frac{1}{3}\hat{e}_3\right] = \frac{44}{9}$$

(b) The magnitude of $t_i^{(\hat{n})}$ is given by

$$\left|t_i^{(\hat{n})}\right| = \sqrt{16 + \frac{100}{9}} = 5.2$$

(c) The angle between $t_i^{(\hat{n})}$ and $\hat{n}$ is given by

$$t_i^{(\hat{n})}.\hat{n} = \left|t_i^{(\hat{n})}\right|\left|\hat{n}\right|\cos\theta \text{ i.e. } \frac{44}{9} = 5.2\cos\theta$$

i.e. $\cos\theta = \left(\frac{44}{9}\right)\Big/5.2 = 0.94$ i.e. $\theta = 20^\circ$

Exp 2 Show that the Cauchy stress quadric for a state of stress represented by

$$\Sigma = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

is an ellipsoid (the stress ellipsoid) when a,b,c are all of the same sign.

Solution. The equations of the Cauchy stress quadric is given by

$$\sigma_{ij}\zeta_i\zeta_j = \pm k^2$$

or, in matrix form

$$[\zeta_1, \zeta_2, \zeta_3]\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \pm k^2$$

$$\text{i.e. } [a\zeta_1, b\zeta_2, c\zeta_3]\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \pm k^2$$

$$\text{i.e. } a\zeta_1^2 + b\zeta_2^2 + c\zeta_3^2 = \pm k^2$$

i.e. $\frac{\zeta_1^2}{bc} + \frac{\zeta_2^2}{ac} + \frac{\zeta_3^2}{ba} = \pm\frac{k^2}{abc}$ is the required quadric.

Exp 3 The stress tensor at a point P is given with respect to the axes $Ox_1x_2x_3$ by the values

$$\sigma_{ij} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

Determine the principal stress values and the principal directions represented by the axes $Ox_1^*x_2^*x_3^*$.

Solution The principal stress values are given by

$$|\sigma_{ij} - \sigma\delta_{ij}| = 0$$

or, in matrix form
$$\begin{vmatrix} 3-\sigma & 1 & 1 \\ 1 & 0-\sigma & 2 \\ 1 & 2 & 0-\sigma \end{vmatrix} = 0$$

or, upon expansion

$$(\sigma+2)(\sigma-4)(\sigma-1) = 0$$

$$\text{i.e.} \quad \sigma = -2, \sigma = 4, \sigma = 1$$

∴ The principal stress values are

$$\sigma_{(1)} = 4, \qquad \sigma_{(2)} = 1, \qquad \sigma_{(3)} = -2$$

The principal directions are given by $\left(\sigma_{ij} - \sigma_{(k)}\delta_{ij}\right) n_i^{(k)} = 0$

Let the x_1^* axis be the direction of $\sigma_{(1)}$ and let $n_i^{(1)}$ be the direction cosines of this axis then we have

$$(3-4)n_1^{(1)} + n_2^{(1)} + n_3^{(1)} = 0$$

$$n_1^{(1)} - 4n_2^{(1)} + 2n_3^{(1)} = 0$$
$$n_1^{(1)} + 2n_2^{(1)} - 4n_3^{(1)} = 0$$

So that $n_1^{(1)} = 2n_2^{(1)} = 2n_3^{(1)}$. Since $n_i n_i = 1$, therefore

$$n_1^{(1)} = \frac{2}{\sqrt{6}}, \qquad n_2^{(1)} = \frac{1}{\sqrt{6}}, \qquad n_3^{(1)} = \frac{1}{\sqrt{6}}$$

Likewise let x_2^* be assoiciated with $\sigma_{(2)}$. Then we have

$$2n_1^{(2)} + n_2^{(2)} + n_3^{(2)} = 0$$
$$n_1^{(2)} - n_2^{(2)} + 2n_3^{(2)} = 0$$
$$n_1^{(2)} + 2n_2^{(2)} - n_3^{(2)} = 0$$
$$\therefore \text{Solving } n_1^{(2)} = -n_2^{(2)} = -n_3^{(2)}$$

and since $n_i n_i = 1$ therefore $n_1^{(2)} = \frac{1}{\sqrt{3}}, \qquad n_2^{(2)} = -\frac{1}{\sqrt{3}} = n_3^{(2)}$

Finally, let x_3^* be associated with $\sigma_{(3)}$. Thus we have

$$\begin{aligned}5n_1^{(3)} + n_2^{(3)} + n_3^{(3)} &= 0\\ n_1^{(3)} + 2n_2^{(3)} + 2n_3^{(3)} &= 0\\ n_1^{(3)} + 2n_2^{(3)} + 2n_3^{(3)} &= 0\end{aligned}$$

so that $n_1^{(3)} = 0, n_2^{(3)} = \frac{1}{\sqrt{2}}, n_3^{(3)} = -\frac{1}{\sqrt{2}}$

Exp 4. The state of stress throughout a Continuum is given w.r.t the cartesion axes $ox_1x_2x_3$ by the array

$$\Sigma = \begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}$$

Determine the body force components when the equilibrium equations are to be satisfied every where.

Solutions The equilibrium equations are

$$\sigma_{ij,j}+\rho b_i=0$$

i.e.

$$\begin{aligned}\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho b_1 &= 0\\ \frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho b_2 &= 0\\ \frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho b_3 &= 0\end{aligned}$$

i.e.

$$3x_2 + 10x_2 + \rho b_1 = 0$$

$$2+\rho b_2=0$$

$$\rho b_3=0$$

Thus the equilibrium equations are satisfied only when

$$b_1 = -\frac{1}{\rho}13x_2,\ b_2 = -\frac{2}{\rho},\ b_3 = 0$$

∴ The reqd. body force components are

$$-\frac{1}{\rho}13x_2, -\frac{2}{\rho}, 0.$$

Exp 5. Split the stress tensor $\sigma_{ij} = \begin{pmatrix} 12 & 4 & 0 \\ 4 & 9 & -2 \\ 0 & -2 & 3 \end{pmatrix}$

into its spherical and deviator parts and show that the first invariant of the deviator is zero.

Solution: We know that

$$\sigma_{ij} = \sigma_M \delta_{ij} + s_{ij}$$

where σ_M is the mean normal stress,

$\sigma_M \delta_{ij}$ is the spherical stress tensor

and s_{ij} is the deviator stress tensor.

Here $\sigma_M = \sigma_{kk}/3$

$$= \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$
$$= 8$$

and $$(\sigma_{ij}) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 12-8 & 4 & 0 \\ 4 & 9-8 & -2 \\ 0 & -2 & 3-8 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 0 \\ 4 & 1 & -2 \\ 0 & -2 & -5 \end{pmatrix}$$

And first invariant of the deviator stress tensor

$$S_{ii} = S_{11} + S_{22} + S_{33} = 4 + 1 - 5 = 0$$

Exp 6 Evaluate the stress invariants for the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Solution The stress invariants are given by

$$I_\Sigma = \sigma_{ii} = \sigma_{11} + \sigma_{22} + \sigma_{33} = 20$$

$$II_\Sigma = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}\sigma_{12} - \sigma_{23}\sigma_{23} - \sigma_{32}\sigma_{32}$$

$$= 36 + 48 + 48 + -9 = 123$$

$$III_\Sigma = |\sigma_{ij}| = 216.$$

Example 7

The stress matrix of a point p in a material is given by

$$(\delta_{ij}) = \begin{pmatrix} x_3x_1 & x_3^2 & 0 \\ x_3^2 & 0 & -x_2 \\ 0 & -x_2 & 0 \end{pmatrix}$$

Find the stress vector of the point Q(1, 0, –1) on the surface $x_2^2 + x_3^2 = sy$.

Solve : Here the stress vector is required in the surface $(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2 = 0$. Hence the unit normal is defined by

$$\hat{n}^\tau = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \frac{\partial\phi}{|\partial x_i|}\hat{e}_i^\tau$$

$$= \frac{\partial\phi}{\partial x_1}\hat{e}_1^\tau + \frac{\partial\phi}{\partial x_2}\hat{e}_2^\tau + \frac{\partial\phi}{\partial x_3}\hat{e}_3^\tau$$

$$= \hat{e}_1^\tau - 2x_2\,\hat{e}_2^\tau - 2x_3\,\hat{e}_3^\tau$$

At Q(1, 0, –1) we get

$$\nabla\phi = \hat{e}_1^\tau + 2\hat{e}_3^\tau \qquad \text{and} \qquad |\nabla\phi| = \sqrt{5}$$

$$\therefore \hat{n}^\tau = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{5}}\left(\hat{e}_1^\tau + 2\hat{e}_3^\tau\right)$$

Therefore from the relation

$$\hat{t}_i^{(n^\tau)} = \sigma_{ij}\, n_j$$

$$\left(\hat{t}_1^{(n^\tau)}, \hat{t}_2^{(n^\tau)}, \hat{t}_3^{(n^\tau)}\right) = \frac{1}{\sqrt{5}}(1,-1,0)$$

Thus $\hat{t}^{(n^\tau)} = \frac{1}{\sqrt{5}}\left(\hat{e}_1^\tau - \hat{e}_2^\tau\right)$

Example 8

Determine the principal stress values and principal direction for the stress sensor

$$\sigma_{ij} = \begin{pmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{pmatrix}$$

Solve : The principal stress values are obtained from

$$\left|\sigma_{ij} - \sigma\,\delta_{ij}\right| = 0$$

$$\text{i.e.} \begin{vmatrix} \tau-\delta & \tau & \tau \\ \tau & \tau-\delta & \tau \\ \tau & \tau & \tau-\delta \end{vmatrix} = 0$$

$$\text{i.e.}\ \sigma^2 - 4\sigma\tau + 7^2 = 0$$

$$\therefore \sigma = 0,\quad 0,\quad 3\tau$$

Let $\sigma_{(1)} = 3\tau \quad \sigma_{(2)} = \sigma_{(3)} = 0$

The principal stress direction are obtained from the relation

$$\left(\sigma_{ij} - \sigma_{(k)}\delta_{ij}\right) n_j^{(k)} = 0$$

where $n_i^{(1)}\, n_i^{(1)} = 1,\ k = 1,2,3$

When $\delta = \delta_{(1)} = 3\tau$ then the principal direction are obtained from the equation

$$\left(\sigma_{ij} - 3\tau\,\delta_{ij}\right) n_j^{(1)} = 0$$

$$\text{i.e.}\ -2\tau\, n_1^{(1)} + \tau n_2^{(1)} + \tau n_3^{(1)} = 0$$

$$\tau\, n_1^{(1)} - 2\tau n_2^{(1)} + \tau n_3^{(1)} = 0$$

$$\tau\, n_1^{(1)} + \tau n_2^{(1)} - 2\tau n_3^{(1)} = 0$$

Solving we get

$$n_1^{(1)} = n_2^{(1)} = n_3^{(1)}$$

With the help of the identify we have

$$n_1^{(1)} = n_2^{(1)} = n_3^{(1)} = \frac{1}{\sqrt{3}}$$

When $\sigma = \sigma_{(2)} = 0$ then the principal direction are obtained from the equation

$$\left(\sigma_{ij} - \delta_{(k)}\, \delta_{ij}\right) n_j^{(k)} = 0 \text{ where } k = 2, 3$$

i.e. $-n_1^{(k)} + n_2^{(k)} + \tau n_3^{(k)} = 0$

$$\tau n_1^{(k)} + \tau n_2^{(k)} + n_3^{(k)} = 0$$

$$\tau n_1^{(k)} + \tau n_2^{(k)} + \tau n_3^{(k)} = 0$$

i.e.

$$n_1^{(k)} + n_2^{(k)} + n_3^{(k)} = 0$$

$$n_1^{(k)} + n_2^{(k)} + n_3^{(k)} = 0$$

$$n_1^{(k)} + n_2^{(k)} + n_3^{(k)} = 0,$$

where k = 2, 3 which together with the identify $n_i^{(k)} n_i^{(k)} = 1$, are in sufficient to determine the 2nd and 3rd principal direction.

Thus one principal direction is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ serve as other two principal axes.

Example 9

The state of stress throughout a continuum is given with respect to the cartesian axes $0X_1X_2X_3$ by the away

$$\Sigma = \begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}$$

Determine the stress vector at his point $P\left(2, 1, \sqrt{3}\right)$ of the plane that is tangent to the cylindrical surface

$$x_2^2 + x_3^2 = 4 \text{ at p.}$$

See ans. to Example 4.

Example 10

The stress tensor at a point is given as

$$\sigma_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & \delta_{22} & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

With δ_{22} unspecified. Determine δ_{22} so that the stress vector on some plane at the point will be zero. Give the unit normal for this traction free plane.

Solution

We have the relation

$$t_i^{(n^\tau)} = \sigma_{ij} n_j$$

Here $t_i^{(n^\tau)} = \sigma_{ij} n_j = 0$ (given)

$$\text{i.e. } \begin{pmatrix} 0 & 1 & 2 \\ 1 & \delta_{22} & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the components equation are given by

$$0 + n_2 + 2n_3 = 0$$

$$n_1 + \delta_{22} n_2 + n_3 = 0$$

$$2n_1 + n_2 + 0 = 0$$

$$\text{i.e. } n_2 + 2n_3 = 0$$

$$n_1 + \delta_{22} . n_2 + n_3 = 0$$

$$2n_1 + n_2 = 0$$

Solving we get

$$\sigma_{22} = 1$$

$$\text{and } n_1 = n_3,\ n_2 = -2n_1$$

Again we have the identify

$$n_i n_i = 1 \text{ i.e. } n_1^2 + n_2^2 + n_3^2 = 1$$

$$\text{i.e. } n_1 = \frac{1}{\sqrt{6}}$$

$$\text{Thus } n_1 = n_3 = \frac{1}{\sqrt{6}} \text{ and } n_2 = -\frac{2}{\sqrt{6}}$$

$$\text{Thus } n^\tau = \frac{1}{\sqrt{6}} n_1^\tau - \frac{2}{\sqrt{6}} 2^\tau + \frac{1}{\sqrt{6}} 3^\tau = \frac{1}{\sqrt{6}} \left(\hat{e}_1^\tau - 2\hat{e}_2^\tau + \hat{e}_3^\tau \right)$$

Example 11

The state of stress of a point is given by the stress sensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

Where a, b, c are constants and σ is some stress value. Determine the constants a, b, c so that the stress vector on the octahedral plane vanishes.

Solution :

(Def^n $\div$ Octahadral plane is that plane whose normal makes equal angles with positive direction with co-ordinator axes. For this plane $n_1 = n_2 = n_3 = \frac{1}{\sqrt{3}}$)

We have the relation

$$t_i^{(n^\tau)} = \sigma_{ij} n_j$$

$$\text{Here } t_i^{(n^\tau)} = \sigma_{ij} n_j = 0$$

$$\text{and } n^\tau = \frac{1}{\sqrt{3}}\left(\hat{e}_1^\tau + \hat{e}_2^\tau + \hat{e}_3^\tau\right)$$

$$\begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The components equations are

$$\frac{1}{\sqrt{3}}(\sigma + a\sigma + b\sigma) = 0$$

$$\frac{1}{\sqrt{3}}(a\sigma + \sigma + c\sigma) = 0$$

$$\frac{1}{\sqrt{3}}(b\sigma + c\sigma + \sigma) = 0$$

i.e. $1 + a + b = 0$

$a + 1 + c = 0$

$b + c + 1 = 0$

Solving there three equations whenever

$$a = b = c = -\frac{1}{2}$$

Example 12 :

Show that $\delta_{ij}\delta_{ik}\delta_{kj}$ is an invariant of the stress tensor.

Solution :

By using the transformation law where are,

$$\delta'_{ij}\,\delta'_{ik}\,\delta'_{kj} = a_{ip}a_{jq}\delta_{pq}a_{ir}a_{ks}\delta_{rs}a_{km}a_{jn}\delta_{mn}$$

$$= \left(a_{ip}a_{ir}\right)\left(a_{jq}a_{jn}\right)\left(a_{ks}a_{km}\right)\delta_{pq}\delta_{rs}\delta_{mn}$$

$$= \delta_{pr}\delta_{qn}\delta_{sm}\delta_{pq}\delta_{rs}\delta_{mn}$$

$$= \left(\delta_{pr}\delta_{pq}\right)\left(\delta_{qn}\delta_{mn}\right)\left(\delta_{sm}\delta_{rs}\right)$$

$$= \delta_{rq}\delta_{qm}\delta_{rm} = \delta_{ij}\delta_{ik}\delta_{kj}$$

Supplementary Problems

1. The stress tensor at a point is given as

$$\sigma_{ij} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & \sigma_{22} & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

with σ_{22} unspecified. Determine σ_{22} so that the stress vector on some plane at the point will be zero. Give the unit normal for this traction- free plane.

Ans. $\sigma_{22} = 1, \hat{n} = \hat{e}_1 - 2\hat{e}_2 + \hat{e}_3/\sqrt{6}$

2. Determine the principal stress values for

$$\text{(i)}\ \sigma_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and (ii) } \sigma_{ij} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

and show that both have the same principal directions.

Ans (i) $\sigma_I = 2, \sigma_{II} = \sigma_{III} = 1$ (ii) $\sigma_I = 4, \sigma_{II} = \sigma_{III} = 1$

3. The stress vectors acting on the three coordinate planes are given by $t_i^{(\hat{e}_1)}$, $t_i^{(\hat{e}_2)}$, $t_i^{(\hat{e}_3)}$. Show that the sum of the squares of the magnitudes of these vectors is independent of the orientation of the coordinate planes.

4. The state of stress at a point is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

where a,b,c are constants and σ is some stress value. Determine the constants a,b,c so that the stress vector on the octahedral plane vanishes.

Ans $a = b = c = -\frac{1}{2}$

5. Determine the Cauchy stress quadric at P for the following states of stress:

(i) uniform tension $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$

(ii) uniaxial tension $\sigma_{11} = \sigma$, $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$

(iii) Simple shear $\sigma_{12} = \sigma_{21} = \tau$, $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$

(iv) plane stress with $\sigma_{11} = \sigma_{22} = \sigma$; $\sigma_{12} = \sigma_{21} = \tau$, $\sigma_{33} = \sigma_{31} = \sigma_{23} = 0$

Ans (i) $\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \pm k^2 / \sigma$

(ii) $\zeta_1^2 = \pm k^2 / \sigma$

(iii) $2\tau\zeta_1\zeta_2 = \pm k^2$

(iv) $\sigma\zeta_1^2 + 2\tau\zeta_1\zeta_2 + \sigma\zeta_2^2 = \pm k^2$

6. Determine the principal stress values and principal directions for the stress tenson

$$\sigma_{ij} = \begin{pmatrix} \tau & \tau & \tau \\ \tau & \tau & \tau \\ \tau & \tau & \tau \end{pmatrix}$$

Ans : $\sigma_{(1)} = \sigma_{(2)} = 0$, $\sigma_{(3)} = 3\tau$

7. Prove that $\sigma_{ij}\ \sigma_{ik}\ \sigma_{kj}$ is an invariant of the stress tensor.

8. Show that the normal component of the stress vector on the octahedral plane is equal to one third the first invariant of the stress tensor.

9. The state of stress throughout a continuum is given with respect to the Cartesion axes $Ox_1x_2x_3$ by the array

$$\Sigma = \begin{pmatrix} 3x_1x_2 & 5x_2^2 & 0 \\ 5x_2^2 & 0 & 2x_3 \\ 0 & 2x_3 & 0 \end{pmatrix}.$$

Determine the stress vector at the point $P(2, 1, \sqrt{3})$ of the plane that is tangent to the cylindrical surface

$x_2^2 + x_3^2 = 4$ at P.

Ans $\vec{t}^{(\hat{n})} = \frac{5}{2}\hat{e}_1 + 3\hat{e}_2 + \sqrt{3}\hat{e}_3$

10. Determine the principal deviator stress values for the stress tensor

$$\sigma_{ij} = \begin{pmatrix} 10 & -6 & 0 \\ -6 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ans $s_I = 9$, $s_{II} = -3$, $s_{III} = -6$.

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Unit II
Analysis of Strain

2.1 Lagrangian and Eulerian descriptions.

When a continuum undergoes deformation (or flow), the particles of the continuum move along various paths in space. This motion may be expressed by equations of the form

$$x_i = x_i(X_1, X_2, X_3, t) \qquad ...(2.1)$$

Which give the present location x_i of the particle that occupied the point (x_1, x_2, x_3) at time $t = 0$. Also, (2.1) may be interpreted as a mapping of the initial configuration into the current configuration. It is assumed that such a mapping is one-to-one and continuous, with continuous partial derivatives to whatever order is required. The description of motion or deformation expressed by (2.1) is known as the Lagrangian description.

If on the other hand, the motion or deformation is given through equations of form

$$X_i = X_i(x_1, x_2, x_3, t) \qquad ...(2.2)$$

in which the independent variables are the coordinates x_i and t, the description is known as the Eluerian description. This description may be viewed as one which provides a tracing to its original position of the particle that now occupies the location (x_1, x_2, x_3). If (2.2) is a continuous one-to-one mapping with continuous partial derviatives, then the two mappings are the unique inverses of one another. A necssary and sufficient condition for the inverse functions to exit is that the Jacobian

$$J = \left|\frac{\partial x_i}{\partial X_j}\right| \qquad ...(2.3)$$

should not vanish.

2.2. Deformation and Displacement gradients

Partial defferentiation of (2.1) with recpect to X_j produces the tensor $\frac{\partial x_i}{\partial X_j}$ which is called the **material deformation gradient** and is represented in matrix form as

$$\left[\frac{\partial x_i}{\partial X_j}\right] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \qquad ...(2.4)$$

Partial differentiation of (2.2) with respect to x_j produces the tensor $\left[\frac{\partial X_i}{\partial x_j}\right]$ which is called the spatial deformation gradient and is presented in matrix form as

$$\left[\frac{\partial X_i}{\partial x_j}\right]=\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}\left[\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}\right]=\begin{bmatrix}\frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3}\\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3}\\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3}\end{bmatrix} \quad ...(2.5)$$

The material and spatial deformation tensors are interrelated through the well known chain rule for partial differentiation

$$\frac{\partial x_i}{\partial X_j}\frac{\partial X_j}{\partial x_k}=\frac{\partial X_i}{\partial x_j}\frac{\partial x_j}{\partial X_k}=\delta_{ik} \quad ....(2.6)$$

Partial differentiation of the displacement vector u_i with respect to the coordinates produces either the material displacement gradient $\frac{\partial u_i}{\partial X_j}$ or the spatial displacement gradient $\frac{\partial u_i}{\partial x_j}$. We know that the displacement can be given by

$$u_i = x_i - X_i \quad ...(2.7)$$

Thus the material displacement gradient is given by

$$\frac{\partial u_i}{\partial X_j}=\frac{\partial x_i}{\partial X_j}-\delta_{ij} \quad ...(2.8)$$

and the spatial displacement gradient is given by

$$\frac{\partial u_i}{\partial x_j}=\delta_{ij}-\frac{\partial X_i}{\partial x_j} \quad ....(2.9)$$

In matrix forms they are expressed as

$$\left[\frac{\partial u_i}{\partial X_j}\right]=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\left[\frac{\partial}{\partial X_1},\ \frac{\partial}{\partial X_2},\ \frac{\partial}{\partial X_3}\right]=\begin{bmatrix}\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3}\\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3}\\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3}\end{bmatrix} \quad ...(2.10)$$

and $$\left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\left[\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} \quad ...(2.11)$$

2.3. Deformation tensors, Finite strain tensors.

In Fig. 2.1 the initial (underformed) and final (deformed) configurations of a continuum are reffered to the superimposed rectangular Cartesian coordinate axes $OX_1X_2X_3$ and $ox_1x_2x_3$. The neighbouring particles which occupy points P_o and Q_o before deformation, move to points P and Q respectively in the deformed configuration.

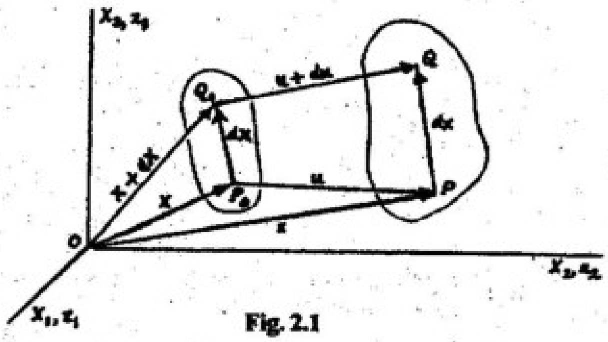


Fig. 2.1

The square of the differential element of length between P_o and Q_o is

$$(dX)^2 = dX_i dX_i = \delta_{ij} dX_i dX_j \quad ...(2.12)$$

Again, the distance element dX_i is seen to be

$$dX_i = \frac{\partial X_i}{\partial x_j} dx_j \quad ...(2.13)$$

so that the squared length $(dX)^2$ in (2.12) may be written as

$$(dX)^2 = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j} dx_i dx_j = C_{ij} dx_i dx_j \quad \cdots\cdots(2.14)$$

in which the second-order tensor

$$C_{ij} = \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}$$

is known as **Cauchy's deformation tensor**.

In the deformed configuration, the square of the differential element of length between P and Q is

$$(dx)^2 = dx_i dx_i = \delta_{ij} dx_i dx_j \quad ...(2.15)$$

Again, $$dx_i = \frac{\partial x_i}{\partial X_j} dX_j$$

so that the squared length $(dx)^2$ in (2.15) may be written as

$$(dx)^2 = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} dX_i dX_j = G_{ij} dX_i dX_j \qquad ...(2.16)$$

in which the second-order tensor

$$G_{ij} = \frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j}$$

is known as Green's deformation tensor.

The difference $(dx)^2$-$(dX)^2$ for two neighbouring particles of a continuum is used as the measure of deformation that occurs in the neighbourhood of the particles between the initial and final configurations. If this difference is identically zero for all neighbouring particles of a continuum, a rigid displacement is said to occur. Using (2.16) and (2.12) this difference may be expressed in the form

$$(dx)^2 - (dX)^2 = \left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) dX_i dX_j = 2L_{ij} dX_i dX_j \qquad ...(2.17)$$

in which the second-order tensor

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right) \qquad ....(2.18)$$

in called the Lagrangian (or Green's) finite strain tensor. Using (2.15) and (2.14), the same difference may be expressed as

$$(dx)^2 - (dX)^2 = \left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) dx_i dx_j = 2E_{ij} dx_i dx_j \qquad ...(2.19)$$

in which the second-order tensor

$$E_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial X_k}{\partial x_i}\frac{\partial X_k}{\partial x_j}\right) \qquad ...(2.20)$$

is called the Eulerian (or Almansi's) finite strain tensor.

Substituling (2.8) into (2.18) and after some simple algebraic manipulation the Lagrangian finite strain tensor can be expressed in the form

$$L_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right) \qquad ...(2.21)$$

Similarly, substituling (2.9) into (2.20), the Eulerian finite strain tensor may be written as

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) ...(2.22)$$

2.4 Small deformation theory, Infinitesimal strain tensors

The small deformation theory of continuum mechanics has its basic candition that the displacement gradients be small campared to unity. The fundamental measure of deformation is the difference $(dx)^2 - (dX)^2$ which may be expressed in terms of the displacement gradients by inserting (2.21) and (2.22) into (2.17) and (2.19) respectively. If the displacement gradients are small, the finite strain tensor in (2.17) and (2.19) reduce to infinitesimal strain tensor, and the resulting equations represent small deformations.

In (2.21), if the displacement gradient components $\frac{\partial u_i}{\partial X_j}$ are each small compared to unity, the product terms are negligible and may be dropped. The resulting tensor is the Lagrangian infinitesimal strain tensor, which is denoted by

$$l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \qquad ...(2.23)$$

Likewise for $\frac{\partial u_i}{\partial x_j} \ll 1$ in (2.22), the product terms may be droped to yield Eulerian infinitesimal strain tensor, which is denoted by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad ...(2.24)$$

If both the displacement gradients and displacements themselves are small, there is very little difference in the material and spatial coordinates of a continuum particle. Accordingly, we may consider

$$l_{ij} = \varepsilon_{ij}$$

if both the displacement and displacement gradients are sufficiently small.

2.5 Relative displacements, Linear rotation tensor, Rotation vector.

In Fig 2.2 the displacements of two neighbouring particles are represented by the vectors $u_i^{(P_o)}$ and $u_i^{(Q_o)}$. The vector

$$du_i = u_i^{(Q_o)} - u_i^{(P_o)} \qquad .....(2.25)$$

is called the relative displacement vector of the particle originally at Q_o with respect to the particle originally at P_o. Again, the relative displacement vector can be written as

$$du_i = \left(\frac{\partial u_i}{\partial X_j}\right)_{P_o} dX_j \qquad ...(2.26)$$

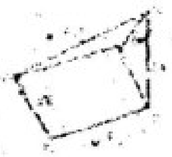


Fig. 2.2

Here the parentheses on the partial derivatives are to emphasize the requirement that the derivatives are to be evaluated at point P_o. Equation (2.26) is the Lagrangian form of the relative displacement vector.

The unit relative displacement vector is defined by

$$\frac{du_i}{dX} = \frac{du_i}{\partial X_j}\frac{dX_j}{dX} = \frac{\partial u_i}{\partial X_j} v_j \quad ...(2.27)$$

where dX is the magnitude of the differential distance vector dX_i and v_i is a unit vector in the directions of dX_i so that $dX_i = v_i\, dX$.

Again, the material displacement gradient $\frac{\partial u_i}{\partial X_j}$ may be decomposed uniquely into a symmetric and an antisymmetric part, the relative displacement vector du_i may be written as

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_i}{\partial X_j}\right)\right] dX_j \quad ..(2.28)$$

$$= [l_{ij} + W_{ij}] dX_j$$

where $l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)$ is called the linear Lagrangian strain tensor and

$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)$ is called the linear Lagrangian rotation tensor.

In a displacement for which the strain tensor l_{ij} is identically zero in the vicinity of the point P_o, the relative displacement at that point will be an infinitesimal rigid body rotation. This infinitesimal rotation may be expressed by the rotation vector $\omega_i = \frac{1}{2} e_{ijk} W_{kj}$ ...(2.29)

in terms of which the relative displacement is given by the expression

$$du_i = e_{ijR}\, \omega_j dX_k \quad ...(2.30)$$

Accordingly the Eulerian description of the relative displacement vector is given by

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j$$

and the unit relative displacement vector is given by

$$du_i = \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dx} = \frac{\partial u_i}{\partial x_j}\mu_j \quad ...(2.31) \quad \text{where } dx_i = \mu_i dx$$

Decomposition of the Eulerian displacement gradient $\frac{\partial u_i}{\partial x_j}$ results in the expression

$$du_i = \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j$$

$$= \left[\varepsilon_{ij} + w_{ij}\right] d_{xj} \qquad ...(2.32)$$

where $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is called the Eulerian linear strain tensor and

$w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ is the linear Eulerian rotation.

The linear Eulerian rotation vector is defined by

$$\omega_i = \frac{1}{2} e_{ijk} w_{kj} \qquad ...(2.33)$$

in terms of which the relative displacement is given by the expression

$$du_i = e_{ijk}\,\omega_j dx_k \qquad ...(2.34)$$

2.6 Interpretation of the linear strain tensors

For small deformation theory, the finite Lagrangian strain tensor L_{ij} may be replaced by the lianer Lagrangian strain tensor l_{ij} as

$$(dx)^2 - (dX)^2 = (dx - dX)\ (dx + dX) = 2l_{ij} dX_i dX_j \qquad ...(2.35)$$

Since $dx \approx dX$ for small deformations, this equation may be put in the form

$$\frac{dx - dX}{dX} = l_{ij}\frac{dX_i dX_j}{dXdX} = l_{ij} v_i v_j \qquad ...(2.36)$$

The left hand side of (2.36) is recognized as the change in length per unit original length of the differential element and is called the normal strain for the line element originally having direction cosines $\frac{dX_i}{dX}$.

When (2.36) is applied to the differential line element P_oQ_o, located with respect to the set of local axes at P_o as shown in Fig 2.3, the result will be the normal strain for that

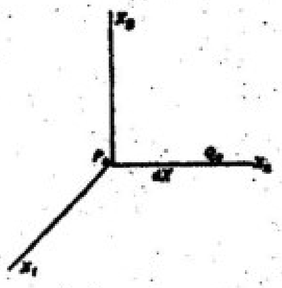


Fig. 2.3

element. Because P_oQ_o here lies along the X_2 axis

$$\frac{dX_1}{dX} = \frac{dX_3}{dX} = 0, \qquad \frac{dX_2}{dX} = 1$$

and therefore (2.36) becomes

$$\frac{dx - dX}{dX} = l_{22} = \frac{\partial u_2}{\partial X_2} \quad ...(2.37)$$

Similarly, for elements originally situated along the X_1 and X_3 axes,

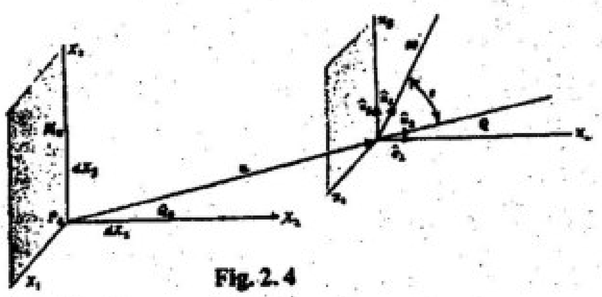


Fig. 2.4

(2.36) yields normal strain values l_{11} and l_{33} respectively. In general, therefore, the diagonal terms of the linear strain tensor represent normal strains in the coordinate directions.

The physical interpretation of the off-diagonal terms of l_{ij} may be obtained by a consideration of the line elements originally located along two of the coordinate axes. In Fig 2.4 the line elements P_oQ_o and P_oM_o originally along the X_2 and X_3 axes, respectively become after deformation the line elements PQ and PM with respect to the parallel set of local axes with origin at P. The original right angle between the line elements becomes the angle θ. Form (2.26) and the assumption of small deformation theory, a first order approximation gives the unit vector at P in the direction of Q as

$$\hat{n}_2 = \frac{\partial u_1}{\partial X_2}\hat{e}_1 + \hat{e}_2 + \frac{\partial u_3}{\partial X_2}\hat{e}_3 \quad ..(2.38)$$

and, for the unit vector at P in the direction of M, as

$$\hat{n}_3 = \frac{\partial u_1}{\partial X_3}\hat{e}_1 + \frac{\partial u_2}{\partial X_3}\hat{e}_2 + \hat{e}_3 \quad ...(2.39)$$

Therefore $$\cos\theta = \hat{n}_2 \cdot \hat{n}_3 = \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \quad ..(2.40)$$

or, neglecting the product term which is of higher order,

$$\cos\theta = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} = 2l_{23} \quad ...(2.41)$$

Again, taking the change in the right angle between the elements as

$$\gamma_{23} = \Pi/2 - \theta,$$

and remembering that for the linear theory γ_{23} is very small, it follows that

$$\gamma_{23} \approx \sin\gamma_{23} = \sin\left(\Pi/2 - \theta\right) = \cos\theta = 2l_{23} \quad ...(2.42)$$

Therefore the off diagonal terms of the linear strain tensor represent one-half the angle change between two line elements originally at right angles to one another. These strain components are called shearing strains.

A similar interpretation may be made for the linear Eulerian strain tensor ε_{ij}.

For those deformations in which the assumption $l_{ij} = \varepsilon_{ij}$ is valid, no distinction is made between the Eulerian and Lagrangian interpretations.

2.7 Strain quadric of Cauchy

For a set of rotated axes x_i' having the transformation matrix $[b_{ij}]$ with respect to the set of local unprimed axes X_i at point P_o as shown in Fig 2.5(a), the components of L'_{ij} and l'_{ij} are given by

$$L'_{ij} = b_{ip} b_{jq} L_{pq}, \quad ...(2.42)$$

$$l'_{ij} = b_{ip} b_{jq} l_{pq} \quad ...(2.43)$$

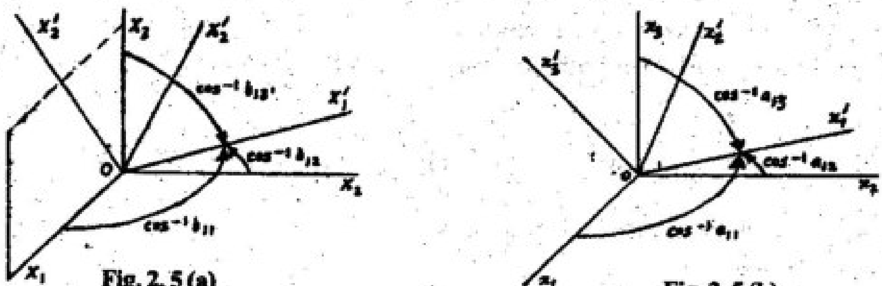


Fig. 2.5 (a) Fig. 2.5 (b)

Likewise, for the rotated axes x_i' the transformation matrix $[a_{ij}]$ in Fig 2.5(b), the components of E'_{ij} and ε'_{ij} are given by

$$E'_{ij} = a_{ip} a_{jq} E_{pq} \quad ...(2.44)$$

$$\varepsilon'_{ij} = a_{ip} a_{jq} \varepsilon_{pq} \quad ...(2.45)$$

The Lagrangian and Eulerian linear strain quadrics may be given with reference to local Cartesion coordinates η_i and ζ_i at the points P_o and P respectively as shown in Fig 2.6. Thus, the equation of the Lagrangian strain quadric is given by

$$l_{ij}\eta_i\eta_j = \pm h^2 \quad ...(2.46)$$

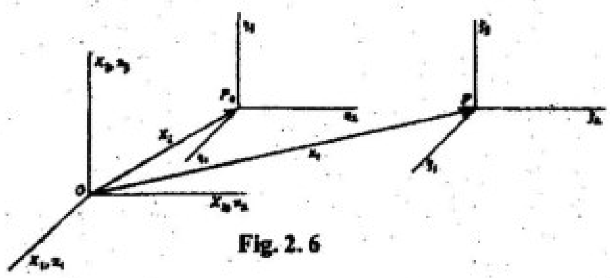


Fig. 2. 6

and the equation of the Eulerian strain quadric is given by

$$\varepsilon_{ij}\zeta_i\zeta_j = \pm g^2 \qquad ...(2.47)$$

Two important properties of the Lagrangian (Eulerian) linear strain quadric are

1. The normal strain with respect to te original (final) length of a line element is inversely proportional to the distance squared from the origin of the quadric P_o (P) to a point on its surface.

2. The relative displacement of the neighbouring particle located at Q_o (Q) per unit original (final) length is parallel to the normal of the quadric surface at the point of intersection with the line through P_oQ_o (PQ)

2.8 Principal strains, Strain invariants, Cubical dilatation

Physically, a principal direction of the strain tensor is one for which the orientation of an element at a given point is not altered by a pure strain deformation. The principal strain value is simply the unit relative displacement (normal strain) that occurs in the principal direction.

For the Lagrangian strain tensor l_{ij}, the unit relative displacement vector may be written as

$$\frac{du_i}{dX} = (l_{ij} + W_{ij})v_j \qquad ...(2.48)$$

Calling $l_i^{(\hat{n})}$ the normal strain in the direction of the unit vector n_i, (2.48) yields for pure strain $(W_{ij} = 0)$ the relation

$$l_i^{(\hat{n})} = l_{ij}n_j \qquad ...(2.49)$$

If the direction n_i is a principal direction with a principal strain value l then

$$l_i^{(\hat{n})} = ln_i = l\delta_{ij}n_j \qquad ...(2.50)$$

From (2.49) and (2.50) we have

$$(l_{ij} - l\delta_{ij})n_j = 0 \qquad ...(2.51)$$

which together with the condition n_in_i=1 on the unit vectors n_i provide the necessary equation for determining the principal strain value l and its direction cosines n_i. Nontrivial solutions of (2.51) exist if and only if the determinant of coefficients vanishes.

Therefore

$$\left| l_{ij} - l\delta_{ij} \right| = 0 \qquad ...(2.52)$$

which upon expansion yields the characteristic equation of l_{ij}, the cubic

$$l^3 - I_L l^2 + II_L l - III_L = 0 \qquad ...(2.53)$$

where $\quad I_L = l_{ii}$

$$II_L = \frac{1}{2}\left(l_{ii} l_{jj} - l_{ij} l_{ij}\right)$$

$$III_L = \left| l_{ij} \right| \qquad ...(2.54)$$

are the first, second and third Lagrangian strain invariants respectivelly. The roots of (2.53) are the principal strain values denoted by $l_{(1)}$, $l_{(2)}$ and $l_{(3)}$.

The first invarient of Lagrangian strain tensor may be expressed in terms of the principal strains as

$$I_L = l_{ii} = l_{(1)} + l_{(2)} + l_{(3)} \qquad ...(2.55)$$

and has an important physical interpretation.

To see this, consider a differential rectangular parallelopiped whose edges are parallel to the principal strain directions as shown in Fig 2.7.

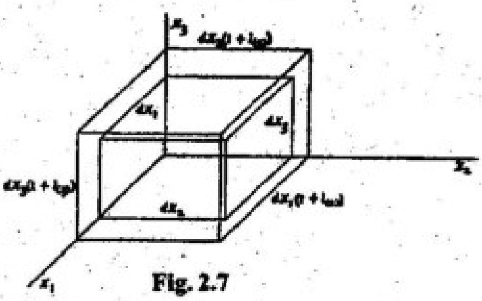


Fig. 2.7

The change in volume per unit original volume of the element is called the Cubical dilatation and is given by

$$D_o = \frac{\Delta V_o}{V_o} = \frac{dX_1\left(1 + l_{(1)}\right) dX_2 \left(1 + l_{(2)}\right) dX_3 \left(1 + l_{(3)}\right) - dX_1 dX_2 dX_3}{dX_1 dX_2 dX_3} \qquad ...(2.56)$$

For small strain theory, the first-order approximation of this ratio is the sum

$$D_o = l_{(1)} + l_{(2)} + l_{(3)} = I_L \qquad ...(2.57)$$

With regard to the Eulerian strain tensor ε_{ij} and its associated relative displacement vector $\varepsilon_i^{(\hat{n})}$, the principal directions and principal strain values $\varepsilon_{(1)}$, $\varepsilon_{(2)}$, $\varepsilon_{(3)}$ are determined in exactly in the same way as their Lagrangian courterparts. The Eulerian strain inveriants may be expressed in terms of the principal strains as

$$
\begin{aligned}
I_{(E)} &= \varepsilon_{(1)} + \varepsilon_{(2)} + \varepsilon_{(3)} \\
II_E &= \varepsilon_{(1)}\varepsilon_{(2)} + \varepsilon_{(2)}\varepsilon_{(3)} + \varepsilon_{(3)}\varepsilon_{(1)} \\
III_E &= \varepsilon_{(1)}\varepsilon_{(2)}\varepsilon_{(3)}
\end{aligned}
\qquad ...(2.58)
$$

The cubical dilatation for the Eulerian description is given by

$$\frac{\Delta v}{v} = D = \varepsilon_{(1)} + \varepsilon_{(2)} + \varepsilon_{(3)}. \qquad ...(2.59)$$

2.9. Spherical and deviator strain tensors

The Lagrangian and Eulerian linear strain tensor may each be split into spherical and deviator tensors in the same manner in which the stress tensor decomposition was carried out in Unit I. If Lagrangian and Eulerian deviator tensor components are denoted by d_{ij} and ε_{ij} respectively, the resolution expressions are

$$l_{ij} = d_{ij} + \delta_{ij}\frac{l_{kk}}{3} \qquad ...(2.60)$$

and $$\varepsilon_{ij} = e_{ij} + \delta_{ij}\frac{\varepsilon_{kk}}{3} \qquad ...(2.61)$$

The deviator tensor are associated with shear deformation for which the cubical dilatation vanishes. Therefore, the first invariants d_{ii} and ε_{ii} of the deviator strain tensors are identically zero.

2.10. Compatibility equations for linear strains.

If the strain components ε_{ij} are given explicitly as functions of the coordinates, the six independent equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad ...(2.62)$$

may be viewed as a system of six partial differential equations for determining the three displacement components u_i. If the displacement components u_i are to be single-valued and continuous, some conditions must be imposed upon the strain components. The necessary and sufficient conditions for such a displacement field are expressed by the equations

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_m} + \frac{\partial^2 \varepsilon_{km}}{\partial x_i \partial x_j} - \frac{\partial^2 \varepsilon_{ik}}{\partial x_j \partial x_m} - \frac{\partial^2 \varepsilon_{jm}}{\partial x_i \partial x_k} = 0 \qquad ...(2.63)$$

There are eighty-one equation in all in (2.63) but only six are distinct. Theresix equations are

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$$

$$\frac{\partial^2\varepsilon_{22}}{\partial x_3^2}+\frac{\partial^2\varepsilon_{33}}{\partial x_2^2}=2\frac{\partial^2\varepsilon_{23}}{\partial x_2\partial x_3}$$

$$\frac{\partial^2\varepsilon_{33}}{\partial x_1^2}+\frac{\partial^2\varepsilon_{11}}{\partial x_3^2}=2\frac{\partial^2\varepsilon_{31}}{\partial x_3\partial x_1}$$

$$\frac{\partial}{\partial x_1}\left(-\frac{\partial\varepsilon_{23}}{\partial x_1}+\frac{\partial\varepsilon_{31}}{\partial x_2}+\frac{\partial\varepsilon_{12}}{\partial x_3}\right)=\frac{\partial^2\varepsilon_{11}}{\partial x_2\partial x_3} \qquad \text{...(2.64)}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial\varepsilon_{23}}{\partial x_1}-\frac{\partial\varepsilon_{31}}{\partial x_2}+\frac{\partial\varepsilon_{12}}{\partial x_3}\right)=\frac{\partial^2\varepsilon_{22}}{\partial x_3\partial x_1}$$

$$\frac{\partial}{\partial x_3}\left(\frac{\partial\varepsilon_{23}}{\partial x_1}+\frac{\partial\varepsilon_{31}}{\partial x_2}-\frac{\partial\varepsilon_{12}}{\partial x_3}\right)=\frac{\partial^2\varepsilon_{33}}{\partial x_1\partial x_2}$$

Compatibilily equations may also be written in terms of the Lagrangian linear strain tensor l_{ij}. For plane strain parallel to the $x_1 x_2$ plane, the six equations in (2.64) reduce to the single equation

$$\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x_1\partial x_2}$$

SOLVED PROBLEMS

Exp 1. The Lagrangian description of a deformation is given by

$$x_1 = X_1 + X_3(e^2-1) \qquad x_2 = X_2 + X_3(e^2-e^{-2}) \qquad x_3 = e^2 X_3$$

where e is a constant show that the Jacobian J does not vanish and determine the Eulerian equations describing the motion.

Solution:

$$J=\left|\frac{\partial x_i}{\partial X_j}\right|=\begin{vmatrix}\frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3}\\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3}\\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3}\end{vmatrix}=\begin{vmatrix}1 & 0 & e^2-1\\ 0 & 1 & e^2-e^{-2}\\ 0 & 0 & e^2\end{vmatrix}=e^2\neq 0$$

and the reqd. Eulerian equations are

$$X_1 = x_1 - x_3\left(1-e^{-2}\right)$$
$$X_2 = x_2 - x_3\left(1-e^{-4}\right)$$
$$X_3 = e^{-2}x_3.$$

Exp 2 For superimposed material and spatial axes, the displacement vector of a body is given by

$$\vec{u} = 4X_1^2\hat{e}_1 + X_2X_3^2\hat{e}_2 + X_1X_3^2\hat{e}_3.$$

Determine the displaced location of the particle originally at (1,0,2).

Solution: The particle was originally at the point (1,0,2).

$\therefore$ The original position vector of the particle is $\bar{X} = \hat{e}_1 + 2\hat{e}_3$.

Its displacement is $\quad \bar{u} = \bar{x} - \bar{X}$

i.e. $\quad \bar{x} = \bar{u} + \bar{X}$

$$= 5\hat{e}_1 + 6\hat{e}_3.$$

Exp 3 A continuum material undergoes the deformation $x_1=X_1$, $x_2=X_2+AX_3$, $x_3=x_3+AX_2$, where A is a constant. Determine the Lagrangian finite strain tensor.

Solution The Lagrangian finite strain tensor is given by

$$L_{ij} = \frac{1}{2}\left[\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right]$$

Now, $\left[\frac{\partial x_k}{\partial X_i}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}$

$$\therefore \left[\frac{\partial x_k}{\partial X_i}\right]\left[\frac{\partial x_k}{\partial X_j}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & A \\ 0 & A & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix}$$

$$\therefore 2L_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1+A^2 & 2A \\ 0 & 2A & 1+A^2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e. $$L_{ij} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & A^2 & 2A \\ 0 & 2A & A^2 \end{bmatrix}$$

Exp 4 A displacement field is defined by

$$x_1 = X_1 - CX_2 + BX_3, \quad x_2 = CX_1 + X_2 - AX_3, \quad x_3 = -BX_1 + AX_2 + X_3 .$$

Show that this displacement represents a rigid body rotation only if the constants A,B,C are small. Determine the rotation vector $\bar{\omega}$ for the infinitesimal rigid body rotation.

Solution The Lagrangian finite strain tensor is given by

$$L_{ij} = \frac{1}{2}\left(\frac{\partial x_k}{\partial X_i}\frac{\partial x_k}{\partial X_j} - \delta_{ij}\right)$$

$$\therefore \left[\frac{\partial x_k}{\partial X_i}\right] = \begin{bmatrix} 1 & -C & B \\ C & 1 & -A \\ -B & A & 1 \end{bmatrix}$$

$$\therefore 2L_{ij} = \begin{bmatrix} 1 & -C & B \\ C & 1 & -A \\ -B & A & 1 \end{bmatrix}\begin{bmatrix} 1 & C & -B \\ -C & 1 & A \\ B & -A & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore L_{ij} = \frac{1}{2}\begin{bmatrix} B^2 + C^2 & -AB & -AC \\ -AB & A^2 + C^2 & -BC \\ -AC & -BC & A^2 + B^2 \end{bmatrix}$$

When the constants A,B,C are very small, then the products of the constants are neglected and we have

$L_{ij} = 0$ i.e. the displacement represents a rigid body rotation.

The rotation vector is given by

$$\bar{\omega} = \frac{1}{2}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial X_1} & \dfrac{\partial}{\partial X_2} & \dfrac{\partial}{\partial X_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = A\hat{e}_1 + B\hat{e}_2 + C\hat{e}_3 .$$

Exp 5 Under the restriction of small deformation theory, the displacement field is given by

$$\bar{u} = (x_1 - x_3)^2\hat{e}_1 + (x_2 + x_3)^2\hat{e}_2 - x_1x_2\hat{e}_3$$

determine the change in length per unit length (normal strain) in the direction of

$$\hat{v} = \frac{(8\hat{e}_1 - \hat{e}_2 + 4\hat{e}_3)}{9}$$

at point P(0,2-1). Also, determine the linear strain tensor, the linear rotation tensor and the rotation vector at P.

Solution. The normal strain is given by $$\frac{dx-dX}{dX} = l_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX} = l_{ij}v_i v_j$$

Now, $l_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$, under the restriction of small deformation theory.

$$\therefore \text{At } P(0,\ 2,\ -1), \left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{At } P, l_{ij} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$\therefore$ The normal strain is

$$\left[\frac{8}{9}, \frac{-1}{9}, \frac{4}{9}\right]\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{8}{9} \\ \frac{-1}{9} \\ \frac{4}{9} \end{bmatrix} = -\frac{6}{81}$$

$$\text{At } P, \left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\therefore \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \quad \text{and} \quad w_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)$$

$$\text{i.e. } \varepsilon_{ij} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\text{and} \quad w_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

The rotation vector is

$$\bar{\Omega} = \frac{1}{2}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ (x_1 - x_2)^2 & (x_2 + x_3)^2 & -x_1 x_2 \end{vmatrix} = -\hat{e}_1$$

$\therefore$ The rotation vector $\bar{\Omega}$ has the components (-1, 0, 0).

Exp 6 A linear deformation is specified by

$$u_1 = 4x_1 - x_2 + 3x_3,\ u_2 = x_1 + 7x_2,\ u_3 = -3x_1 + 4x_2 + 4x_3$$

Determine the principal strains and the principal deviator strains for this deformation.

Solution : The principal strains are obtained from

$$|\varepsilon_{ij} - \varepsilon\delta_{ij}| = 0$$

$$\text{Now,}\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\text{and}\quad \left[\frac{\partial u_i}{\partial x_j}\right] = \begin{bmatrix} 4 & -1 & 3 \\ 1 & 7 & 0 \\ -3 & 4 & 4 \end{bmatrix}$$

$$\therefore \varepsilon_{ij} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

Hence principal strain values are obtained from

$$\begin{vmatrix} 4-\varepsilon & 0 & 0 \\ 0 & 7-\varepsilon & 2 \\ 0 & 2 & 4-\varepsilon \end{vmatrix} = 0$$

Expanding this we have

$$\varepsilon = 8,4,3 \qquad \text{i.e.}\quad \varepsilon_{(1)} = 8,\quad \varepsilon_{(2)} = 4,\quad \varepsilon_{(3)} = 3.$$

Which gives the principal strains and the principal deviator strains are given by

$$e_{(n)} = \varepsilon_{(n)} - \frac{1}{3}\varepsilon_{ii}$$

$$\therefore \qquad e_{(1)} = 8 - \frac{1}{3}(4 + 7 + 4) = 3$$

$$e_{(2)} = 4 - 5 = -1$$
$$e_{(3)} = 3 - 5 = -2$$

i.e. $e_{(1)} = 3, e_{(2)} = -1, e_{(3)} = -2$

Example 7

A displacement field is given by

$$\bar{x} = X_1X_3^2\hat{e}_1^\tau + X_1^2X_2\hat{e}_2^\tau + X_2^2X_3\hat{e}_3^\tau$$

Determine independently the material deformation gradient F and material displacement gradient J and show that F − J = 1.

Solution :

The material deformation gradient is defined by $F = \left(\dfrac{\delta x_i}{\delta x_j}\right)$

We know the displacement vector

$$u_i = x_i - X_i$$
$$\therefore x_i = u_i + X_i$$
$$x_1 = X_1X_3^2 + X_1$$
$$x_2 = X_1^2X_2 + X_2$$
$$x_3 = X_2^2X_3 + X_3$$

$$\therefore F = \left(\frac{\delta x_i}{\delta X_j}\right) = \begin{pmatrix} \dfrac{\delta x_1}{\delta X_1} & \dfrac{\delta x_1}{\delta X_2} & \dfrac{\delta x_1}{\delta X_3} \\ \dfrac{\delta x_2}{\delta X_1} & \dfrac{\delta x_2}{\delta X_2} & \dfrac{\delta x_2}{\delta X_3} \\ \dfrac{\delta x_3}{\delta X_1} & \dfrac{\delta x_3}{\delta X_2} & \dfrac{\delta x_3}{\delta X_3} \end{pmatrix}$$

$$= \begin{pmatrix} X_3^2+1 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2+1 & 0 \\ 0 & 2X_2X_3 & X_2^2+1 \end{pmatrix}$$

The displacement gradient is denoted by

$$J = \left(\frac{\partial u_i}{\partial X_j}\right) = \begin{pmatrix} X_3^2 & 0 & 2X_1X_3 \\ 2X_1X_2 & X_1^2 & 0 \\ 0 & 2X_2X_3 & X_2^2 \end{pmatrix}$$

$$\therefore F - J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Example 8 : A displacement field is given $x_1 = X_1 + AX_2$, $x_2 = X_2 + AX_3$, $x_3 = X_3 + AX_1$. Calculate the Lagrangian and Eulerian lionear strain tensor, compare them for the component, when A is small.

Solution : The displacement is defined by

$$ui = xi - Xi$$

$$u_1 = AX_2,\ u_2 = AX_3,\ u_3 = AX_1$$

Now Lograngian linear straen tensor is defined by

$$l_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta X_j} + \frac{\delta u_j}{\delta X_i}\right)$$

$$\left(\frac{\delta u_i}{\delta X_j}\right) = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{pmatrix}$$

$$l_{ij} = \frac{1}{2}\left\{\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & A \\ A & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & A \\ A & 0 & 0 \\ 0 & A & 0 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{pmatrix}$$

The Eulerian linear strain tensor defined by

$$e_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right)$$

From given conditions

$$x_1 = X_1 + AX_2$$

$$x_2 = X_2 + AX_3$$

$$x_3 = X_3 + AX_1$$

$$\therefore X_1 = x_1 - AX_2 = x_1 - Ax_2 + A_1^2 x_3 - A_1^3 X_1$$

$$\therefore (1 + A^3) X_1 = x_1 - Ax_2 + A^2 x_3$$

$$\therefore X_1 = \frac{x_1 - Ax_2 + A^2x_3}{1+A^3}$$

Similarly, $X_2 = \frac{x_2 - Ax_3 + A^2x_1}{1+A^3}$

$$X_3 = \frac{x_3 - Ax_1 + A^2x_2}{1+A^3}$$

Here, $u_1 = x_1 - X_1 = \frac{A(A^2x_1 + x_2 - Ax_3)}{1+A^3}$

$$u_2 = \frac{A(A^2x_2 + x_3 - Ax_1)}{1+A^3}, \qquad u_3 = \frac{A(A^2x_3 + x_1 - Ax_2)}{1+A^3}$$

$$\therefore \left(\frac{\delta u_i}{\delta u_j}\right) = \begin{pmatrix} \frac{A^3}{1+A^3} & \frac{A^3}{1+A^3} & \frac{A^2}{1+A^3} \\ -\frac{A^2}{1+A^3} & \frac{A^3}{1+A^3} & \frac{A}{1+A^3} \\ \frac{A}{1+A^3} & -\frac{A^2}{1+A^3} & \frac{A^3}{1+A^3} \end{pmatrix}$$

Hence

$$e_{ij} = \frac{1}{2}\left\{\begin{pmatrix} \frac{A^3}{1+A^3} & \frac{A}{1+A} & -\frac{A^2}{1+A^3} \\ -\frac{A^2}{1+A^3} & \frac{A^3}{1+A^3} & \frac{A}{1+A^3} \\ \frac{A}{1+A^3} & -\frac{A^2}{1+A^3} & \frac{A^3}{1+A^3} \end{pmatrix} + \begin{pmatrix} \frac{A^3}{1+A^3} & \frac{-A^2}{1+A^3} & \frac{A}{1+A^3} \\ \frac{A}{1+A^3} & \frac{A^3}{1+A^3} & -\frac{A^2}{1+A^3} \\ \frac{-A^2}{1+A^3} & -\frac{A^2}{1+A^3} & \frac{A^3}{1+A^3} \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} \frac{2A^3}{1+A^3} & \frac{A-A^2}{1+A^3} & \frac{A-A^2}{1+A^3} \\ \frac{A-A^2}{1+A^3} & \frac{2A^3}{1+A^3} & -\frac{A-A^2}{1+A^3} \\ \frac{A-A^2}{1+A^3} & \frac{A-A^2}{1+A^3} & \frac{2A}{1+A^3} \end{pmatrix}$$

When A is small then A^2 and higher powers of A may be neglected

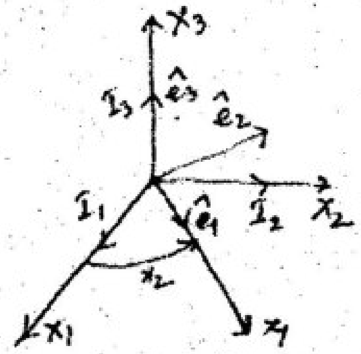
$$e_{ij} = \frac{1}{2}\begin{pmatrix} 0 & A & A \\ A & 0 & A \\ A & A & 0 \end{pmatrix} = l_{ij}$$

Example 9 : With respect to rectangular Cartesian material co-ordinates X_i a displacement field is given by

$$U_1 = AX_2X_3,\ U_2 = AX_1X_3,\ U_3 = 0,$$

where A is a constant. Determine the displacement components for cylindrical spatial co-ordinates x_i if the two systems have a common origin.

Solution :



From the geometry of the a:.es the traris formation tensor

$$\alpha_{pk} = \hat{e_p} \cdot \hat{I_k} \text{ is}$$

$$\alpha_{pk} = \begin{pmatrix} \cos x_2 & \sin x_2 & 0 \\ -\sin x_2 & \cos x_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and from the inverse form

$U_p = \alpha_{pk} U_k$. Thus since Cartesian and cylindridal co-ordinates are related through the equations

$$X_1 = x_1 \cos x_2,$$
$$X_2 = x_1 \sin x_2$$
$$X_3 = x_3$$

$$\therefore u_1 = (-\cos x_2)AX_2X_3 + (\sin x_2)AX_1AX_3$$
$$= (-\cos x_2)AX_3x_1 \sin x_2 + (\sin x_2)AX_3x_1 \cos x_2 = 0$$
$$x_2 = (\sin x_2)AX_2X_3 + (\cos x_2)AX_1X_3$$
$$= (\sin^2 x_2)AX_1x_3 + (\cos^2 x_2)AX_1x_3$$
$$= Ax_1x_3$$
$$U_3 = 0$$

This displacement is that of a circular shaft in torsion.

Supplementary problems

1. For superimposed material and spatial axes a continuum body undergoes displacement

$$\vec{u} = (3X_2 - 4X_3)\hat{e}_1 + (2X_1 - X_3)\hat{e}_2 + (4X_2 - X_1)\hat{e}_3.$$

Determine the displaced position of the position vector of points C(2,6,3) which is parallel to the vector joining A(1,0,3) and B(3,6,6). show that two vectors remain parallel after deformation.

2. A displacement field is given by $x_1=X_1+AX_2$, $x_2=X_2+AX_3$, $x_3=X_3+AX_1$ where A is a constant. Calculate the Lagrangian linear strain tensor and the Eulerian linear strain tensor and also compare them for the case when A is small.

3. A continuum body undergoes the deformation

$$x_1=X_1+2X_3, \quad x_2=X_2-2X_3, \quad x_3=X_3-2X_1+2X_2$$

Determine the Lagrangian and Eulerian finite strain tensors

Ans $L_G = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $E_A = \frac{1}{9}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

4. A certain homogeneous deformation field results in the finite strain tensor

$$L_{ij} = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 1 & -2 \\ -2 & -2 & 6 \end{pmatrix}$$

Determine the principal strains, strain invariants and principal directions.

Ans Principal strains: 8, 2, -2, principal invariants : 8, -4, -32 and principal direction

$$[a_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}}, & -\frac{1}{\sqrt{6}}, & \frac{2}{\sqrt{3}} \end{bmatrix}$$

5. The state of strain throughout a continuum is given by

$$\varepsilon_{ij} = \begin{pmatrix} x_1^2 & x_2^2 & x_1x_3 \\ x_2^2 & x_3 & x_3^2 \\ x_1x_3 & x_3^2 & 5 \end{pmatrix}$$

Are the compatibility equations for strain satisfied? **Ans** yes.

6. A displacement field is given by

$$u_1 = 3x_1x_2^2, \quad u_2 = 2x_3x_1, \quad u_3 = x_3^2 - x_1x_2$$

Determine the strain tensor ε_{ij} and check whether or not the compatibility conditions are satisfied.

Ans (i)

$$\varepsilon_{ij} = \begin{pmatrix} 3x_2^{\,2} & 3x_1x_2 + x_3 & -x_2/2 \\ 3x_1x_2 + x_3 & 0 & x_1/2 \\ -x_2/2 & x_1/2 & 2x_3 \end{pmatrix}$$

(ii) yes.

7. With respect to rectangular Cartesian material coordinates X_i, a displacement field is given by $U_1 = -AX_2X_3, U_2 = AX_1X_3, U_3 = 0$ where A is a constant. Determine the displacement components for cylindrical spatial coordinates x_i if the two systems have a common origin.

Ans $u_1 = 0, u_1 = Ax_1x_3, u_3 = 0.$

●●●

Unit III
Motion and Flow

3.1. Material derivatives

The motion of a continuum may be expressed either in terms of material coordinates (Lagrangian description) by

$$x_i = x_i(X_1, X_2, X_3, t) \qquad ...(3.1)$$

or, by the inverse of these equations in terms of spatial coordinates (Eulerian description) as

$$X_i = X_i(x_1, x_2, x_3, t) \qquad ...(3.2)$$

The necessary and sufficient condition for the inverse function (ii) to exist is that the Jacobian

$$J = \left|\frac{\partial x_i}{\partial X_j}\right|$$

should not vanish.

Physically, the Lagrangian description fixes attention on specific particles of the continuum, where as the Eulerian description concerns itself with a particular region of the space occupied by the continuum. Since (i) and (ii) are the inverses of one another, any physical quantity of the continuum that is expressed with respect to a specific particle may also be expressed w.r.t. the particular location in space occupied by the particle.

The time rate of change of any property of a continuum w.r.t. specific particles of the moving continuum is called the material derivative(or, substantial, or comoving or convective derivative) of that property. The instantaneous position x_i of a particle is itself a property of the particle. The material derivative of the particle's position is the instantaneous velocity of the particle. The velocity vector is defined by,

$$v_i = \frac{dx_i}{dt} = \dot{x}_i \qquad ...(3.3)$$

In general, if $P_{ij}...$ is any scalar, vector or tensor property of a continuum that may be expressed as a point function of the coordinates, and if the Lagrangian description is given by $P_{ij}...... = P_{ij}......(\bar{X}, t)$ ...(3.4)

the material derivative of the property is expressed by

$$\frac{dP_{ij...}}{dt} = \frac{\partial P_{ij}...(\bar{X}, t)}{\partial t} \qquad ...(3.5)$$

When the property $P_{ij}...$ is expressed by the spatial description in the form

$$P_{ij}...... = P_{ij}.....(\bar{x}, t) \qquad ...(3.6)$$

the material derivative is given by

$$\frac{dP_{ij}...(\bar{x}, t)}{dt} = \frac{\partial P_{ij}...(\bar{x}, t)}{\partial t} + \frac{\partial P_{ij}...(\bar{x}, t)}{\partial x_k}\frac{dx_k}{dt} \qquad ...(3.7)$$

where the second term on the right arises because the specific particles are changing position in space. The first term on the right side of (3.7) give the rate of change at a particular location and is called the local rate of change and the second term is called the convective rate of change since it expresses the contribution due to the motion of the particles in the variable field of the property.

From (3.3), the material derivative (3.7) may be written as

$$\frac{dP_{ij}...(\bar{x},t)}{dt} = \frac{\partial P_{ij}...(\bar{x},t)}{\partial t} + v_k \frac{\partial P_{ij}...(\bar{x},t)}{\partial x_k} \qquad ...(3.8)$$

which suggests the introduction of the material derivative operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} \qquad ...(3.9)$$

which is used in taking the material derivatives of quantities expressed in spatial coordinates.

3.2. Velocity, Acceleration, Instantaneous velocity field.

The velocity vector is defined by

$$v_i = \frac{dx_i}{dt}$$

Again,
$$v_i \equiv \frac{dx_i}{dt} = \frac{d(u_i + X_i)}{dt} = \frac{du_i}{dt} \qquad ...(3.10)$$

since X is independent of time. In (3.10), if the displacement is expressed in the Lagrangian form $u_i = u_i(\bar{X},t)$, then

$$v_i = \dot{u}_i = \frac{du_i(\bar{X},t)}{dt} = \frac{\partial u_i(\bar{X},t)}{\partial t} \qquad ...(3.11)$$

If, on the other hand, the displacement is in the Eulerian form i.e. $u_i = u_i(\bar{x},t)$, then

$$v_i(\bar{x},t) \equiv \dot{u}_i(\bar{x},t) = \frac{du_i(\bar{x},t)}{dt}$$

$$= \frac{\partial u_i(\bar{x},t)}{\partial t} + v_k(\bar{x},t)\frac{\partial u_i(\bar{x},t)}{\partial x_k} \qquad ...(3.12)$$

The function $v_i = v_i(\bar{x},t)$ is said to specify the instantaneous velocity field.

The material derivative of the velocity is the acceleration. In Lagrangian form,

$$a_i \equiv \dot{v}_i = \frac{dv_i(\bar{X},t)}{dt} = \frac{\partial v_i(\bar{X},t)}{\partial t} \qquad ...(3.13)$$

and in Eulerian form,

$$a_i \equiv \dot{v}_i \equiv \frac{dv_i(\bar{x},t)}{dt} = \frac{\partial v_i(\bar{x},t)}{\partial t} + v_k(\bar{x},t)\frac{\partial v_i(\bar{x},t)}{\partial x_K} \qquad ...(3.14)$$

3.3 Path lines, stream lines, steady motion

A path line is the curve or path followed by a particle during its motion or flow, The differential equation of path line is $dx_i = v_i dt$

or, $\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} = dt$ where the velocity components v_1, v_2, v_3 are functions of space coordinates.

A stream line is the curve whose tangent at any point is in the direction of the velocity at that point. Thus the stream lines at time t are the curves that are tangents to the velocity field. Hence the integral curves to $v_i = kdx_i$ are the stream lines at time t.

The motion of a continuum is termed steady motion if the velocity field is independent of time so that $\frac{\partial v_i}{\partial t} = 0$. For steady motion, stream lines and path lines coincide.

3.4 Rate of deformation, Vorticity, Natural strain increments

The velocity gradient tensor $\frac{\partial v_i}{\partial x_i}$ may be decomposed into its symmetric and skew-symmetric part as

$$\frac{\partial v_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

$$= D_{ij} + V_{ij} \qquad ...(3.15)$$

This decomposition is valid upon even if v_i and $\frac{\partial v_i}{\partial x_j}$ are finite quantities. The symmetric tensor

$$D_{ij} = D_{ji} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad ...(3.16)$$

is called the rate of deformation tensor and the skew symmetric tensor

$$V_{ij} = -V_{ji} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \qquad ...(3.17)$$

is called the vorticity or spin tensor.

The rate of deformation tensor is easily shown to be the material derivative of the Eulerian linear strain tensor.

Thus if in the equation

$$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad ...(3.18)$$

the differentiation with respect to the coordinates and time are interchanged, the equation takes the

form $$\frac{d\varepsilon_{ij}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = D_{ij} \qquad ...(3.19)$$

Similarly, the vorticity tensor may be shown to be equal to the material derivative of the Eulerian linear rotation tensor and the result is expressed by the equation

$$\frac{dw_{ij}}{dt} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) = V_{ij} \qquad ...(3.20)$$

The equation (3.19) may be rewritten as

$$d\varepsilon_{ij} = D_{ij}dt \qquad ...(3.21)$$

The left side of (3.21) represents the components known as the natural strain increments.

3.5. Physical interpretation of the Rate of deformation and Vorticity tensor.

In Fig 3.1 the velocities of the neighbouring particles at points P and Q in a moving continuum are given by v_i and $v_i + dv_i$ respectively. The relative velocity of the particle at Q with respect to the one at P is therefore

$$dv_i = \frac{\partial v_i}{\partial x_j} dx_j \qquad ...(3.21)$$

in which the partial derivatives are to be evaluated at P. In terms of D_{ij} and V_{ij}, (3.21) becomes

$$dv_i = (D_{ij} + V_{ij})dx_j \qquad ...(3.22)$$

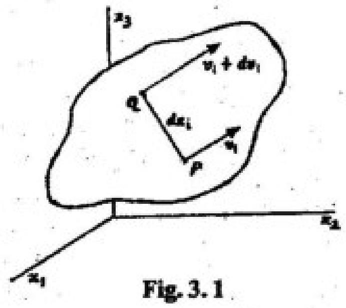


Fig. 3.1

If the rate of deformation tensor is identically zero $(D_{ij} \equiv 0)$,

$$dv_i = V_{ij}\ dx_j \qquad ...(3.23)$$

and the motion in the neighbourhood of P is a rigid body rotation. For this reason a velocity field is said to be irrotational if the vorticity tensor vanishes everywhere within the field.

Associated with the vorticity tensor, the vector defined by

$$q_i = e_{ijk} v_{k,j} \qquad ...(3.24)$$

is known as the vorticity vector. The vector defined as one-half the vorticity vector,

$$\Omega_i = \frac{1}{2}q_i = \frac{1}{2}e_{ijk}v_{k,j} \qquad ..(3.25)$$

is called the rate of rotation vector. For rigid body rotation, such as that described by (3.23), the relative velocity for a neighbouring particle separated from P by dx_i is given as

$$dv_i = e_{ijk}\Omega_j dx_k \qquad ...(3.26)$$

The components of the rate of deformation tensor have the following physical interpretation. The diagonal elements of D_{ij} are called as the stretching or rate of extension components. Thus for pure deformation, from (3.22)

$$dv_i = D_{ij}dx_j \qquad ...(3.27)$$

and, since the rate of change of length of the line element dx_i per unit instantaneous length is given by

$$d_i^{(v)} = \frac{dv_i}{dx} = D_{ij}\frac{dx_j}{dx} = D_{ij}v_j \quad ...(3.28)$$

the rate of deformation in the direction of unit vector v_i is

$$d = d_i^{(v)}v_i = D_{ij}v_i v_j \qquad ...(3.29)$$

From (3.29), if v_i is the direction of a coordinate axis,say $\hat{e}_2$,

$$d = d_{22} \qquad ...(3.30)$$

The off-diagonal terms of D_{ij} are shear rates, being a measure of the rate of change between directions at right angles.

3.6. Material derivatives of Volume, Area and Line Elements.

In the motion from some initial configuration at time t=0 to the present configuration at time t, the continuum particles which occupied the differential volume element dv_o in the initial state now occupy the differential volume element dv. If the initial volume element is taken as the rectangular parallelopiped shown in Fig 3.2 and

$$dV_0 = dX_1 dX_2 dX_3 \qquad ...(3.31)$$

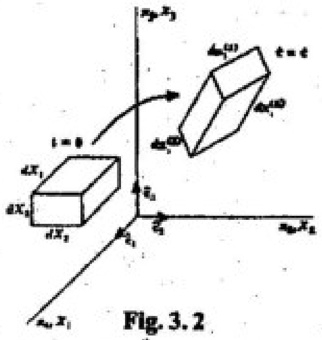


Fig. 3.2

Due to motion, the parallelopiped is moved and distored, but because the motion is assumed continuous the volume elements does not break up. The "line of particles" that formed dX_1 now form the diferential line segment $dx_i^{(1)} = \frac{\partial x_i}{\partial X_1}dX_1$. Similarly, dX_2 becomes $dx_i^{(2)} = \frac{\partial x_i}{\partial X_2}dX_2$ and dX_3 becomes $dx_i^{(3)} = \frac{\partial x_i}{\partial X_3}dX_3$.

Therefore the differential volume element dV is a skewed parallelopiped having edges $dx_i^{(1)}$, $dx_i^{(2)}$, $dx_i^{(3)}$ and a volume is given by

$$dv = e_{ijk}\frac{\partial x_i}{\partial X_1}\frac{\partial x_j}{\partial X_2}\frac{\partial x_k}{\partial X_3}dX_1dX_2dX_3 = JdV_o \qquad ....(3.32)$$

where $J = \left|\frac{\partial x_i}{\partial X_j}\right|$ is the Jacobian

Now,
$$\frac{d}{dt}(dv) = \frac{d}{dt}(J dV_0) = \frac{dJ}{dt} dV_o \qquad ...(3.33)$$

since dV_o is time independent, so that $\frac{d}{dt}(dV_o) = 0$.

The material derivative of the Jacobian J is shown to be

$$\frac{dJ}{dt} = J\frac{\partial v_i}{\partial x_i} \qquad ...(3.34)$$

and hence (3.34) may be put in th e form

$$\frac{d}{dt}(dv) = \frac{\partial v_i}{\partial x_i} dv \qquad ...(3.35)$$

For the initial configuration of a continuum, a differential element of area having the magnitude ds_o may be represented in terms of its unit normal vector n_i by the expression $ds_o n_i$. For the current configuration of the continuum in motion, the particles initially making up the area ds_o now fill an area element represented by the vector $ds\, n_i$ or ds_i. It may be shown that

$$\frac{d}{dt}(ds_i) = \frac{\partial v_j}{\partial x_j} ds_i - \frac{\partial v_j}{\partial x_i} ds_j \qquad ...(3.36)$$

The material derivative of the squared length of the differential line element dx_i may be calculated as

$$\frac{d}{dt}(dx)^2 = \frac{d}{dt}(dx_i dx_i) = 2\frac{d(dx_i)}{dt} dx_i \qquad ...(3.37)$$

However, Since $dx_i = \frac{\partial x_i}{\partial X_j} dX_j$,

$$\frac{d}{dt}(dx_i) = \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_j}\right) dX_j = \frac{\partial v_i}{\partial X_j} dX_j = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} dX_j$$

$$= \frac{\partial v_i}{\partial x_k} dX_k. \qquad ...(3.38)$$

and (3.37) becomes

$$\frac{d}{dt}(dx)^2 = 2\frac{\partial v_i}{\partial x_k} dx_k dx_i \qquad ...(3.39)$$

$$= \frac{\partial v_i}{\partial x_k} dx_k dx_i + \frac{\partial v_k}{\partial x_i} dx_i dx_k$$

$$= \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) dx_i dx_k$$

$$= D_{ik} dx_i dx_k \quad \text{...(3.40)}$$

[since the right side in the indicial form of (3.39) is symmetric in i and k]

3.7 Material derivative of Volume, Surface and Line elements.

Some properties of a continuum are defined as integrals over a finite portion of the continuum. Let P_{ij}... be any scalar, vector or tensor property and represented by the volume integral

$$P_{ij}...(t) = \int_v P^*_{ij}...(\bar{x}, t) dV \quad \text{...(3.41)}$$

where V is the volume that the considered part of the continuum occupies at time t. The material derivative of $P_{ij}...(t)$ is

$$\frac{d}{dt}\left[P_{ij}...(t)\right] = \frac{d}{dt} \int_v P^*_{ij}...(\bar{x}, t) dV \quad (3.42)$$

Since, the differentiation is with respect to a definite portion of the contiuum(i.e. a specific mass system), the operations of differentiation and integration may be interchanged. Therefore

$$\frac{d}{dt} \int_v P^*_{ij}....(\bar{x}, t) dv = \int_v \frac{d}{dt} \left[P^*_{ij}....(\bar{x}, t) dv\right]$$

$$= \int_v \left[\frac{d P^*_{ij}....(\bar{x}, t)}{dt} + P^*_{ij}....(\bar{x}, t) \frac{\partial v_p}{\partial x_p} \right] dv \quad \text{...(3.43)}$$

Again, we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_p \frac{\partial}{\partial x_p}$$

From (3.43) we have

$\therefore$

$$\frac{d}{dt} \int_v P^*_{ij}...(\bar{x}, t) dV = \int_v \left[\frac{\partial P^*_{ij}....(\bar{x}, t)}{\partial t} + \frac{\partial}{\partial x_p} \left\{ v_p P^*_{ij}....(\bar{x}, t) \right\} \right] dV \quad \text{...(3.44)}$$

By using Gauss' theorem, the second term of right hand integral of (3.44) becomes

$$\int_v \frac{\partial}{\partial x_p} \left\{ v_p P^*_{ij}....(\bar{x}, t) \right\} dV = \int_s v_p \left[P^*_{ij}....(\bar{x}, t) \right] dS_p$$

Hence (3.44) becomes

$$\frac{d}{dt}\int_V P_{ij}^{*}....(\bar{x},t)dV = \int_V \frac{\partial P_{ij}^{*}....(\bar{x},t)}{\partial t}dV + \int_s v_p\left[P_{ij}^{*}....(\bar{x},t)\right]dS_p \quad ...(3.45)$$

This equation states that the rate of increase of the property $P_{ij}....(t)$ in that portion of the continuum instantaneously occupying V is equal to the sum of the amount of the property created within V plus the flux $v_p\left[P_{ij}^{*}....(\bar{x},t)\right]$ through the bounding surface S of V.

Similarly, for any tensorial property of a continuum represented by the surface integral

$$Q_{ij}....(t) = \int_s Q_{ij}^{*}...(\bar{x},t)dS_p \quad ...(3.46)$$

where s is the surface occupied by the considered part of the continuum at time t, then

$$\frac{d}{dt}\left[Q_{ij}.....(t)\right] = \frac{d}{dt}\int_s Q_{ij}^{*}.....(\bar{x},t)ds_p = \int_s \frac{d}{dt}\left[Q_{ij}^{*}.....(\bar{x},t)ds_p\right]$$

$$= \int_s\left[\frac{dQ_{ij}^{*}.....(\bar{x},t)}{dt} + \frac{\partial v_q}{\partial x_q}Q_{ij}^{*}.....(\bar{x},t)\right]dS_p - \int_s\left[Q_{ij}^{*}.....(\bar{x},t)\frac{\partial v_q}{\partial x_q}dS_p\right] \quad ...(3.47)$$

For properties expressed in line intergral form such as

$$R_{ij}....(t) = \int_C R_{ij}^{*}.....(\bar{x},t)dx_p \quad ...(3.48)$$

the material derivative is given by

$$\frac{d}{dt}\int_C R_{ij}.....(t)dx_p = \int_c \frac{d}{dt}\left[R_{ij}^{*}.....(\bar{x},t)dx_p\right] \quad ...(3.49)$$

Differentiating the right hand integral of (3.49), we have

$$\frac{d}{dt}\int_C R_{ij}....(t)dx_p = \int_C \frac{d\left[R_{ij}^{*}....(\bar{x},t)\right]}{dt}dx_p + \int_C \frac{\partial v_p}{\partial x_q}\left[R_{ij}^{*}....(\bar{x},t)dx_q\right] \quad ....(3.50)$$

3.8 Conservation of Mass, Continuity equation

Associated with every material continuum there is the property known as mass. The amount of mass in that portion of the continuum occupying the spatial volume V at time t is given by the integral

$$m = \int_V \rho(\bar{x},t)dV \quad ....(3.51)$$

in which $\rho(\bar{x},t)$ is called the mass density.

The law of conservation of mass requires that the mass of a specific portion of a continuum remain constant and hence that the material derivative of (3.51) is zero i.e

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho(\bar{x},t)dV = \int_V\left[\frac{d\rho}{dt} + \rho\frac{\partial v_k}{\partial v_k}\right]dV = 0 \quad ...(3.52)$$

Since V is arbitrary therefore

$$\frac{d\rho}{dt} + \rho\frac{\partial v_k}{\partial x_k} = 0 \quad ...(3.53)$$

This equation is called the **continuity equation.** Using the material derivative, we get from (3.53)

$$\frac{\partial \rho}{\partial t} + (\rho v_k),k = 0 \qquad \left[\because \frac{d}{dt} \equiv \frac{d}{dt} + v_k\frac{\partial}{\partial x_k}\right] \quad ...(3.54)$$

For an incomoressible continuum the mass denisity of each particle is independent of time, so that $\frac{d\rho}{dt} = 0$ and (3.53) yields

$$v_{k,k}=0$$

The velocity field $\bar{v}(\bar{x},t)$ of an incompressible continuum can therefore be expressed by the equation

$$v_i = e_{ijk}S_{k,j}$$

in which $\bar{s}(\bar{x},t)$ is called the vector potential of $\bar{v}$.

The continuity equation can also be expressed in the Lagrangian form. The conservation of mass requires that

$$\int_{V_0} \rho_0(\bar{X},0)dV_0 = \int_V \rho(\bar{X},t)dV \quad ...(3.57)$$

where the integrals are taken over the same particles, i.e. V is the volume now occupied by the material which occupied V_0 at time t=0. Now,

$$\int_{V_0} \rho_0(\bar{X},0)dV_0 = \int_{V_0} \rho(\bar{x}(\bar{X},t),t)JdV_0 = \int_{V_0} \rho(\bar{X},t)JdV_0 \quad ...(3.58)$$

since the relationship holds for any volume V_0, it follows that

$$\rho_0=\rho J$$

i.e. the product ρJ is independent of time since V is arbitrary, or that

$$\frac{d}{dt}(\rho J) = 0 \quad ...(3.59)$$

This equation is the Lagrangian differential form of the continuity equation.

3.9 Linear momentum principle, Equation of motion, Equilibrium equation.

Consider a moving continuum which occupies the volume V at time t.

Also, let b_i be the body forces per unit mass and $t_i^{(\hat{n})}$ be the stress vector acting on the differential element ds of the bounding surface. The velocity field $v_i = \frac{du}{dt}$ is prescribed throughout the region occupied by the continuum. For this situation, the total linear momentum of the mass system within V is given by

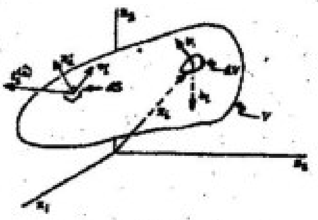


Fig. 3. 3

$$P_i(t) = \int_V \rho v_i dV \quad ...(3.60)$$

Based upon Newton's second law the principle of linear momentum states that the time rate of change of the linear momentum of an arbitrary portion of a continuum is equal to the resultant force acting upon the considered portion. Therefore, if the internal forces between particles of the continuum in Fig 3.3 obeys Newton's third law of action and reaction, the momentum principle for this mass system is expressed by

$$\int_s t_i^{(\hat{n})} ds + \int_V \rho b_i dV = \frac{d}{dt}\int_V \rho v_i dV$$

$$\text{i.e.} \int_s \sigma_{ji} n_j ds + \int_V \rho b_i dV = \frac{d}{dt}\int_V \rho v_i dV \qquad \left[\because t_i^{(\hat{n})} = \sigma_{ji} n_j\right]$$

$$\text{i.e.} \int_V (\sigma_{ji,j} + \rho b_i) dV = \frac{d}{dt}\int_V \rho v_i dV \quad ...(3.61)$$

[By using divergence theorem of Gauss]

Again, $$\frac{d}{dt}\int_V \rho v_i dV = \frac{d}{dt}\int_{V_o} \rho v_i J dV_o = \int_{V_o}\left[v_i \frac{d(\rho J)}{dt} + \rho J \frac{dv_i}{dt}\right] dV_o$$

$$= \int_V \frac{dv_i}{dt} \rho dV \quad ....(3.62)$$

Using (3.62) in (3.61) we have

$$\int_V (\sigma_{ji,j} + \rho b_i - \rho \dot{v}_i) dV = 0 \quad ...(3.63)$$

Since V is arbitrary, therefore

$$\sigma_{ji,j} + \rho b_i = \rho \dot{v}_i \quad ...(3.64)$$

These equations are called **equations of motion.**

In case of static equilibruim, The acceleration components vanishes therefore (3.64) become

$$\sigma_{ji,j} + \rho b_i = 0 \quad ...(3.65)$$

These are the equilibrium equations, used extensively in solid mechanies.

3.10 Moment of momentum (Angular Momentum) principle.

The moment of momentum principle is simply the moment of linear momentum with respect to some point. Thus for the continuum shown in Fig 3.3, the total moment of momentum (or, angular momentum) with respect to the origin is

$$N_i(t) = \int_V e_{ijk} x_j \rho v_k dV \qquad ...(3.66)$$

where x_i is the position vector of the volume element dV. The moment of momentum principle states that the time rate of change of the angular momentum of any portion of a continuum with respect to an arbitrary point is equal to the resultant moment with respect to that point of the body and surface forces acting a the considered portion of the continuum. According for the continuum of fig (3.3) the moment of momentum principal is expressed by

$$\int_S \varepsilon_{ijk} x_j t_k^{(\hat{n})} ds + \int_V \varepsilon_{ijk} x_j \rho b_k dV = \frac{d}{dt}\int_V \varepsilon_{ijk} x_j \rho v_k dV \quad ....(3.67)$$

Equation (3.67) is valid for those continua in which the forces between particles are equal, opposite and collinear, and in which distributed moments are absent.

The moment of momentum principle does not furnish any new differential equation of motion.

3.11 Conservation of mass, continuity equation.

Associated with every material continuum there is the property known as mass. The amount of mass in that portion of the continuum accupying the spatial volume V at time t is given by the integral.

$$m = \int_V \rho(x,t) dV \rightarrow (3.1)$$

in which $\ell(x,t)$ is a continuous function of the co-ordinates called the mass density. The law of conservation of mass requires that the mass of a specific portion of the continuum remain constant, and hence that the material derivative of (3, 1) be zero. Therefore the rate of change of m in (3, 1) is

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho(x,t) dv = \int_V \left[\frac{dp}{dt} + \rho \frac{\delta v_k}{\delta x_k}\right] dv = 0 \rightarrow (3.2)$$

Since this equation holds form arbitrary volume V, the integrand must vanish, or

$$\frac{d\rho}{dt} + \rho V_{k,k} = 0$$

$$\text{or, } \frac{d\rho}{dt} + \rho(\nabla, V) = 0 \rightarrow (3.3)$$

This equation is called the continuity equation, using the material derivative operator if may be put into the alternative form

$$\frac{\delta\rho}{\delta t}+(\rho,\vartheta_k),k=0$$

$$\text{or, } \frac{\delta\rho}{\delta t}+,\nabla.(ev)=0\rightarrow(3.4)$$

For an incompressible continuum the mass density of each particle in independent of time, so that $\frac{\delta\rho}{\delta t}=0$ and (3.2) yields the result

$$v_{k,k}=0\rightarrow(3.5)$$

The velocity field $v(x_i t)$ of an incompressible continuum can therefore be exprienced by the equation

$$v_i=\sigma_{ijk}S_{k,j}\rightarrow(3.6)$$

or, which $s(x_i t)$ is called the vector potential of υ.

The continuety equation may also be expressed in the Lagrangian, or material form. The conservation of mass requires that

$$\int_{V_0}\rho_0(X,0)dV_0=\int_{V_0}(x(x,t),t)JdV_0$$

$$=\int_{V_0}\rho(X_1 t)JdV_0\rightarrow(3.8)$$

Since this relationship must hold for any volume V_0, if follows that

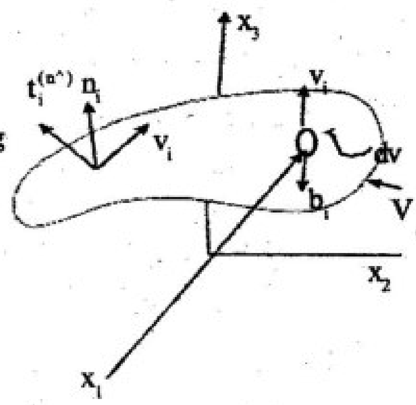
$$\rho_0=\rho J\rightarrow(3.9)$$

Which implies that the product ρJ is independent of time since V is arbitrary, or that

$$\frac{d}{dt}(PJ)=0\rightarrow(3.10)$$

Equation (3.10) is the Lagrungian differential form of the contineuity equation.

3.12 Linear momentum principle, Equation of motion, Equilibrium equations :



A moving continuum which occerpies the volume V at time t is shown in the fig. Body force b_i per unit mass are given. On the differential element ds of the bounding surface, the stress vector is $t_i^{(n^\wedge)}$. The velocity field $v_i=\frac{du_i}{df}$ is prescribed throughout the region occupied by the continuum. For this situation, the total linear momentum if the mass system within V is given by

$$P_i(t) = \int_V \rho v_i dV \rightarrow (3.11)$$

Based upon Newton's second law, the principle of linear momentum states that the time rate of change of an arbitrary portion of a continuum is equal to the resuetant force acting upon the considered portion. Therefore.if the interal force between particles of the continuum in Fig 5. obey Newton's 3rd law of action and reaction, the momentum principle for this mass system is expressed

$$\int_S t_i^{(n^\wedge)} ds + \int_V \rho b_i dV = \frac{d}{dt}\int_V P v_i dV \rightarrow (3.12)$$

upon substituting $t_i^{(n^\wedge)} = \delta_{ji} n_j$ into the first integral of (3.12) and converting the resulting surface integral by the divergience theorem of Gauss (3.12) becomes

$$\int_V (\delta_{ji,j} + \rho b_i) dV = \frac{d}{df}\int_V \rho v_i dV \rightarrow (3.13)$$

In calculating the material derivative is (3.13) the continuity equation in the form given by (3.10) may be used. Thus

$$\frac{d}{dt}\int_V \rho v_i dV = \frac{d}{dt}\int_{V_0} \rho v_i J dV_0$$

$$= \int_{V_0}\left[v_i \frac{d}{dt}(\rho J) + \rho J \frac{dv_i}{dt}\right] dV_0$$

$$= \int_V \frac{dv_i}{dt} \rho dV \frac{dv_i}{dt} \rightarrow (3.14)$$

Replacing the RHS of (3.13) by the R.H.S of (3.14) and collecting terms results in the linear momentum principle in integral form.

$$\int_V \left(\sigma_{ji,j} + \rho b_i - \rho v_i\right) dV = 0 \rightarrow (3.15)$$

Since the volume V is arbitrary, the integrand of (3.15) must vanish. The resulting equations

$$\sigma_{ji,j} + \rho b_i - \rho v_i \rightarrow (3.16)$$

are known as the equation of motion.

The important case of static equilibrium, in which the acveleration components vanish, is given at once from (3.16) as

$$\sigma_{ji,j} + \rho b_i = 0 \rightarrow (3.17)$$

These are equilibrium equations, used extensively in solid mechanis.

3.13 Moment of momentum (Angular momentum) principle :

The moment of momentum is, as the name implies, simply the moment of linear momentum with respect to some point. Thus for the continuum shown in Fig 5, the total moment of momentum or angular momentum as it is often called, ... to the origin is

$$N_i(t) = k\int_V \in_{ijk.}{}^{k}\ j\rho v_k \quad / \rightarrow (3.18)$$

in which x_j is the positive vector of the volume element dV. The moment of momentum principle states that the time rate of change of the angular momentum of any position of a continuum with respect to an arbitrary point is equal to the resuetant moment (with respect to that point) of the body and surface force deting on the considered portion of the continuum. Accordingly for the of fig. 5 the moment of momentum principle is expressed in integral form by

$$\int_S \in_{ijk} x_j t_k^{(n^\wedge)} ds + \int_V \in_{ijk} x_j \rho b_k dV$$

$$= \frac{d}{dt}\int_V \in_{ijk} x_j \rho v_k dV \rightarrow (3.19)$$

Equation (3.19) is valid for those continuum in which the forces between particles are equal, opposite and collinear, and in which distributed moments are absent.

The moment of momentum principle does not furnish any new differential equation of motion. If the substitutions $t_k^{(n^\wedge)} = \sigma_{pk} n_p$ is made in (3.19) and the symmetry of the stress tensor assuemed, the equation is satisfied identically by using the relationship given in (3.16). If stress symmetry is not assumed, such symmetry may be shown to follow directly from (3.19), which upon substitation of $t_k^{(n^\wedge)} = \sigma_{pk} n_p$ reduces to

$$\int_V \in_{ijk} \sigma_{jk} dV = 0 \rightarrow (3.20)$$

Since the volume V is arbitrary

$$\in_{ijk} \delta_{jk} = 0 \rightarrow (3.21)$$

Which by expansion demonstrates that $\delta_{jk} = \delta_{kj}$

3.14 Energy Equation

If mechanical quantities only are considered, the principle of conservation of energy for the continuum of Fig (3.3)may be derived directly from the equation of motion given by (3.64). To accomplish this, the scalar product between (3.16) and the velocity v_i is first computed, and the result integrated over the volume V. Thus

$$\int_V \rho v_i \dot{v}_i dV = \int_V v_i \sigma_{ji,j} dV + \int_V \rho v_i b_i dV \qquad ...(3.68)$$

But $\int_V \rho v_i \dot{v}_i dV = \frac{d}{dt}\int_V \rho \frac{v_i v_i}{2} dV = \frac{d}{dt}\int_V \frac{\rho v^2}{2} dV = \frac{dK}{dt}$ ...(3.69)

which represents the time rate of change of the Kinetic energy K in the continuum. Also,

$$v_i \sigma_{ji,j} = (v_i \sigma_{ji})_{,j} - v_{i,j}\sigma_{ji}$$
$$= (v_i \sigma_{ji})_{,j} - (D_{ij} + V_{ij})\sigma_{ji}$$
$$= (v_i \sigma_{ji})_{,j} - D_{ij}\sigma_{ji} \qquad [\because V_{ij}\sigma_{ji} = 0]$$

Thus (3.68) may be writen in the form

$$\frac{dk}{dt} = \int_V [(v_i \sigma_{ji})_{,j} - D_{ij}\sigma_{ji}] dV + \int_V \rho b_i v_i dV$$

i.e. $$\frac{dk}{dt} + \int_V D_{ij}\sigma_{ji} dV = \int_V (v_i \sigma_{ji})_{,j} dV + \int_V \rho b_i v_i dV$$

$$= \int_S v_i t_i^{(\hat{n})} ds + \int_V \rho b_i v_i dV \qquad ...(3.70)$$

[Using divergence theorem of Gauss]

This equation is known as the **energy equation for a continuum.** This equation relates the time rate of change of total machanical energy of the continuum on the left side to the rate of work done by the surface and body forces, on the right hand side of the equation. The integral on the left side is known as the time rate of change of intrnal mechanical energy, and written as $\frac{dU}{dt}$. Therefore, (3.70) may be written as

$$\frac{dk}{dt} + \frac{dU}{dt} = \frac{đW}{dt} \qquad ...(3.71)$$

Where $\frac{đW}{dt}$ represents the rate of work and the symbol $đ$ is used to indicate that this quantity is not an exact differential.

If both mechanical and non-mechanical energies are to be considered, the principle of conservation of energy in its most general form must be used. In this form the conservation principle states that the time rate of change of the kinetic plus the internal energy is equal to the sum of the rate of work plus all other energies supplied to or, removed form the continuum per unit time. Such energies supplied may include thermal energy, chemical energy, or electromagnetic energy. Here, only mechanical and thermal energies are considered, and the energy principle takes on the form of the well-known **first law of thermodynamics.**

For a thermo nechanical continuum, the time rate of change of internal energy may be expressed as

$$\frac{dU}{dt} = \frac{d}{dt}\int_V \rho u dV = \int_V \rho \dot{u} dV \qquad ...(3.72)$$

where u is called the specific internal energy. Also, if the vector c_i is defined as the heat flux per unit area per unit time by conduction, and z is taken as the radiant heat constant per unit mass per unit time, the rate of increase of total heat into the continuum is given by

$$\frac{dQ}{dt} = -\int_S c_i n_i ds + \int_V \rho z dV \qquad ...(3.73)$$

Therefore the energy principle for a thermomechanical continuum is given by

$$\frac{dk}{dt} + \frac{dU}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} \qquad ...(3.74)$$

or, in terms of energy integrals as

$$\frac{d}{dt}\int_V \rho \frac{v_i v_i}{2} dV + \int_V \rho \dot{u} dV = \int_S t_i^{(\hat{n})} v_i ds + \int_V \rho v_i b_i dV + \int_V \rho z dz - \int_S c_i n_i ds \qquad ...(3.75)$$

Converting the surface integrals in (3.75) to volume integrals by the divergence theorem of Gauss, and again using the fact that v is arbitrary, leads to the local form of the energy equation

$$\frac{d}{dt}\left(\frac{v^2}{2} + u\right) = \frac{1}{\rho}(\sigma_{ij} v_i)_{,j} + b_i v_i - \frac{1}{\rho} c_{i,i} + z \qquad ...(3.76)$$

Within the arbitrary small volume element for which the local energy equation (3.76) is valid the balance of momentum given by (3.64) must also hold. Therefore by taking the scalar product between (3.64) and the velocity $\rho \dot{v}_i v_i = v_i \sigma_{ji,j} + \rho v_i b_i$ and, after some simple manupulations, substracting this product from (3.76), the result is the reduced, but highly useful form of the local energy equation,

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij} D_{ij} - \frac{1}{\rho} c_{i,i} + z \qquad ...(3.77)$$

This equation expresses the rate of change of internal energy as the sum of the stress power plus the heat added to the continuum.

Solved Problems

Exp 1. A velocity field is given by $v_1 = \frac{x_1}{(1+t)}$, $v_2 = \frac{2x_2}{(1+t)}$, $v_3 = \frac{2x_3}{(1+t)}$. Determine the acceleration components for this motion.

Solution: The acceleration components are defined by

$$a_i = \dot{v}_i = \frac{dv_i}{dt} = \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}$$

$$\therefore a_1 = \frac{\partial v_1}{\partial t} + v_k \frac{\partial v_1}{\partial x_k}$$

$$= \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}$$

$$= -\frac{x_1}{(1+t)^2} + \frac{x_1}{(1+t)^2} = 0$$

$$\therefore a_2 = \frac{\partial v_2}{\partial t} + v_k \frac{\partial v_2}{\partial x_k}$$

$$= -\frac{2x_2}{(1+t)^2} + \frac{4x_2}{(1+t)^2} = \frac{2x_2}{(1+t)^2}$$

$$\therefore a_3 = \frac{\partial v_3}{\partial t} + v_k \frac{\partial v_3}{\partial x_k}$$

$$= -\frac{3x_3}{(1+t)^2} + \frac{9x_3}{(1+t)^2} = \frac{6x_3}{(1+t)^2}$$

Exp 2 The motion of a continuum is given by

$$x_1 = A + \frac{e^{-B\lambda}}{\lambda} \sin \lambda(A + \omega t)$$

$$x_2 = -B - \frac{e^{-B\lambda}}{\lambda} \cos \lambda(A + \omega t), \qquad x_3 = X_3.$$

Show that the particle paths are circles and that the velocity magnitude is constant. Also, determine the relationship between X_1 and X_2 and the constants A and B.

Solution By given condition

$$x_1 - A = \frac{e^{-B\lambda}}{\lambda} \sin \lambda(A + \omega t)$$

$$x_2 + B = -\frac{e^{-B\lambda}}{\lambda} \cos \lambda(A + \omega t),$$

Squaring and adding, t is eliminated and the path lines are the circles

$$(x_1 - A)^2 + (x_1 - B)^2 = e^{-2B\lambda} / \lambda$$

$$v_1 = \frac{dx_1}{dt} = \omega e^{-B\lambda}\cos\lambda(A+\omega t)$$

$$v_2 = \frac{dx_2}{dt} = \omega e^{-B\lambda}\sin\lambda(A+\omega t)$$

$$v_3 = \frac{dx_3}{dt} = 0$$

$$\therefore v^2 = v_1^2 + v_2^2 + v_3^2 = \omega^2 e^{-2B\lambda}$$

Finally, when $t=0$, $x_i = X_i$ and so

$$X_1 = A + \left(e^{-B\lambda}/\lambda\right)\sin\lambda A$$
$$X_2 = B - \left(e^{-B\lambda}/\lambda\right)\cos\lambda A$$

Exp 3 A velocity field is described by $v_1 = \frac{x_1}{1+t}$, $v_2 = \frac{2x_2}{1+t}$, $v_3 = \frac{3x_3}{1+t}$. Determine the stream lines and path lines of the flow and show that they coincide.

Solution The differential equations of stream lines are

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3}$$

For the given flow, this equation becomes

$$\frac{dx_1}{x_1} = \frac{dx_2}{2x_2} = \frac{dx_3}{3x_3}$$

Integrating and using the conditions $x_i = X_i$ at $t=0$, the equations of stream lines are

$$\left(\frac{x_1}{X_1}\right)^2 = \frac{x_2}{X_2}, \left(\frac{x_1}{X_1}\right)^3 = \frac{x_3}{X_3}, \left(\frac{x_2}{X_2}\right)^3 = \frac{x_3}{X_3}$$

Again, the differential equations of path lines are

$$\frac{dx_i}{dt} = v_i$$

i.e. $\frac{dx_i}{v_i} = dt$

i.e. $\frac{dx_1}{v_1} = dt, \quad \frac{dx_2}{v_2} = dt, \quad \frac{dx_3}{v_3} = dt$

Integrating the above expressions we have $x_1 = X_1(1+t)$, $x_2 = X_2(1+t)^2$, $x_3 = X_3(1+t)^2$.

Elimination of t from these equations gives the path lines which are identical with the stream lines presented above.

Exp 4 For the steady velocity field $\bar{v} = 3x_1^2x_2\hat{e}_1 + 2x_2^2x_3\hat{e}_2 + x_1x_2x_3^2\hat{e}_3$, determine the rate of extension at P(1,1,1) in the direction of $\hat{v} = \frac{(3\hat{e}_1 - 4\hat{e}_3)}{5}$.

Also, determine the of shear at P between the orthogonal directions $\hat{v}$ and $\hat{\mu} = \frac{(4\hat{e}_1 + 3\hat{e}_3)}{5}$.

Solution. The velocity gradient is

$$\left[\frac{\partial v_i}{\partial x_j}\right] = \begin{bmatrix} 6x_1x_2 & 3x_1^2 & 0 \\ 0 & 4x_2x_3 & 2x_2^2 \\ x_2x_3^2 & x_1x_3^2 & 2x_1x_2x_3 \end{bmatrix}$$

Now, $\frac{\partial v_i}{\partial x_j} = D_{ij} + V_{ij}$, where $D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ is the rate of deformation tensor and $V_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$ is the vorticity or spin tensor.

Now, at P(1,1,1),

$$[D_{ij}] = \begin{bmatrix} 6 & 1\cdot5 & 0\cdot5 \\ 1\cdot5 & 4 & 1\cdot5 \\ 0\cdot5 & 1\cdot5 & 2 \end{bmatrix}$$

The rate of extension is $d = D_{ij}v_iv_j$

i.e. $$d = \left[\tfrac{3}{5}, \; 0, \; -\tfrac{4}{5}\right]\begin{bmatrix} 6 & 1\cdot5 & 0\cdot5 \\ 1\cdot5 & 4 & 1\cdot5 \\ 0\cdot5 & 1\cdot5 & 2 \end{bmatrix}\begin{bmatrix} \tfrac{3}{5} \\ 0 \\ -\tfrac{4}{5} \end{bmatrix} = \frac{74}{25}$$

and the rate of shear is $\dot{\gamma}_{\mu v} = 2D_{ij}\mu_iv_j$

i.e. $$\dot{\gamma}_{\mu v} = \left[\tfrac{4}{5}, \; 0, \; \tfrac{3}{5}\right]\begin{bmatrix} 12 & 3 & 1 \\ 3 & 8 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} \tfrac{3}{5} \\ 0 \\ -\tfrac{4}{5} \end{bmatrix} = \frac{89}{25}$$

Exp 5 Show that the material from $\frac{d(\rho J)}{dt} = 0$ of the continuity equation and the spatial form

$\frac{d\rho}{dt} + \rho v_{k,k} = 0$ are equivalent.

Solution: $\frac{d(\rho J)}{dt} = 0 \Leftrightarrow \frac{d\rho}{dt} J + \rho \frac{dJ}{dt} = 0$

$$\Leftrightarrow \frac{d\rho}{dt} J + \rho J \frac{\partial v_k}{\partial x_k} = 0 \qquad \left[\frac{dJ}{dt} = J v_{k,k}\right]$$

$$\Leftrightarrow J\left(\frac{d\rho}{dt} + \rho \frac{\partial v_k}{\partial x_k}\right) = 0$$

$$\Leftrightarrow \frac{d\rho}{dt} + \rho \frac{\partial v_k}{\partial x_k} = 0 \qquad [\because J \neq 0]$$

Exp 6 At a certain point in a continuum the rate of deformation and stress tensors are given by

$$D_{ij} = \begin{pmatrix} 1 & 6 & 4 \\ 6 & 3 & 2 \\ 4 & 3 & 5 \end{pmatrix} \quad \text{and} \quad \sigma_{ij} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & -2 & 7 \\ -1 & 7 & 8 \end{pmatrix}$$

Determine the value λ of the stress power $D_{ij}\sigma_{ij}$ at the point.

Solution Multiplying each element of D_{ij} by its counterpart in σ_{ij} and adding we get the value λ of the stress power $D_{ij}\sigma_{ij}$ as $\lambda = 4 + 0 - 4 + 0 - 6 + 14 - 4 + 14 + 40 = 58$

Exp 7 If $\sigma_{ij} = -p\delta_{ij}$ where p is a positive constant, show that the stress power may be expressed by the equation $D_{ij}\sigma_{ij} = \frac{p}{\rho}\frac{d\rho}{dt}$.

Solution We know

$$\frac{\partial v_i}{\partial x_j} = D_{ij} + V_{ij}$$

$$\therefore D_{ij} = v_{i,j} - V_{ij}$$

Thus $D_{ij}\sigma_{ij} = v_{i,j}\sigma_{ij} - V_{ij}\sigma_{ij}$

$$= v_{i,j}\sigma_{ij} \qquad [\because V_{ij}\sigma_{ij} = 0]$$

$$= v_{i,j}(-p\delta_{ij})$$

$$= -pv_{i,i}$$

$$= \frac{p}{\rho}\frac{d\rho}{dt} \qquad \left[\begin{array}{l}\because \text{By continuity equation} \\ \frac{d\rho}{dt} + \rho v_{i,i} = 0\end{array}\right]$$

Exp 8 A two dimensional incompressible flow is given by $v_1 = A(x_1^2 - x_2^2)/r^4, v_2 = A(2x_1x_2)/r^4, v_3 = 0$ where $r^2 = x_1^2 + x_2^2$. Show that the continuity equation is satisfied by the motion.

Solution For incompressible flow, the equation of continuity is

$$v_{i,i} = 0 \qquad \text{i.e. } v_{1,1} + v_{2,2} = 0$$

Here $v_{1,1} = \dfrac{\partial v_1}{\partial x_1} = A\left[\dfrac{-4x_1(x_1^2 - x_2^2)}{r^6} + \dfrac{2x_1}{r^4}\right]$

and $\quad v_{2,2} = \dfrac{\partial v_2}{\partial x_2} = A\left[\dfrac{2x_1}{r^4} - \dfrac{8x_1x_2^2}{r^6}\right]$

Adding, $v_{1,1} + v_{2,2} = 0$

Exp 9 For a continuum whose constitutive equations are $\sigma_{ij} = (-p + \lambda^* D_{kk})\delta_{ij} + 2\mu^* D_{ij}$, determine the equations of motion in terms of the velocity v_i.

Solution The equation of motion is

$$\begin{aligned}
\rho \dot{v}_i &= \sigma_{ij,j} + \rho b_i \\
&= \rho b_i + [(-p + \lambda^* D_{kk})\delta_{ij} + 2\mu^* D_{ij,j}] \\
&= \rho b_i - p_{,j}\delta_{ij} + \lambda^* D_{kk,j}\delta_{ij} + 2\mu^* D_{ij,j} \\
&= \rho b_i - p_{,i} + \lambda^* v_{k,ki} + 2\mu^* \tfrac{1}{2}(v_{i,jj} + v_{j,ij}) \\
&= \rho b_i - p_{,j} + (\lambda^* + \mu^*) v_{j,ij} + \mu^* v_{i,jj} \qquad [\because D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})]
\end{aligned}$$

Exp 10 Show that $q_i = e_{ijk}V_{kj}$ and that $2V_{ij} = e_{ijk}q_k$.

Solution The velocity vector is defined by

$$\begin{aligned}
\bar{q} &= \bar{\nabla}_x \times \bar{v} \\
\text{i.e. } q_i &= e_{ijk} v_{k,j} \\
&= e_{ijk}\left[D_{kj} + V_{kj}\right]
\end{aligned}$$

where D_{kj} is the rate of deformation tensor and V_{kj} is the velocity or spin tensor.

$$q_i = e_{ijk} V_{kj} \qquad [\text{since } e_{ijk} D_{kj} = 0]$$

$$\therefore e_{irs} q_i = e_{irs} e_{ijk} V_{kj}$$
$$= (\delta_{rj}\delta_{sk} - \delta_{rk}\delta_{sj}) V_{kj}$$
$$= \delta_{rj}\delta_{sk} V_{kj} - \delta_{rk}\delta_{sj} V_{kj}$$
$$= V_{sr} - V_{rs}$$
$$= V_{sr} + V_{sr} \qquad [\because V_{rs} = -V_{sr}]$$
$$= 2V_{sr}$$

Example 11 : A continuum motion is given by

$$x_1 = Xe^t + X_3(e^t - 1),\ x_2 = X_2 + X_3(e^t - e^{-t}),\ X_3 = X_3.$$

Show that the jacobian J does not vanish for this motion and obtain the velocity acceleration components.

Solve :

We have

$$j = \left|\frac{\partial x_i}{\partial x_j}\right| = \begin{vmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{vmatrix}$$

$$= \begin{vmatrix} e^t & 0 & e^t - 1 \\ 0 & 1 & e^t - e^{-t} \\ 0 & 0 & 1 \end{vmatrix}$$

$$= e^t \neq 0$$

Now $u_i = x_i - X_i$

In material form

$$u_1 = x_1 - X_1 = (X_1 + X_3)(e^t - 1)$$
$$u_2 = X_3 (e^t - e^{-t})$$
$$u_3 = 0$$

$$\text{and } v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t}$$

$$v_1 = \frac{\partial u_1}{\partial t} = (X_1 + X_3)e^t$$

$$v_2 = \frac{\partial u_2}{\partial t} = X_3(e^t + e^{-t})$$

$$v_3 = \frac{\partial u_3}{\partial t} = 0$$

In spatial form

$$u_1 = x_1 - X_1 = (1 - e^{-t})(x_1 + x_3)$$

$$u_2 = x_3(e^t - e^{-t}), u_3 = 0$$

$$\text{and } v_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + v_k \frac{\partial u_i}{\partial x_k}$$

$$v_1 = \frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x_1} + v_2 \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_3}$$

$$\Rightarrow v_1 = e^{-t}(x_1 + x_3) + (1 - e^{-t})v_1$$

$$\Rightarrow v_1 = x_1 + x_3$$

$$v_2 = x_3(e^t + e^{-t})$$

$$v_3 = 0$$

Again

$$a_i = \frac{dv_i}{dt} = \frac{\partial v_i}{dt} + \vartheta_k \frac{\partial v_k}{\partial x_k}$$

$$\therefore a_1 = \frac{dv_i}{\partial t} + v_k \frac{\partial v_1}{\partial x_k}$$

$$= \frac{dv_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}$$

$$= x_1 + x_3$$

$$a_2 = x_3\ (e^t - e^{-t})$$

$$a_3 = 0$$

Supplementary Problems

1. A continuum motion is given by $x_1 = X_1e^t + X_3(e^t - 1)$, $x_2 = X_2 + X_3(e^t - e^{-t})$, $x_3 = X_3$. Show that the Jacobian J does not vanish for this motion and obtain the velocity components.

Ans. $v_1 = (X_1 + X_3)e^t$, $v_2 = X_3(e^t + e^{-t})$, $v_3 = 0$

$v_1 = x_1 - x_3$, $v_2 = x_3(e^t + e^{-t})$, $v_3 = 0$

2. A velocity field is specified in Lagrangian form by $v_1 = -X_2 e^{-t}$, $v_2 = -X_3$, $v_3 = 2t$. Determine the acceleration components in Eulerian form.

Ans. $a_1 = e^{-t}(x_2 + tx_3 - t^3)$, $a_2 = 0$, $a_3 = 0$.

3. Show that for the flow $v_i = \frac{x_i}{1+t}$ the streamlines and pathlines coincide.

4. A steady velocity field is given by $v_1 = 2x_3$, $v_2 = 2x_3$, $v_3 = 0$. Determine the principal directions and principal values of the rate of deformation tensor for this motion.

Ans.

$$[a_{ij}] = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

and principal values are $\left(\sqrt{2}, \quad 0, \quad -\sqrt{2}\right)$

5. Calculate the second material derivative of the scalar product of two line elements i.e. determine $\frac{d^2(dx^2)}{dt^2}$.

Ans. $$\frac{d^2(dx^2)}{dt^2} = 2\left[\frac{dD_{ij}}{dt} + D_{kj}\frac{\partial v_k}{\partial x_i} + D_{ik}\frac{\partial v_k}{\partial x_j}\right]dx_i dx_j$$

6. Show that $\frac{d(l_n J)}{dt} = \operatorname{div}\bar{v}$.

7. Show that for steady motion $\left(\frac{\partial v_i}{\partial t} = 0\right)$ of a continuum the stream lines and path lines coincide.

8. Taking the material derivative of ds_i in its cross product form $ds_i = e_{ijk} dx_j^{(2)} dx_k^{(3)}$, show that

$$\frac{ds_p}{dt} = \left(\frac{\partial v_q}{\partial x_q}\right)ds_p - \left(\frac{\partial v_q}{\partial x_p}\right)ds_p$$

9. Determine the form of the continuity equation for an irrotational motion.

10. Show that the velocity field $v_i = \frac{Ax_i}{r^3}$, where $x_i x_i = r^2$ and A is an arbitrary constant, satisfies the continuity equation for an incompressible flow.

11. Determine the material rate of change of kinetic energy of the continuum which occupies the volume V and give the meaning of the resulting integrals.

12. Show that the constitutive equation $\sigma_{ij}=\lambda^*\delta_{ij}D_{pp}+2\mu^*D_{ij}$ may split into the equivalent equations $\sigma_{ii}=(3\lambda^*+2\mu^*)D_{ii}$ and $S_{ij}=2\mu^* D'_{ij}$ where S_{ij} and D'_{ij} are the deviators tensors of stress and rate of deformation respectively.

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Unit IV
Equations of Elasticity

4.1. Generalized Hooke's low, Strain Energy Function.

In classical linear elasticity theory, it is assumed that the dispalcements and displacements gradients are sufficiently small that no distinction need be made between the Lagrangian and Eulerian descriptions. Accordingly, the linear strain tensor is given by the equivalent expressions

$$l_{ij} = \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad ...(4.1)$$

Here it is further assumed that the deformation processes are adiabatic (no heat loss or gain) and isothermal (constant temperature) unless specifically stated otherwise.

The constitutive equations for a linear elastic solid relate the stress and strain tensors through the relationship

$$\sigma_{ij} = C_{ijkm} e_{km} \quad ...(4.2)$$

which is known as the generalized Hooke's low. In (4.2) the tensor of elastic constants C_{ijkm} has 81 components. However, due to symmetry of both stress and strain tensors, there are at most 36 distinct elastic constants. For writing Hooke's low in terms of these 36 components, the double indexed system of stress and strain components is often replaced by a single indexed system having a range of 6. Thus in the notation

$$\begin{aligned} &\sigma_{11} = \sigma_1 \quad , \quad \sigma_{23} = \sigma_{32} = \sigma_4 \\ &\sigma_{22} = \sigma_2 \quad , \quad \sigma_{31} = \sigma_{13} = \sigma_5 \\ &\sigma_{33} = \sigma_3 \quad , \quad \sigma_{12} = \sigma_{21} = \sigma_6 \end{aligned} \quad ...(4.3)$$

and

$$\begin{aligned} &\varepsilon_{11} = \varepsilon_1 \quad , \quad 2\varepsilon_{23} = 2\varepsilon_{32} = \varepsilon_4 \\ &\varepsilon_{22} = \varepsilon_2 \quad , \quad 2\varepsilon_{31} = 2\varepsilon_{13} = \varepsilon_5 \\ &\varepsilon_{33} = \varepsilon_3 \quad , \quad 2\varepsilon_{12} = 2\varepsilon_{21} = \varepsilon_6 \end{aligned} \quad ...(4.4)$$

Hooke's law may be written as

$$\sigma_k = C_{KM}\varepsilon_M (K, M = 1,2,3,4,5,6) \quad ..(4.5)$$

Where C_{km} represents 36 elastic constants.

When thermal effects are neglected the energy balance equation may be written as

$$\frac{du}{dt} = \frac{1}{\rho}\sigma_{ij}D_{ij} = \frac{1}{\rho}\sigma_{ij}\dot{\varepsilon}_{ij} \quad ...(4.6)$$

The internal energy in this case is purely mechanical and is called the strain energy (per unit mass). from (4.6),

$$du = \frac{1}{\rho}\sigma_{ij} d\varepsilon_{ij} \quad ...(4.7)$$

and if u is considered a function of the nine strain components, $u = u(\varepsilon_{ij})$, its differential is given by

$$du = \frac{\partial u}{\partial \varepsilon_{ij}} d\varepsilon_{ij} \qquad ...(4.8)$$

Companing (4.7) and (4.8) we observe that

$$\frac{1}{\rho}\sigma_{ij} = \frac{\partial u}{\partial \varepsilon_{ij}} \qquad ...(4.9)$$

The strain energy density u^* (per unit volume) is defined as

$$u^* = \rho u \qquad ...(4.10)$$

and since ρ may be considered a constant in the small strain theory, u^* has the property that

$$\sigma_{ij} = \rho\frac{\partial u}{\partial \varepsilon_{ij}} = \frac{\partial u^*}{\partial \varepsilon_{ij}} \qquad ..(4.11)$$

Furthermore, the zero state of strain energy may be chosen arbitrarily ; and the stress must vanish with the strains , the simplest form of strain energy function that leads to a linear stress-strain relation is the quadratic form

$$u^* = \frac{1}{2}C_{ijkm}\varepsilon_{ij}\varepsilon_{km} \qquad ...(4.12)$$

From (4.2), this equation may be written as

$$u^* = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad ...(4.13)$$

In the single indexed system of symbols, (4.12) becomes

$$u^* = \frac{1}{2}C_{km}\varepsilon_K\varepsilon_M \qquad ...(4.14)$$

In which $C_{kM}=C_{Mk}$. Because of the symmetry on C_{kM}, the number of independent elastic constants is at most 21 if a strain energy function exists.

4.2. Isotropy, Anisotropy, Elastic symmetry.

If the elastic properties are independent of the reference system used to describe it, a material is said to be elastically isotropic. A material that is not isotropic is called anisotropic. Since the elastic properties of a Hookian solid are expressed through the coefficients C_{kM}, a general anisotropic body will have an elastic constant matrix of the form

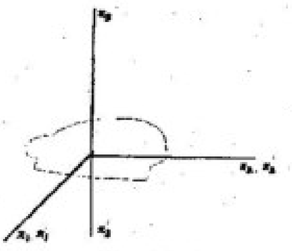

Fig. 4.1

$$[C_{KM}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \quad ...(4.15)$$

When a strain energy function exists for the body, $C_{KM} = C_{MK}$, and the 36 constants in (4.15) are reduced to 21.

A plane of elastic symmetry exits at a point where the elastic constants have the same values for every pair of coordinate systems which are the reflected images of one another with respect to the plane.

The axes of such coordinate system are referred to as "equivalant elastic directions". If the x_1x_2 plane is one of elastic symmetry, the constants C_{KM} are invariant under the coordinate transformation

$$x_1' = x_1, \quad x_2' = x_2, \quad x_3' = -x_3 \quad ...(4.16)$$

as shown in Fig 4.1. The tranformation matrix of (4.16) is given by

$$[a_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad ..(4.17)$$

The elastic matrix for a material having x_1x_2 as a plane of symmetry is

$$[C_{km}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{21} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{31} & C_{32} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{54} & C_{55} & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & C_{66} \end{bmatrix} \quad ...(4.18)$$

The 20 constants in (4.18) are reduced to 13 when a strain energy function exists.

If a material possesses three mutually perpendicular planes of elastic symmetry, the material is called orthotropic and its elastic matrix is of the form

$$[C_{km}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \qquad ...(4.19)$$

having 12 independent constants, or 9 if $C_{KM} = C_{MK}$.

An axis of elastic symmetry of order N exits at a point when there are sets of equivalent elastic directions which can be superimposed by a rotation through an angle of $2\Pi/N$ about the axis.

4.3. Isotropic madia, Elastic Constants

Bodies which are elastically equivalent in all directions possess complete symmetry and are termed isotropic.

Every plane and every axis is one of elastic symmetry in this case. For isotropy, the number of elastic constants reduce to 2, and the elastic matrix is symmetric regardless of the existence of a strain energy function. Choosing as the two independent constants the well-known Lame' constants, λ and μ, the matrix (6.19) reduces to the isotropic elastic form

$$[C_{km}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \qquad ...(4.20)$$

In terms of λ and μ, Hooke's law (4.2) for an isotropic body is written as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \qquad ...(4.21)$$

This equation may be inverted to express the strains in terms of stresses as

$$\varepsilon_{ij} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk} + \frac{1}{2\mu}\sigma_{ij} \qquad ...(4.22)$$

For a simple uniaxial state of stress in the x_1 direction, engineering constants E and ν may be introduced through the relationships $\sigma_{11} = E\varepsilon_{11}$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$. The constant E is known as young's modulus and ν is called Poisson's ratio. In terms of these elastic constants. Hooke's law for isotropic bodies becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \qquad ...(4.23)$$

or, when inverted

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \qquad ...(4.24)$$

From a consideration of a uniform hydrostatic pressure state of stress, it is possible to define the bulk modulus,

$$K = \frac{E}{3(1-2\nu)} \quad \text{or} \quad K = \frac{3\lambda + 2\mu}{3} \qquad ...(4.25)$$

which relates the pressure to the cubical dilatation of a body so loaded. For a so called state of pure shear, the shear modulus G relates the shear components of stress and strain. G is actually equal to μ and the expression

$$\mu = G = \frac{E}{2(1+\nu)} \qquad ...(4.26)$$

may be easily established.

4.4 Elastostatic problems, Elastodynamic problems–

In an elastostatic problem of a homogreueous isolopic body, certain field equations, namely

(a) Equilibrium equations,

$$\sigma_{ji,j} + \rho b_i = 0 \rightarrow (4.27)$$

(b) Hooke's law

$$\sigma_{ij} = \lambda\delta_{ij} \in_{kk} + 2u \in_{ij} \rightarrow (4.28)$$

(c) Strain – displacement relations,

$$\in_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \rightarrow (4.29)$$

must be satisfied at all interior points of the body. Also prescribed conditions on stress and displacement must be satisfied on the bunding surface of the body.

The boundary value problems of elasticity are usually classified according to boundary condition into problems for which

(1) displacements are prescribed everywhere on the boundary.

(2) stresses (surface tractions) are prescribed everywhere on the boundary.

(3) displacements are prescribed over a portion of the boundary stresser are prescribed over the remaining part.

For all three categories the body forur are assumed to be given throughout the contunuum.

For those problems in which boundary displacement components are given everywhere by an equation of the form

$$u_i = g_i(X) \rightarrow (4.30)$$

the strain–displacement relation (4.29) may be substitufed into stooke's law (4.28) and the result in turn sub stituted into (4.27) to produce the governing equations.

$$\mu U_{i,jj} + (\lambda + \mu) u_{j,ji} + \rho b_i = 0 \rightarrow (4.31)$$

which are called the Navier-Cauchy equations. The solution of this type of problem is therefore given in the form of the displacement vector u_i, satisfying (4.31) throughout the continuum and fulfilling (4.30) on the boundary.

For those problems in which surface transactions are prescribed everywhere on the boundary by equation of the form

$$t_i^{(n^*)} = \sigma_{ij} n_j \rightarrow (4.32)$$

The equation of compatibility may be combined with stooke's law (4.24) and the equilibrium equation (4.27) to produce the governing equations,

$$\sigma_{ij,kk} + \frac{1}{1+v} \delta_{kk,ij} + \rho(bi, j + bj, i)$$

$$+ \frac{v}{1-v} \delta_{ij} b_{k,k} = 0 \rightarrow (4.33)$$

which are called the Beltrami-Michell equations of compatibility.

The solution for this type of problem is given by specifying the stress tensor which satisfies (4.33) throughout the continuum and fulfills (4.32) on the boundary.

In the formulation of elastodynamils problems, the equilibrium equation (4.27) must be replaced by the equation of motion.

$$\delta_{ij,j} + \rho b_i = \rho U^{\cdot}{}_i \rightarrow (4.34)$$

and initial conditions as well as boundary conditions must be specified. In terms of the displacement field u_i, the governing equation here, analogous to (4.31) in the elastostatic case is

$$\mu U_{i,jj} + (\lambda + \mu)\, u_{j,ji} + \rho b_i = \rho \ddot{u}_i \rightarrow (4.35)$$

solutions of (4.35) appear in the form $u_i = u_i(x, t)$ and must satisfy notonly initial conditions on the motion, usually expressed by equations such as

$$u_i = u_i(x,0) \text{ and } \dot{u}_i = \dot{u}_i = (x,0) \rightarrow (4.36)$$

but also boundary conditions, either on the displacements

$$ui = g_i(x, t) \rightarrow (4.37)$$

or on the surface tractions

$$t_i^{(n^*)} = t_i^{(n^*)}(x, t) \rightarrow (4.38)$$

Solved Problems

Exp 1 Show that $K = \frac{E}{3(1-2\nu)} = \frac{3\lambda + 2\mu}{3}$, assuming a state of uniform compressive body stress $\sigma_{ij} = -p\delta_{ij}$.

Solution Hooke's law for isotropic body becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right) \qquad \text{...(i)}$$

or when inverted

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \qquad \text{...(ii)}$$

When $\sigma_{ij} = -p\delta_{ij}$ then (ii) becomes

$$\varepsilon_{ij} = \frac{1+\nu}{E}(-p\delta_{ij})\frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

and so $\sigma_{ii} = \frac{1}{E}[-3p(1+\nu)+9\nu p] = \frac{p}{E}[3(2\nu-1)]$

Now, Bulk modulus can be defined by

k= ratio of pressure to volume change

$$= -\frac{p}{\varepsilon_{ii}} = \frac{E}{3(1-2\nu)}$$

Again, in terms of Lame' constants λ and μ Hooke's law for an isotropic body may be written as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

$$\therefore \sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii}$$

But $\quad \sigma_{ij} = -p\delta_{ij} \;\therefore\; \sigma_{ii} = -3p$

Hence $\quad (3\lambda + 2\mu)\varepsilon_{ij} = -3p$

$$\Rightarrow -\frac{p}{\varepsilon_{ii}} = \frac{3\lambda + 2\mu}{3} \quad \Rightarrow k = \lambda + \frac{2}{3}\mu$$

Thus $\quad k = \frac{E}{3(1-2\nu)} = \lambda + \frac{2}{3}\mu$

Exp 2 Express the ingineering constants ν and E in terms of the Lame' constants λ and μ.

Solution We know the bulk modulus and k is defined as

$$k = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)} \qquad ...(1)$$

Where E is Young's modulus and υ is Poisson's ratio. From (i) we have

$$E = (1-2\nu)(3\lambda + 2\mu) \qquad ...(2)$$

Again the shear modulus G is defin s

$$G = \mu = \frac{E}{2(1+\nu)} \qquad ...(3)$$

From (iii),

$$E = 2\mu(1+\nu) \qquad ...(4)$$

From (ii) and (iv)

$$(1-2\nu)(3\lambda + 2\mu) = 2\mu(1+\nu) \; ...(5)$$

Soliving we get,

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \qquad ...(6)$$

Substituting ν from (v), we have

$$E = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$$

Exp 3 For uniaxial state of stress in the x_i direction, show that Hooke's law

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

may be expresed as

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right)$$

Deduce that

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

Solution The generalized Hooke's law is defined as

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km} \quad ... (i)$$

In terms of Lame' constants λ and μ, Hooke's law for an isotropic body is given by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \quad ...(ii)$$

For a simple uniaxial state of stress in the x_i direction engineering constants E and ν may be introduced through the relationship $\sigma_{11} = E\varepsilon_{11}$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$.

The constant E is known as Young's modulus and ν is called Poisson's ratio. In terms of λ and μ ; E and ν may be given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda+\mu} \quad ...(iii) \qquad \nu = \frac{\lambda}{2(\lambda+\mu)} \quad ...(iv)$$

From (iii)

$$\lambda = \frac{\mu(2\mu - E)}{E - 3\mu} \quad ...(v)$$

From (iv),

$$\mu = \frac{\lambda(1-2\nu)}{2\nu}$$

Substituting the value of μ in (V) ,

$$\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

Thus,

$$\mu = \frac{E}{2(1+\nu)}$$

substituting these values of λ and μ in (ii),

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right] \quad ...(vi)$$

From (vi)

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk} \quad ...(vii)$$

From (vi)

$$\sigma_{ii} = \frac{E}{1-2\nu}\varepsilon_{kk} \quad \text{or,} \quad \varepsilon_{kk} = \frac{1-2\nu}{E}\sigma_{ii}$$

Substituting ε_{kk} in (vii), we have

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

Exp 4 Find the inversion formula of $\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$.

Solution Hooke's law for an isotropic body is given by

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \quad ...(i)$$

Putting i=j, we have

$$\sigma_{ii} = (3\lambda + 2\mu)\,\varepsilon_{ii} \quad ...(ii)$$

Again $2\mu\varepsilon_{ij} = \sigma_{ij} - \lambda\delta_{ij}\varepsilon_{kk}$

i.e $\quad 2\mu\varepsilon_{ij} = \sigma_{ij} - \lambda\delta_{ij}\dfrac{\sigma_{kk}}{(3\lambda+2\mu)}$

i.e $\quad \varepsilon_{ij} = \dfrac{1}{2\mu}\sigma_{ij} - \dfrac{\lambda}{2\mu(3\lambda+2\mu)}\delta_{ij}\sigma_{kk}$

provided $\mu \neq 0$, $3\lambda + 2\mu \neq 0$

Supplementary problems

1.Show that the shear modulus

$$G = \mu = \frac{E}{2(1+\nu)}.$$

2. Find the inversion formula of

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon_{kk}\right)$$

3. When a material is said to be elastically isotropic? What will be the number of elastic constants if $C_{KM}=C_{MK}$.

4. Write the generalized Hooke's law in tensor form. What is the number of elastic constants?

5. Assuming a state of uniform compressive stress $\sigma_{ij} = -p\delta_{ij}$, prove that the bulk modulus is given by $k = \dfrac{3\lambda+2\mu}{2}$ and shear modulus $G = \mu = \dfrac{E}{2(1+\nu)}$.

6. For an orthotropic elastic continuum find the elastic coefficient matrix.

•••

UNIT V
Equations of Fluid Dynamics

5.1. Fluid pressure, Viscous stress tensor, Barotropic flow

In any fluid is rest the stress vector $t_i^{(\hat{n})}$ on an arbitrary surface element is collinear with the normal $\hat{n}$ of the surface and equal in manitude for every direction at a given point. Thus

$$t_i^{(\hat{n})} = \sigma_{ij} n_j = -p_o n_i \qquad ...(5.1)$$

In which p_o is the stress magnitude, or hydrostatic pressure. The negative sign indicates a compressive stress for a positive value of the presssure. Here every diection is a principal direction, and from (5.1)

$$\sigma_{ij} = -p_o \delta_{ij} \qquad ...(5.2)$$

which repressents a spherical state of stress referred to as hydrostatic pressure. From (5.2), the shear stress components are observed to be zero in a fluid at rest.

For a fluid in motion, the shear stress components are usually not zero, and in this case we have

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad ...(5.3)$$

where τ_{ij} is called the viscous stress tensor and p is the pressure.

A perfect or inviscid fluid is one for which τ_{ij} is taken identically zero even when motion is present.

From (5.3), the mean normal stress is given by

$$\frac{1}{3}\sigma_{ii} = -p + \frac{1}{3}\tau_{ii} \qquad ...(5.4)$$

For a fluid at rest, τ_{ij} vanishes and p reduces to p_o which in this case is equal to the negative of the mean normal stress. For an incompressible fluid, The thermodynamic pressure is not defined separately from the mechanical conditions so that p must be considered as an independent mechanical variable in such fluids.

In a compressible fluid, the pressure p.the density ρ and the absolute temperature T are related through a kinetic equation of static having the form

$$p = p(\rho, T) \qquad ...(5.5)$$

An example of such an equation of static is the well-known ideal gas law

$$p = \rho RT \qquad ...(5.6)$$

where R is the gas constant. If the changes of state of a fluid obeys an equation of state that does not contain the temperature, i.e. $p=p(\rho)$, such changes are termed barotropic. An isothermal prosses for a perfect gas is an example of a special case which obeys the barotropic assumption.

5.2. Constitutive equations,Stokesion fluids, Newtonian fluids

In developing constitutivere relations for fluids, It is generally assumed that the viscous stress tensor τ_{ij} is a function of the rate of deformation tensor D_{ij}. If the functional relationship is a nonlinear one, as expressed symbolically by

$$\tau_{ij} = f_{ij}(D_{pq}) \qquad ...(5.7)$$

The fluid is called a stokesion fluid. When the function is a linear one of the form

$$\tau_{ij} = K_{ijpq} D_{pq} \qquad ....(5.8)$$

where the constants K_{ijpq} are called viscosity coefficients, the fluid is known as a Newtonian fluid. Some authors classify fluids simply as Newtonian and Non-Newtonain.

The constitutive equation for an isotropic homogeneous Newtonain fluid is of form

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij} \qquad ...(5.9)$$

where λ^* and μ^* are velocity coefficients of the fluid. From (5.9), the mean normal stress is given by

$$\frac{1}{3}\sigma_{ii} = -p + \frac{1}{3}(3\lambda^* + 2\mu^*)D_{ii}$$

$$= -p + k^* D_{ii} \qquad ...(5.10)$$

where $k^* = \lambda^* + \frac{2}{3}\mu^*$ is called the coefficient of bulk scosity. The condition that

$$k^* = \lambda^* + \frac{2}{3}\mu^* = 0 \qquad ...(5.11)$$

is known as **Stoke's condition,** and guarantees that the pressure p is defined as the average of the normal stress for a compressible fluid at rest. In this way the thermodynamic pressure is defined in terms of the machanical stresses.

In terms of the deviator components

$s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{kk}/3$ and $D'_{ij} = D_{ij} - \delta_{ij} D_{kk}/3$, equation (5.9) above may be rewritten in the form

$$s_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk} = -p\delta_{ij} + \delta_{ij}\left(\lambda^* + \frac{2}{3}\mu^*\right)D_{ii} + 2\mu^* D'_{ij} \qquad ...(5.12)$$

Therefore in view of the relationship (5.10), equation (5.12) may be expressed by the pair of equations

$$s_{ij} = 2\mu^* D'_{ij} \quad , \quad \sigma_{ii} = -3p + 3K^* D_{ii} \qquad ...(5.13)$$

The first of which relates the shear effects in the fluid and the second gives the volumetric relationship.

5.3. Navier Stokes-Duhem equations

Cauchy's first equation of motion of the continuum is

$$\sigma_{ij},_j + \rho b_i = \rho \dot{v}_i \qquad ...(5.14)$$

where σ_{ij} is stress tensor, b_i the body force, v_i the velocity and

$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}$. Again, the constitutive equation for an isotropic homogeneous Newtonian fluid is

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} D_{kk} + 2\mu^* D_{ij} \qquad ...(5.15)$$

where $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the rate of deformation tensor. Thus $D_{kk} = v_{k,k}$

Hence we can write

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* \delta_{ij} v_{k,k} + \mu^*(v_{i,j} + v_{j,i})$$

$$\therefore \sigma_{ij,j} = -p_{,j}\delta_{ij} + \lambda^* \delta_{ij} v_{k,kj} + \mu^*(v_{i,jj} + v_{j,ij})$$

$$= -p_{,i} + \lambda^* v_{k,ki} + \mu^*(v_{i,jj} + v_{j,ij})$$

Substituting in (5.14),

$$\rho\dot{v}_i = \rho b_i - p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj} \qquad ...(5.16)$$

These equations are known as Navier-stokes-Duhem equations. In case of incompressible flow $(v_{i,i} = 0)$, (5.16) reduces to

$$\rho\dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj} \qquad ...(5.17)$$

If stokes condition is assumed $\left(\lambda^* = -\frac{2}{3}\mu^*\right)$, (5.16) reduce to Navier stokes equations for compressible flow

$$\rho\dot{v}_i = \rho b_i - p_{,i} + \frac{1}{3}\mu^* v_{j,ji} + \mu^* v_{i,jj} \qquad ...(5.18)$$

The Navier-Stokes equations (5.17), together with the continuity equation

$$\dot{\rho} + \rho v_{i,i} = 0 \qquad ...(5.19)$$

form a complete set of four equations in four unknowns

the pressure p and the three velocity components v_i.

If the Navier-Stokes equations are put into dimensionless form, several ratios of the normalizing parameters appear. One of the most significant and commonly used ratio is the Reynolds numbers R_e which expresses the ratios of inertia to viscous force.

Thus if a flow is characterized by a certain length L, velocity V and density ρ, the Reynolds number is

$$R_e = \frac{VL}{\nu}$$

where $\nu = \frac{\mu^*}{\rho}$ is called the **Kinematic viscosity.**

5.4. Steady flow, Hydrostatics, Irrotational flow

The motion of a fluid is referred to as a steady flow if the velocity components are independent of time. For this situation, the derivative $\frac{\partial v_i}{\partial t}$ is zero, and hence the material derivative of the velocity

$$\frac{dv_i}{dt} \equiv \dot{v}_i = \frac{dv_i}{\partial t} + v_j v_{i,j} \qquad ...(5.20)$$

reduces to simple form

$$\dot{v}_i = v_j v_{i,j} \qquad ...(5.21)$$

A steady flow in which the velocity is zero everywhere, causes the Navier-Stokes equation (5.16) to reduce to

$$\rho b_i = p_{,i} \qquad ...(5.22)$$

which describes the hydrostatic equilibrium situation. If the barotropic condition $\rho = \rho(p)$ is assumed, a pressure function

$$P(\rho) = \int_{p_0}^{p} \frac{dp}{\rho} \qquad ...(5.22)$$

may be defined. Furthermore, If the body force may be prescribed by a potential function

$$b_i = -\Omega_{,i} \qquad ...(5.23)$$

equations (5.22) take on the form

$$(\Omega + P)_{,i} = 0 \qquad ...(5.24)$$

A flow in which the vorticty or spin tensor

$$V_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \qquad ...(5.25)$$

vanishes everywere is called an irrotational flow. The vorticity vector q_i is related to the vorticity tensor by the equation

$$q_i = \varepsilon_{ijk} V_{kj} \qquad ...(5.26)$$

and therefore also vanishes for irrotaltional flow.
Furthermore,

$$q_i = \varepsilon_{ijk} v_{k,j} \qquad ...(5.27)$$

and since $\nabla \times \bar{v} = 0$ is necessary and sufficient for a velocity potential ϕ to exit, the velocity vector for irrotational flow may be expressed by $v_i = -\phi_{,i}$

5.5. Perfect fluids, Bernoulli equation, Circulation

If the viscosity coefficients λ^* and μ^* are zero, the resulting fluid is called on inviscid or perfect (frictionless) fluid and Navier-Stokes-Duhem equations (5.16) reduce to the form

$$\rho \dot{v}_i = \rho b_i - p_{,i} \qquad \text{...(5.28)}$$

which is known as Euler equation of motion.

For a barotropic fluid with conservative body forces this equations becomes

$$\dot{v}_i = -(\Omega + P)_{,i} \qquad \text{...(5.29)}$$

If the Euler equation (5.28) is integrated along a streamline, the result is the well known Bernoulli's equation in the form

$$\Omega + P + v^2/2 + \int \frac{\partial v_i}{\partial t} dx_i = C(t) \qquad \text{...(5.30)}$$

For steady motion, $\frac{\partial v_i}{\partial t} = 0$ and C(t) becomes the Bernoulli's constant C which is, in general, different along different streamlines. If the flow is irrotational as well, a single constant C holds everywhere in the field of flow.

When the only body force present is gravity, the potential Ω=gh where g is the gravitational constant and h is the elevation above some refference level. Thus with $h_p = \frac{P}{g}$ defined as the pressure head, and $\frac{v^2}{2g} = h_v$ defined as the velocity head, Bernoulli's equation requires the total head along any streamline to be constant. For incompressible fluids, the equation takes the form

$$h + h_p + h_v = h + p/\rho g + v^2/2g = \text{Constant} \qquad \text{...(5.31)}$$

By definition, the velocity circulation around a closed path of fluid particles is given by the line integral

$$\Gamma_c = \oint v_i dx_i \qquad \text{...(5.32)}$$

From Stokes theorem, the line integral (5.32) may be converted to the surface integral

$$\Gamma_c = \int_s n_i e_{ijk} v_{k,j} ds \qquad \text{...(5.33)}$$

when n_i is the unit normal to the surface S enclosed by the path. If the flow is irrotational, the circulation is zero.

The material derivative $\frac{d\Gamma_C}{dt}$ of the circulation may be given as

$$\dot{\Gamma}_c = \oint (\dot{v}_i dx_i + v_i dv_i) \qquad \text{...(5.34)}$$

For a barotropic, inviscid fluid with conservative body forces the circulation may be shown to be constant.

This is known as **Kelvin's Theorem of constant circulation.**

Solved Problems

Exp 1 The stress tensor at a given point for a Newtonian fluid with zero vicosity is

$$\sigma_{ij} = \begin{pmatrix} -6 & 2 & -1 \\ 2 & -9 & 4 \\ -1 & 4 & -3 \end{pmatrix}$$

Determine the viscous stress tensor.

Solution The viscous stress tensor τ_{ij} is given by

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$\text{where } p = -\frac{1}{3}\sigma_{ii} = -\frac{1}{3}(\sigma_{11} + . \ \sigma_{22} + \ \sigma_{33}) = \sigma$$

$$\therefore \tau_{ij} = \sigma_{ij} + p\delta_{ij}$$

$$\therefore (\tau_{ij}) = \begin{pmatrix} -6 & 2 & -1 \\ 2 & -9 & 4 \\ -1 & 4 & -3 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ 2 & -3 & 4 \\ -1 & 4 & 3 \end{pmatrix}$$

Exp 2 Show that the constitutive relations for a Newtonian fluid with zero bulk viscosity may be expressed by the pair of equations

$$S_{ij} = 2\mu^* D_{ij} \quad \text{and} \quad -\sigma_{ii} = 3p$$

Solution Given bulk viscosity $k^*=0$

$$\text{i.e.} \quad \lambda^* + \frac{2}{3}\mu^* = 0$$

$$\text{i.e.} \quad \lambda^* = -\frac{2}{3}\mu^*$$

The constitutive equation for a Newtonian fluid is

$$\sigma_{ij} = -p\delta_{ij} + \lambda^*\delta_{ij}D_{kk} + 2\mu^*\delta_{ij}$$

$$= -p\delta_{ij} - \frac{2}{3}\mu^*\delta_{ij}D_{kk} + 2\mu^* D_{ij}$$

$$= -p\delta_{ij} + 2\mu^*\left(D_{ij} - \frac{1}{3}\delta_{ij}D_{kk}\right)\ldots(i)$$

We introduce the deviator components.

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}D_{kk} \quad \text{and} \quad D'_{ij} = D_{ij} - \frac{1}{3}\delta_{ij}D_{kk}$$

Thus (i) becomes

$$S_{ij} + \frac{1}{3}\delta_{ij}\sigma_{kk} = -p\delta_{ij} + 2\mu^*\left(D'_{ij} + \frac{1}{3}\delta_{ij}D_{kk} - \frac{1}{3}\delta_{ij}D_{kk}\right)$$

$$= -p\delta_{ij} + 2\mu^* D_{ij}'$$

This relationship may be expressed by the pair of equations

$$S_{ij} = 2\mu^* D_{ij}' \text{ and } \sigma_{kk} = -3p.$$

Exp 3 Determine the pressure function P(p) for barotropic fluid having the state $p = \rho^k\lambda$ where λ and k are constants.

Solution The pressure function P(p) is defined as

$$P(p) = \int_{p_o}^{p} \frac{dp}{\rho} \qquad \text{Given} \quad p = \lambda\rho^k$$

$$= \int_{p_o}^{p} \frac{dp}{\lambda^{-\frac{1}{k}} p^{\frac{1}{k}}} \qquad \therefore \left(\frac{p}{\lambda}\right)^{\frac{1}{k}} = \rho$$

$$= \lambda^{1/k} \int_{p_o}^{p} \frac{dp}{p^{1/k}} = \frac{k\lambda^{1/k}}{k-1}\left[p^{\frac{k-1}{k}}\right]_{p_0}^{p}$$

$$= \frac{k\lambda^{1/k}}{k-1}\left[p^{\frac{k-1}{k}} - p_o^{\frac{k-1}{k}}\right]$$

$$= \frac{k}{k-1}\left[\frac{p}{\rho} - \frac{p_o}{\rho_o}\right]$$

Exp 4 If a fluid motion is very slow so that higher order terms in the velocity are negligible, show that in a steady incompressible flow with zero body forces the pressure is a harmonic function i.e. $\nabla^2 p = 0$.

Solution for an incompressible flow, the Navier-Stokes equations are

$$\rho\dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj}$$

i.e. $\rho\left(\frac{\partial v_i}{\partial t}+v_j v_{i,j}\right)=\rho b_i - p_{,i}+\mu^* v_{i,jj}$

For creeping flow, this equation become

$$\rho\frac{\partial v_i}{\partial t}=\rho b_i - p_{,i}+\mu^* v_{i,jj}$$

For steady flow, this equation reduces to

$$0=\rho b_i - p_{,i}+\mu^* v_{i,jj}$$

i.e. $p_{,i}=\mu^* v_{i,jj}$ for zero body forces.

Taking the divergence of this equation

$$p_{,ii}=\mu^* v_{i,ijj}$$

$$=0 \qquad [\because \text{ For incompressible flow } v_{i,i}=0]$$

i.e. $\nabla^2 p=0.$

Exp 5 Derive Bernoulli's equation by integrating Euler's equation along a stream line.

Solution Euler's equation of motion

$$\rho\dot{v}_i=\rho b_i - p_{,i} \qquad ...(i)$$

For a barotropic fluid with conservative body forces,

$$b_i=-\Omega_{,i} \quad , \quad p_{,i}=\rho P_{,i}$$

where P is a pressure function and is defined by $P=\int_{P_0}^{P}\frac{dp}{\rho}$.

Thus (i) becomes

$$\rho\dot{v}_i=-\rho(\Omega+P)_{,i}$$

i.e. $\frac{\partial v_i}{\partial t}+v_j v_{i,j}+\Omega_{,i}+P_{,i}=0 \qquad ...(ii)$

Let dx_i be an increment of dicplacement along a stream line. Taking the scaler product of this increment with (ii) we have

$$\frac{\partial v_i}{\partial t}dx_i+v_j v_{i,j}\,dx_i+\Omega_{,i}\,dx_i+P_{,i}\,dx_i=0$$

Integrating,

$$\int\frac{\partial v_i}{\partial t}dx_i+\int v_j v_{i,j}\,dx_i+\int\Omega_{,i}\,dx_i+\int P_{,i}\,dx_i=C(t)$$

i.e. $$\int \frac{\partial v_i}{\partial t} dx_i + \int v_j v_{i,j}\, dx_i + \Omega + P = C(t) \qquad ...(iii)$$

Also, along a stream line

$$dx_i = \left(\frac{v_i}{v}\right) ds$$, where ds is the increment of distance.

$$\therefore v_j v_{i,j}\, dx_i = v_j v_{i,j} \left(\frac{v_i}{v}\right) ds = v_i v_{i,j} \left(\frac{v_j}{v}\right) ds = v_i v_{i,j}\, dx_j = v_i dv_i$$

$$\therefore \int v_j v_{i,j}\, dx_i = \int v_i dv_i = \frac{1}{2} v_i v_i = \frac{1}{2} v^2$$

Thus from (iii) we have

$$\Omega + P + \frac{1}{2} v^2 + \int \frac{\partial v_i}{\partial t} dx_i = C(t)$$

Thus equation is known as Bernoulli's equation after Daniel Bernoulli (1738).

Exp 6 Show that for a barotropic, inviscid fluid with conservative body forces the rate change of circulation is zero.

Solution The velocity circulation around a closed poth of fluid particles is given by the line integral

$$\Gamma_C = \oint_C v_i dx_i$$

The material derivative $\frac{d\Gamma_C}{dt}$ of the circulation is given by

$$\dot{\Gamma}_C = \oint_C [\dot{v}_i dx_i + v_i dv_i]$$

For a barotropic, inviscid fluid with conservative body forces the rate of circulation is given by

$$\dot{\Gamma}_C = \oint_C [-(\Omega_{,i} + p_{,i}) dx_i + v_i dv_i] \qquad [\because \dot{v}_i = -(\Omega + P)_{,i}]$$

$$= \oint_C \left[-d\Omega - dP + d\left(\frac{v^2}{2}\right)\right]$$

$$= -\oint_C \left[d(\Omega + P) - \frac{v^2}{2}\right] = 0$$

This is known as **Kelvin's Circulation Theorem** after Lord Kelvin (1869).

Exp 7 Show that $\frac{1}{\rho}\frac{d\rho}{dt}=0$ is a condition for $-\frac{\sigma_{ii}}{3}=p$ for a Newtonian fluid

Solution The constitutive equation for an isotropic homogeneous Newtonian fluid is.

$$\sigma_{ij}=-p\delta_{ij}+\lambda^*\delta_{ij}D_{kk}+2\mu^*D_{ij}$$

For i=j,

$$\sigma_{ii}=-3p+3\lambda^*D_{kk}+2\mu^*D_{ii}$$

$$=-3p+(3\lambda^*+2\mu^*)D_{ii}$$

$$\Rightarrow -\frac{1}{3}\sigma_{ii}=p-\left(\lambda^*+\frac{2}{3}\mu^*\right)D_{ii}$$

$$=p-\left(\lambda^*+\frac{2}{3}\mu^*\right)v_{i,i} \quad ...(i)$$

Again, the continuity equation is

$$\frac{d\rho}{dt}+\rho b_{i,i}=0$$

i.e. $$\frac{1}{\rho}\frac{d\rho}{dt}+v_{i,i}=0$$

For incompressible flow, $v_{i,i}=0 \quad ...(ii)$

Thus from (i) we have $-\frac{1}{3}\sigma_{ii}=p$

for a Newtonian fluid with the condition that

$$\frac{1}{\rho}\frac{d\rho}{dt}=0 \quad \text{Ans}$$

Exp 8 show that for a perfect fluid with negligible body forces the rate of change of circulation may be given by $-\int_S e_{ijk}\left(\frac{1}{\rho}\right)_{,j}p_{,k}\,ds_i$.

Solution The rate circulation is defined by

$$\Gamma_C=\oint_C v_i dx_i$$

$$\dot{\Gamma}_C=\oint_C(\dot{v}_i dx_i+v_i dv_i)$$

$$= \oint_C \dot{v}_i dx_i + \oint d\frac{1}{2}(v^2)$$

$$= \oint_C \dot{v}_i dx_i \quad ...(i)$$

Now, when the viscosity coefficients λ^* and μ^* are zero, the resulting fluid is called perfect fluid and Navier-stokes equation reduces to

$$\rho\dot{v}_i = \rho b_i - p_{,i}$$

Again, with negligible body forces this equation becomes

$$\rho\dot{v}_i = -p_{,i}$$

$$\text{or,} \qquad \dot{v}_i = -\frac{1}{\rho}p_{,i}$$

Thus from (i) we have

$$\dot{\Gamma}_C = -\oint_C \frac{1}{\rho} p_{,i}\, dx_i$$

Writing in the form of surface integral

$$\dot{\Gamma}_C = -\oint_s n_i e_{ijk} (p_{,i}/\rho)_{,j}\, ds \qquad \left[\because \oint_C F_i dx_i = \int_s n_i e_{ijk} F_{k,j} ds\right]$$

$$= -\int_s \left[e_{ijk} \left\{ \left(\frac{1}{\rho}\right)_{,j} p_{,k} + \frac{1}{\rho} p_{,kj} \right\} ds_i \right]$$

$$= -\int_s e_{ijk} \left(\frac{1}{\rho}\right)_{,j} p_{,k}\, ds_i .$$

Exp 9 Express the continuity equation and the Navier Stokes-Duhem equations in terms of the velocity potential ϕ for an irrotational motion.

Solution The velocity potential ϕ for an irrotational motion is defined by

$$v_i = -\phi_{,i}$$

Now, the equation of continuity is

$$\frac{d\rho}{dt} + \rho v_{i,i} = 0$$

$$\text{i.e.} \quad \dot{\rho} - \rho v_{i,ii} = 0 \quad \text{i.e.} \quad \dot{\rho} - \rho\nabla^2\phi = 0$$

and the Navier-Stokes-Duham equation is

$$\rho \dot{v}_i = \rho b_i - p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj}$$

$$\Rightarrow -\rho \dot{\phi}_{,i} = \rho b_i - p_{,i} - (\lambda^* + \mu^*) \phi_{,jji} - \mu^* \phi_{,ijj}$$

$$\Rightarrow -\rho \left(\frac{\partial \phi_{,i}}{\partial t} + \phi_{,k} \phi_{,ik} \right) = \rho b_i - p_{,i} - (\lambda^* + 2\mu^*) \phi_{,jji}$$

i.e. $$-\rho \nabla \left[\frac{\partial \phi}{\partial t} + \frac{(\nabla \phi)^2}{2} \right] = \rho \vec{b} - \nabla p - (\lambda^* + 2\mu^*) \nabla (\nabla^2 \phi)$$

Exp 10 . Determine the stress $\frac{\sigma_{ii}}{3}$ for an incompressible Stokesion fluid for which $\tau_{ij} = \alpha D_{ij} + \beta D_{ik} D_{kj}$ where α and β are constants.

Solution we know

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

$$= -p\delta_{ij} + \alpha D_{ij} + \beta D_{ik} D_{kj}$$

$$\therefore \sigma_{ii} = 3p + \alpha D_{ii} + \beta D_{ik} D_{ki}$$

$$\therefore \frac{\sigma_{ii}}{3} = -p + \frac{\beta D_{ik} D_{ik}}{3} \qquad [\because D_{ii} = v_{i,i} \quad \text{and} \quad D_{ik} = D_{ki}]$$

Supplementary Problems

1. Determine the pressure function P(p) for a barotropic fluid having the equation of state $p = \lambda \rho^k$ where λ and k are constants.

If the fluid flows from a large closed tank through a thin smooth pipe and if the pressure in the tank is N times the atmospheric pressure, determine the speed of the emerging fluid.

Ans. (i) $P(p) = \frac{k}{k-1} \left(\frac{p}{\rho} - \frac{p_0}{\rho_0} \right)$

(ii) $v_B^2 = \frac{2k}{k-1} \frac{p_B}{\rho_B} \left(N^{\frac{k-1}{k}} - 1 \right)$

2. A barotropic fluid having the equation of state $p = \lambda \rho^k$ where λ and k are constants is at rest in gravity field in the x_3 direction. Determine the pressure in the fluid with respect to x_3 and p_0, the pressure at $x_3 = 0$.

Ans. [illegible]

3. Assuming the constitutive equation

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* D_{kk}\delta_{ij} + 2\mu^* D_{ij}$$

show that the equation of motion is

$$\rho\dot{v}_i = \rho b_i - p_{,i} + (\lambda^* + \mu^*)v_{j,ji} + \mu^* v_{i,jj}$$

Deduce that (i) $\rho\dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj}$ for incompressible flow.

(ii) $\rho\dot{v}_i = \rho b_i - p_{,i} + \frac{1}{3}\mu^* v_{j,ji} + \mu^* v_{i,jj}$ for compressible fluid.

If the fluid motion is very slow, so that higher order terms in the velocity distribution are negligible show that in a steady incompressible flow with zero body forces the pressure is harmonic.

4. At a certain point of an incompressible viscous fluid, the stress matrix is

$$[\sigma_{ij}] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find the pressure and the viscous stress tensor.

Ans $p = 1, [\tau_{ij}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$

5. Show that

$$v_1 = -\frac{2x_1x_2x_3}{(x_1^2 + x_2^2)^2}, \quad v_2 = \frac{(x_1^2 - x_2^2)x_3}{(x_1^2 + x_2^2)^2}, \quad v_3 = \frac{x_2}{(x_1^2 + x_2^2)}$$

are possible velocity components of liquid motion.

Is this motion irrotational?

Ans For possible motion $v_{i,i}=0$, Yes.

6. Prove that liquid motion is possible when velocities at a point (x_1, x_2, x_3) are given by

$$v_1 = \frac{3x_1^2 - r^2}{r^5}, \qquad v_2 = \frac{3x_1x_2}{r^5}, \qquad v_3 = \frac{3x_1x_2}{r^5}$$

Where $r^2 = x_1^2 + x_2^2 + x_3^2$.

7. What is the distinction between a Newtonian and non-Newtonian fluid?

8. For perfect fluid, write Euler equation of motion. Show that for a barotropic fluid with conservative body forces $b_i = -\Omega_{,i}$ and $P(p) = \int \frac{dP}{\rho}$; the equation reduces to

$$\dot{v}_i = -(\Omega + P)_{,i}$$

Integrate this equation along a stream line.

9. For a fluid in motion, express shear stress components σ_{ij} in terms of viscous stress tensor τ_{ij} and the pressure p.

10. With usual notations, for an isotropic homogeneous Newtonian fluid determine the the constitutive equation in the form

$$\sigma_{ij} = -p\delta_{ij} + \lambda^* D_{kk}\delta_{ij} + 2\mu^* D_{ij}$$

From this expression determine the 'stress power' for this fluid.

Question Paper of G.U

1997

1. (a) What is stress quadric of Cauchy ?

(b) Explain small deformation theory?

(c) Define the rate of deformation tensor D_{ij} and vorticity tensor V_{ij}. Show that D_{ij} is symmetric and V_{ij} is skew symmetric.

(d) Determine the form of continuity equation for an irrotational motion.

(e) When a material is said to be elastically isotropic? What will be the number of elastic constants if $C_{KM} = C_{MK}$

(f) For a fluid in motion, express shear stress components σ_{ij} in terms of viscous stress tensor τ_{ij} and the pressure p.

$3 \times 6 = 18$

2. (a) Deduce the expressions for first, second and the third Lagrangian strain invariants. If the Lagrangian and the Eularian deviator tensor components are denoted by d_{ij} and e_{ij} respectively, to show that the resolution expressions are

$$l_{ij} = d_{ij} + \delta_{ij}\frac{l_{kk}}{3}$$

and $\varepsilon_{ij} = e_{ij} + \delta_{ij}\frac{\varepsilon_{kk}}{3}$ 10

(b) The stress tensor at apoint is given as

$$\sigma_{ij} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & \sigma_{22} & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

with σ_{22} unspecified. Determine σ_{22} so that the stress vector on some plane at the point will be zero. 6

3. (a) A continuum body undergoes the displacement

$$\bar{u} = (3\xi_2 - 4\xi_3)\,\hat{e}_1 + (2\xi_1 - \xi_3)\,\hat{e}_2 + (4\xi_2 - \xi_1)\,\hat{e}_3$$

Determine the displaced position of the vector joining particles A(1,0,3) and B(3,6,6) assuming superposed material and spatial axes. For the above displacementfield, determine the displaced position of the position vector of the particle C(2,6,3) which is parallel to the vector joining particles A and B. Show that the two vectors remain parallel after deformation. 4+2+2=8

(b) Define vortex line. Show that the equations of the vortex lines are

$$\frac{dx_1}{q_1} = \frac{dx_2}{q_2} = \frac{dx_3}{q_3}.$$

Show that for the velocity field

$$\bar{v} = (Ax_3 - Bx_2)\,\hat{e}_1 + (Bx_1 - Cx_3)\,\hat{e}_2 + (Cx_3 - Ax_1)\,\hat{e}_3$$

the vortex lines are straight lines and determine their equations. 8

4. (a) Show that $\frac{d}{dt}\left(\frac{q_i}{\rho}\right) = \frac{(\epsilon_{ijk} a_{k,j} + q_j u_{i,j})}{\rho}$ where ρ is the density, a_i the acceleration and q_i the vorticity vector. 6

(b) Develop the Navier equation for plane stress

$$\frac{E}{2(1+\nu)}\nabla^2 u_\alpha + \frac{E}{2(1-\nu)} u_{\beta,\beta\alpha} + \rho b_\alpha = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}$

and show that it is equivalent to the corresponding equation for plane strain

$$\mu \nabla_1{}^2 u_\alpha + (\lambda + \mu) u_{\beta,\beta\alpha} + \rho b_\alpha = 0$$

if $\lambda' = \frac{2\lambda\mu}{\lambda + 2\mu}$ 10

5. (a) Determine the mean normal stress $\frac{\sigma_{ii}}{3}$ for an incompressible non-linear fluid for which $\tau_{ii} = \alpha D_{ij} + \beta D_{ik} D_{kj}$ where α and β are constants. 5

(b) Define creaping flow. For creaping flow show that in a steady incompressible flow with zero body forces, the pressure is a harmonic function. 5

(c) If a fluid moves radially with the velocity $\bar{v} = \bar{v}(r,t)$ where $r^2 = x_i x_i$, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 v) = 0$$ 6

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1.(a) The stress vector acting on the three coordinate planes are given by

$t_i^{(\hat{e}_1)}, t_i^{(\hat{e}_2)}$ and $t_i^{(\hat{e}_3)}$. Show that the sum of the squares of the magnitudes of these vectors is independent of the orientation of the coordinate planes.

(b) Show that for small deformation theory, the Eulerian and the Lagrangian infinitesimal strain tensor are equal.

(c) If J denotes Jacobian of transformation from material to spatial coordinates, show that

$$\frac{d}{dt}(\log J) = \operatorname{div} \vec{v}$$

(d) If $\sigma_{ij} = -p\delta_{ij}$, where p is a positive constant, show that the stress power may be expressed by the equation

$$D_{ij}\sigma_{ij} = \frac{p}{\rho}\frac{d\rho}{dt}$$

(e) What do you mean by strain energy of the elastic body.

(f) Show that the time rate of change of kinetic and internal mechanical energy of a continuum is equal to the rate of work done by the surface and body forces. $3\times 6 = 18$

2. (a) What are principal strains and strain invariants? What is the physical interpretation of first invariant of the Lagrangian strain tensor. 2+2+4=8

(b) What do you mean by the state of stress at a point. 3+5=8

The state of stress at a point is given by the stress tensor

$$\sigma_{ij} = \begin{pmatrix} \sigma & a\sigma & b\sigma \\ a\sigma & \sigma & c\sigma \\ b\sigma & c\sigma & \sigma \end{pmatrix}$$

where a, b, c are constants and σ is some stress value. Determine the constants a, b and c so that the stress vector on the octahedral plane vanishes.

3. (a) State the principle of linear momentum for moving continuum. From this principle, derive the equations of motion and the equillibrium equations. 2+4+2=8

(b) A vector field is given by

$v_1 = 0$

$v_2 = A\,(x_1x_2 - x_3^2)e^{-Bt}$

$v_3 = A\,(x_2^2 - x_1x_3)e^{-Bt}$

where A and B are constants. Determine the velocity gradient for this motion and compute the rate of deformation tensor and the spin tensor for the point $p(1,0,3)$ when $t = 0$. State the condition for which the given velocity field represents a rigid body rotation. 8

4. (a) For uniaxial state of stress in the x_1 – direction, show that the generalized Hooke's law

$$\sigma_{ij} = \lambda\sigma_{ij}e_{kk} + 2\mu e_{ij}$$

may be expressed as

$$\sigma_{ij} = \frac{E}{1+\nu}\left(e_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}e_{kk}\right)$$

Deduce that

$$e_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$ 6+4=10

(b) In a vertical elastic beam deforming under its own weight (acting in the x3– direction), the strain components are found to be

$$e_{11} = e_{22} = -\frac{\lambda}{2(\lambda+\mu)}a(b-x_3)$$

$$e_{33} = a(b-x_3)$$

$$e_{12} = e_{23} = e_{31} = 0$$

where a and b are constants. Find the stress components.

5 (a) What do you mean by Barotropic flow? Show that for a Barotropic, inviscid fluid with conservative body forces, the rate of circulation is zero. Also, prove that for a perfect fluid with negligible body forces the rate of change of circulation may be given by

$$-\int_S \varepsilon_{ijk}\left(\frac{1}{p}\right)_{,j} p_{,k}\, ds_i$$ 2+4+4=10

(b) Show that the velocity field 4+2=6

$$v_1 = -\frac{2x_1x_2x_3}{r^4}$$

$$v_2 = \frac{(x_1^2 - x_2^2)x_3}{r^4}$$

$$v_3 = \frac{x_2}{r^4}$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$ is a possible flow for an incompressible fluid. Is the motion irrotational?

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