M 401

Institute of Distance and Open Learning
Gauhati University

M.A./M.Sc. in Mathematics
Semester 4

Paper I
Graph Theory

Contents :

Math Paper I

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Unit 1

1.1. Introduction : Graph theory is the study of some set of 2-tuples. Graphs arise in many settings and are used to model a wide variety of situations. Lets consider several problems and concentrate on finding models representing these problems rather than worrying about their solutions.

Example.1

Suppose that we are given a collection of intervals on the real line, say $C = \{I_1, I_2, \ldots\ldots, I_k\}$. These intervals may or may not have a nonempty intersection. Suppose that we want a way to display the intersection relationship among these intervals. What form of model will easily display these intersections?

One possible model for representing these intersections is the following:

Let each interval be represented by a circle and draw a line between two circles if, and only if, the intervals that correspond to these circles intersect. For example, consider the set C = {[-4, 2], [0, 1], [-8, 2], [2, 4], [4, 10]}.

The model is shown below:

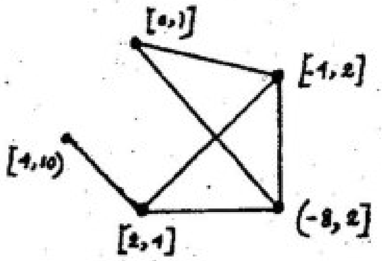


Fig. 1.1.1

Example.2.

Suppose there are three houses (call them h_1, h_2, h_3) and three utility companies (say gas (g), water (w) & electricity (e)). Our problem is to determine if it is possible to connect each of the three houses to each of these three utilities without crossing the service lines that run from the utilities to the houses. We model this puzzle by representing each house and each utility as a circle and drawing line between two circles if there is a service line between the corresponding house and utility. We picture this situation in fig 1.1.2.

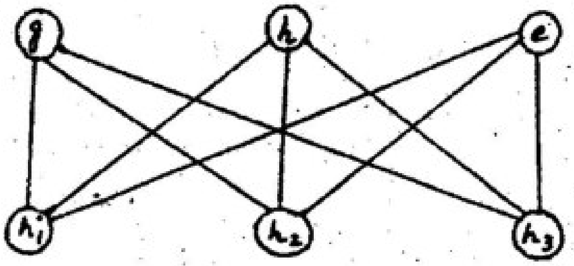


Fig. 1.1.2

Example.3.

Suppose you are the manager of a company that has four job openings (say j_1, j_2, j_3 .. and j_4) and five applicants a_1, a_2, a_3, a_4 and a_5 that some of these applicants are qualified for more than one of your jobs. How do you go about choosing people to fill the jobs so that you will fill as many openings as possible? We picture such a situation in Figure 1.1.3. Again, each job and each applicant can be represented as a circle. This time, a line is drawn from a circle representing an applicant to each of the circles representing the jobs for which the applicant is qualified. Fig. 1.1.3

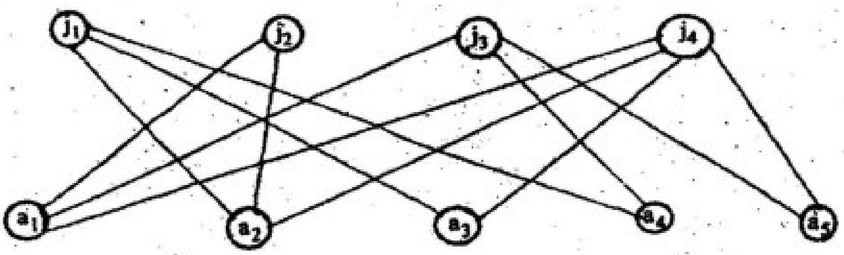


Fig. 1.1.3

A solution to this problem would be a set of four lines joining distinct jobs to distinct applicants, that is, one line joins each job to a distinct example, the lines joining j_1 and a_2, j_2 and a_1, j_3 and a_4 and j_4 and a_4 constitutes a solution to this problem. Since lines only join jobs to applicants, this is clearly the maximum number of lines possible. Can you find another solution? The real problem is, how can we find solution in general?

1.2. Graphs:

Despite the fact that the above problem seems very different, we have used a similar type of diagram to model them. Such a diagram is called a graph.

A graph G consists of a finite nonempty set V=V (G) of p points together with a prescribed set X of q unordered pairs of distinct points of V. Each pair x = {u, v} of points in X is a line of G, and x is said to join u and v. We write x = uv and say that u and v are adjacent points (sometimes denoted u adj v); point u and line x are incident with each other, as are v and x. If two distinct lines x and y are incident with a common point, then they are adjacent lines. A graph with p points and q lines is called a (p, q) graph. The (1,0) graph is trivial.

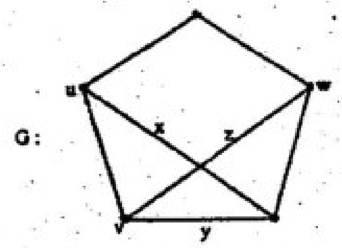


Fig. 1.2.1.A graph to illustrate adjacency.

It is customary to represent a graph by means of a diagram and to refer to it as the graph. Thus, in graph G of Fig. 1.2.1, the points u and v are adjacent but u and w are not; lines x and y are adjacent but x and z are not. Although the lines x and z intersect in the diagram, their intersection is not a point of the graph. There are several variations of graphs which deserve mention. Note that the definition of graph permits no loop, that is, no line joining a point to itself. In a multigraph, no loops are allowed but more than one line can join two points; these are called multiple lines. If both loops and multiple lines are permitted, we have a pseudo graph. Figure 1.2.2 shows a multigraph and a pseudograph with the same "underlying graph", a triangle.

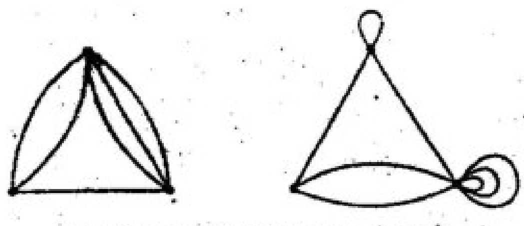


Fig. 1.2.2A multigraph and a pseudograph.

A directed graph or digraph D consists of a finite nonempty set V of points together with a precribed collection X of ordered pairs of distinct points. The elements of X are directed lines or arcs. By definition, a digraph has no loops or multiple arcs. An oriented graph is a digraph having no symmetric pair of directed lines. In fig 1.2.3 all digraphs with three

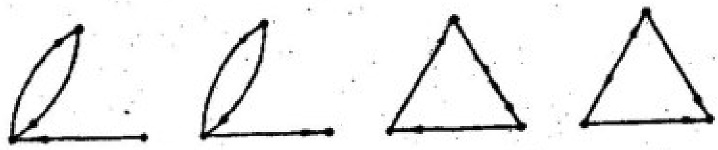


Fig. 1.2.3 The digraphs with three points and three arcs.

ponints and three arcs are shown, the last two are oriented graphs.

A graph G is labeled when the p points are sistinguished from one another by names such as v_1, v_2....., v_p. For exmaple the two graphs G_1 and G_2 of Fig 1.2.4 are labeled but G_3 is not.

Two graphs G and H are isomorphic (written $G \cong H$ or sometimes $G = H$) if there exists a one to one correspondence between their points sets which proserves adjacency. For example G_1 and G_2 of Fig 1.2.4 are isomorphic under the correspondence, $v_i \leftrightarrow u_{ii}$ and incidentally G_3 is isomorphic with each of them. It goes without saying that isomorphism is an equivalence telation of graphs.

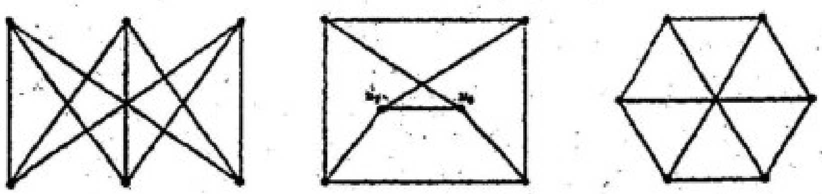
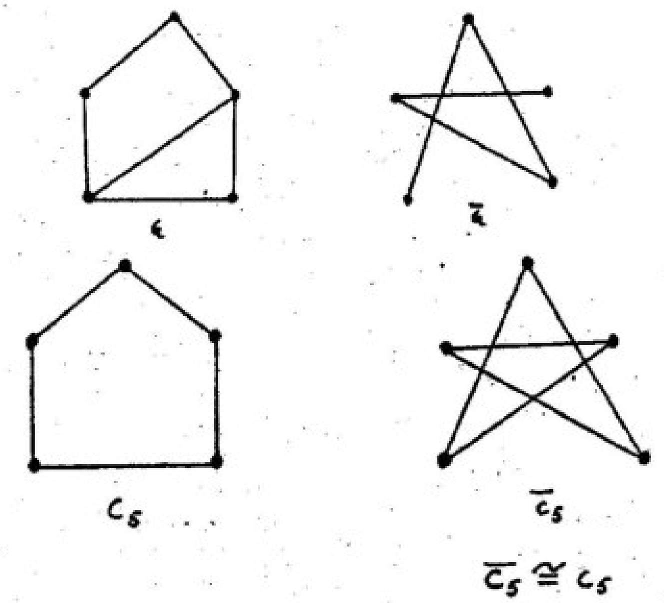


Fig. 1.2.4Labeled and unlabeled graphs.

An invariant of a graphs isomorphic to G. Thus the numbers p and q are certainly invariants. A complete set of invariants determines a graph up to isomorphism. For example the numbers p and q constitute such a set for all graphs with less than four points. No decent complete set of invariants for a graph is known.

Complement $\overline{G} = (V, \overline{X})$ of a graph $G = (V, X)$ has the same point set as G and its line set is the set complement $\overline{X}$ of X; i.e., uv is a line of $\overline{G}$ if and only if uv is not a line of G.

A graph G is said to be **self complementary** if $G \cong \overline{G}$.



1.3. Subgraphs : A subgraph of G is a graph having all of its points and lines in G. If G_1 is subgraph of G, then G is a supergraph of G_1 the induced subgraph <s> if a only if they are adacent in G. In Fig 1.3.1 G_2 is spanning subgraph of G but G_1 is not, is an induced subgraph but G_2 is not.

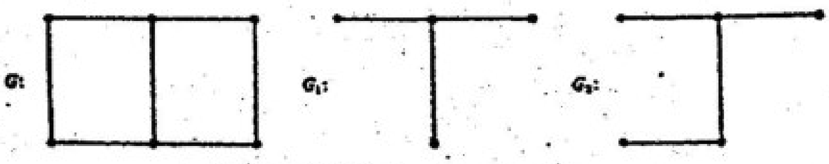


Fig. 1.3.1 A graph and two subgraphs.

The removal of a points v_i from a graph G results in that subgraph G-v_i of G consisting of all points of G except v_i and all lines not incident with v_i. Thus G - v_i is the maximal subgraph of G not containing v_i. On the other hand, the removal ot a line x_j fro G yields the spanning subgraph G- x_j containing all lines of G except x_j. Thus G - x_j is the maximal subgraph of G not containing x_j. The removal of set of points or lines from G is defined by the removal of single elements in succession. On the ther hand, if v_i and v_j are not adjacent in G, the addition of line $v_i v_j$ results in the smallest supergraph of G containing the line $v_i v_j$. These concepts are illustrated in Fig 1.3.2

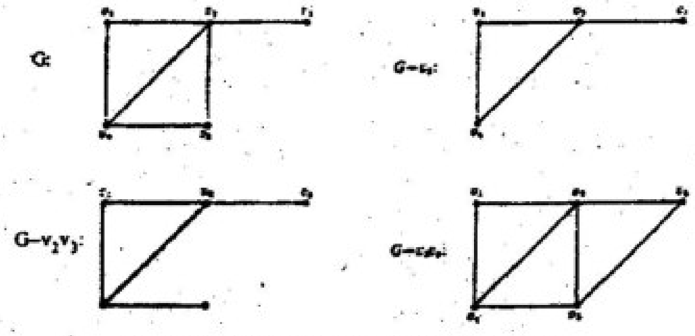


Fig. 1.3.2 A graph plus or minus a specific point or line.

There are certain graphs for which the result of deleting a points or line, or adding a line, is independent of the particular point or line selected. If this is so for a graph, G, we denote the result accordingly by G - v, G - x, or G+ x; see Fig 1.3.3.

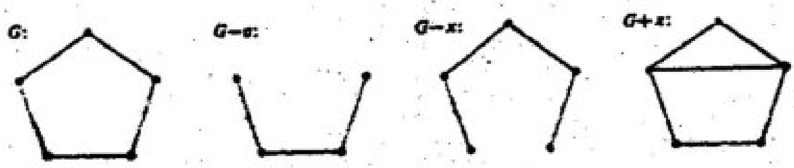
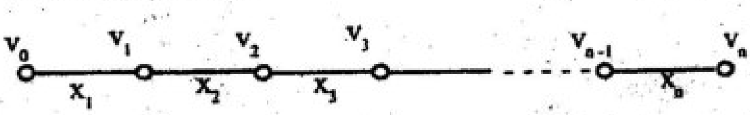


Fig. 1.3.3 A graph plus or minus a point or line.

1.4. Walk, Paths, Cycles and Compontnts :

A **walk** of a graph G is an alternating sequence of points and lines $v_0, x_1, v_1, \ldots\ldots, v_{n-1}x_n, v_n$, beginning and ending with points, in which each line is incident with the two points immediately preceeding and following it. This walk joins v_0 and v_n and may also be denoted $v_0v_1v_2\ldots\ldots v_n$ (the line being evident by context);



it is sometimes called a $v_0 = v_n$ **walk**. It is closed if $v_0 = v_n$ and **open** otherwise. It is a **trail** if all the **lines** are distinct and a **path** if all the points (and thus necessarily all the lines) are **distinct**. If the walk is closed then it is a **cycle** provided its n points are distinct and $n \geq 3$. The vertices v_0 and v_n are called the **end vertices** of the path and the other vertices are called the **inner vertices** of the path. In the labeled graph G of Fig. 11, $v_1 v_2 v_5 v_2 v_3$ is a **walk**, which is not a **trail**. $v_1 v_2 v_5 v_4 v_2 v_3$ is a **trail** which is not a path. $v_1 v_2 v_5 v_4$ is a **path** and $v_2 v_4 v_5 v_2$ is a **cycle**.

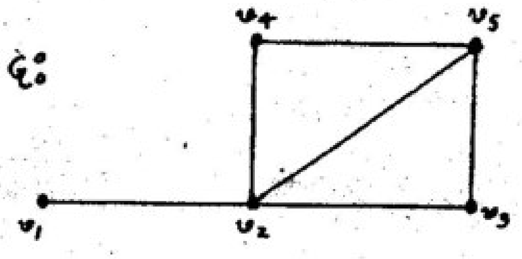


Fig. 1.4.1

We denote by C_n the graph consisting of a cycle with n points and by P_n a path with n points; C_3 is often called a **triangle**.

Ex. In a connected graph, any two longest paths have a common point.

Proof:

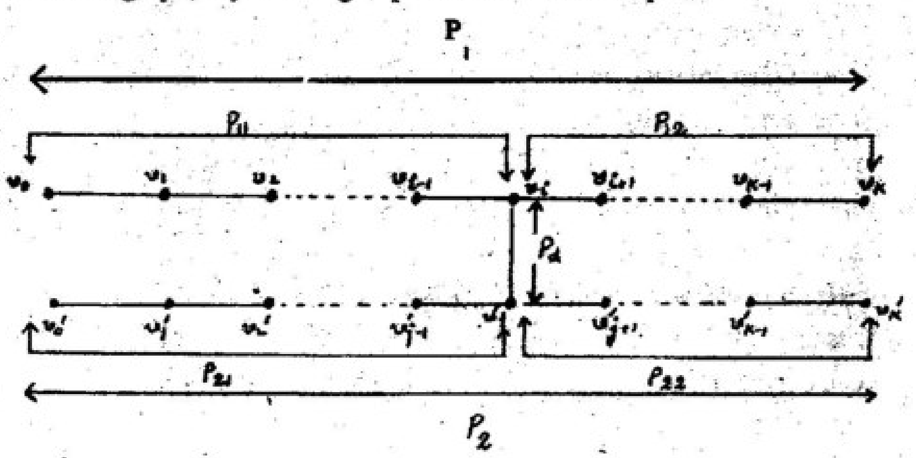


Fig. 1.4.2

Consider any two longest paths P_1 and P_2 in a connected graph G. Let P_1 be denoted by the sequence

$$v_o, v_1, v_2, \ldots\ldots, v_k$$

and P_2 by the sequence

$$v'_o, v'_1, v'_2, \ldots\ldots, v'_k.$$

Assume that P_1 and P_2 have no common point. Since the graph G is connected then for some i, $0 \le i \le k$ and for some j, $0 \le j \le k$, there exists a v_i - v'_j path P_α, such that all points of P_α other that v_i and v'_j are different from those of P_1 and P_2. The paths P_1, P_2 and P_α may be as shown in the Fig.-15.

Let t_1 = length of $v_o - v_i$ path P_{11}

t_2 = length of $v_i - v_k$ path P_{12}

t'_1 = length of $v'_o - v'_j$ path P_{21}

t'_2 = length of $v'_j - v'_k$ path P_{22}

t_α = length of path P_α

Note that

$t_1 + t_2 = t'_1 + t'_2$ = length of a longest path in G, and $t_\alpha > 0$. Without any loss of generality, let

$$t_1 \geq t_2$$

$$t'_1 \geq t'_2 \text{ so that}$$

$$t_1 + t'_1 \geq t_1 + t_2 = t'_1 + t'_2$$

Now it may be verified that the paths P_{11}, P_α and P_{21} together constitute a v_o - v'_o path with its length equal to

$$t_1 + t'_1 + t_\alpha > t_1 + t_2.$$

This contradicts that $t_1 + t_2$ is the length of a longest path in G. This completes the proof.

Two **distinct points** u and v of a graph G are said to be **connected** if there is u – v walk in G. By convention a point is connected to itself. A **graph** is said to be **connected** if every two of its points are connected; otherwise **disconnected**.

With the convention mentioned above, the **relation of connectedness** is an **equivalence relation** on the point set V of a graph. The graphs induced on the equivalence classes of this relation are called the **components** of the graph or equivalently.

A **maximal connected** subgraph of a graph G is called a **component** of G. A component which is K_1 is called a **trivial component**.

A component of G with an odd (even) number of points is called an **odd (even) component** of G.

A disconnected graph has at least two components. The graph of Fig-16 has 10 components.

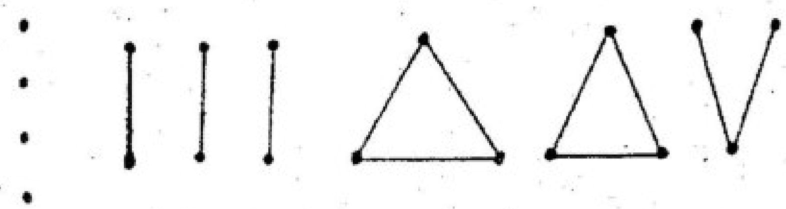


Fig. 1.4.3

The **length** of a walk $v_0 v_1 \ldots\ldots v_n$ is n, the number of occurrences of lines in it.

The **girth** of a graph G denoted by **g (G)**, is the length of a **shortest cycle** (if any) in G: **circumference c** (G), is the length of any **longest cycle**. Note that these terms are undefined if G has no cycle.

The distance d (u, v) between two points u and v in G is the length of a shortest path joining them if any; otherwise $d(u, v) = \infty$. In a connected graph **distance** is a **metric**; that is for all points u, v, w

(i) $d(u, v) \geq 0$ with $d(u, v) = 0$ if and only if $u = v$

(ii) $d(u, v) = d(v, u)$

(iii) $d(u, v) + d(v, w) \geq d(u, w)$.

A **shortest u – v path** is often called a **geodesic**. The diameter **d (G)** of a **connected** graph G is the length of any **longest geodesic**. In the graph G of Fig-14 girth $g = 3$, $c(G) = 4$, $d(G) = 2$.

The **square** G^2 of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever $d(u, v) \leq 2$ in G. The powers $G^3, G^4, \ldots\ldots$ of G are defined similarly.

1.5. Intersection of graphs :

Let S be a set and $F = \{S_1, \ldots\ldots, S_p\}$ a non-empty family of distinct nonempty subsets of S whose union is S, i.e., $S_i \subseteq S$ s.t. $\bigcup_{i=1}^{p} S_i = S$.

The **intersection graph** of F is denoted $\Omega(F)$ and defined by $V(\Omega(F)) = F$ with S_i & S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \phi$. A graph G is an **intersection graph on S** if $\exists$ a family F of subsets of S for which $G \cong \Omega(F)$.

Theorem 1.5.1.: Every graph is an intersection graph.

Proof: For each point v_i of G, let S_i be the union of $\{v_i\}$ with the set of lines incident with v_i.

$S_i = \{v_i\} \cup (v_i x_i)$, if $v_i x_i$ adj

$S_j = \{v_j\} \cup (v_j x_j)$.

If v_i & v_j are adj then S_i & S_j will have that line in common and consequently $S_i \cap S_j \neq \phi$. So we can get a one one correspondence between the points v_i of G and the points S_i of F where $F = \{S_i\}$ which preserves adjacency. Thus $G \cong \Omega(F)$. So G is an intersection graph.

Ex. 1.:

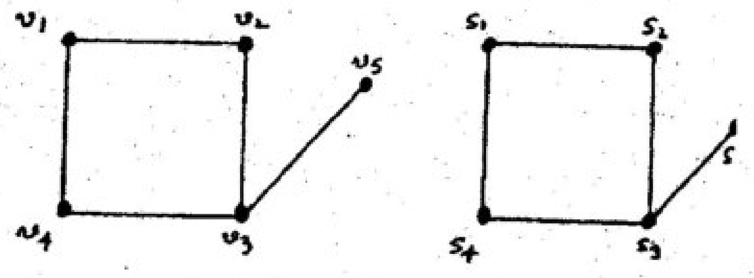


Fig. 1.5.1

$S_1 = \{v_1, v_1v_2, v_1v_4\}$

$S_2 = \{v_2, v_2v_1, v_2v_3\}$

$S_3 = \{v_3, v_2v_3, v_3v_4, v_3v_5\}$

$S_4 = \{v_4, v_4v_1, v_4v_3\}$

$S_5 = \{v_5, v_3v_5\}$

$F = \{S_i\}_{i=1}^{5}$

Ex. 2.:

$S = \{x_1, x_2, x_4\}$

$S_1 = \{x_1, x_2\}, S_2 = \{x_2\},$

$S_3 = \{x_2, x_4\}, S_4 = \{x_4\}.$

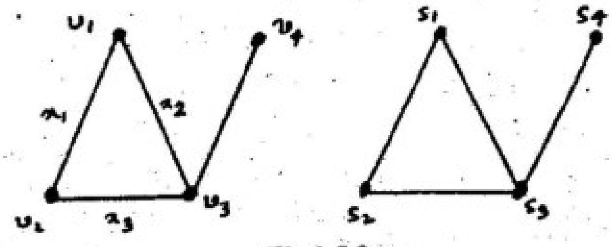


Fig.1.5.2

In view of this theorem, we can define another invariant, the **intersection number w(G)** of a given graph G, which is the minimum number of elements in the set S such that G is an intersection graph on S.

Corollary 1.5.1. (a): If G is connected and $p \geq 3$, then $w(G) \leq q$.

Proof: In this case, the points can be omitted from the sets S_i used in the proof of the theorem 1.3.1, so that $S = X(G)$.

Corollary 1.5.1(b): If G has p_o isolated points and no K_2 component, then $w(G) \leq q + p_0$.

Theorem 1.5.2: Let G be a connected graph with $p>3$ points. Then $w(G)=q$ if and only if G has no triangles.

Proof: We first prove the sufficiency. In view of cor. 1.5.1(a), it is only necessary to show that $w(G) \geq q$ for any connected graph G with at least 4 points having no triangles.

By definition of the intersection number, G is isomorphic with an intersection graph $\Omega(F)$ on a set S with $|S| = w(G)$. For each point v_i of G, let S_i be the corresponding set. Because G has no triangles, no element of S can belong to more than two of the sets S_i (because otherwise those sets will be mutually adjacent and thus forming a triangle in $\Omega(F) \cong G$) and $S_i \cap S_j \neq \phi$ if and only if $v_i v_j$ is a line of G. Thus we can from a $1-1$ correspondence between the lines of G and those elements of S which belong to exactly two sets S_i. Therefore $w(G) = |S| \geq q$ so that $w(G) = q$.

To prove the necessity, let $w(G) = q$ and assume that G has a triangle.

Then let G_1 be a maximal triangle free spanning subgraph of G. By the preceding paragraph $w(G_1) = q_1 = |X(G_1)|$.

Suppose that $G_1 \cong \Omega$ (F), where F is a family of subsets of some set S with cardinality q_1. Let x be a line of G not in G_1 and consider $G_2 = G_1 + x$. Since G_1 is maximal triangle free, G_2 must have some triangle, say $u_1\, u_2\, u_3$, where $x = u_1\, u_3$. Denote by S_1, S_2, S_3 the subsets of S corresponding to u_1, u_2, u_3. Now if u_2 is adjacent to only u_1 & u_3 in G_1, replace S_2 by a singleton chosen from $S_1 \cap S_2$ and add that element to S_3. Otherwise, replace S_3 by the union of S_3 and any element in $S_1 \cap S_2$. In either case this gives a family F′ of distinct subsets of S such that $G_2 \cong \Omega$ (F′). Thus w $(G_2) \le q_1$ while $|X(G_2)| = q_1 + 1$. If $G_2 \cong G$, there is nothing to prove (then G has a triangle and $w(G) \le q_1 < |X(G)| = q_1 + 1$ contradicting the hypothesis that w (G) = q and hence G must not have a triangle). But if $G_2 \ne G$, then let

$$|X(G)| - |X(G_2)| = q_o.$$

It follows that G is an intersection graph on a set with $q_1 + q_o$ elements. However, $q_1 + q_o = q - 1$. Thus w (G) < q completing the proof.

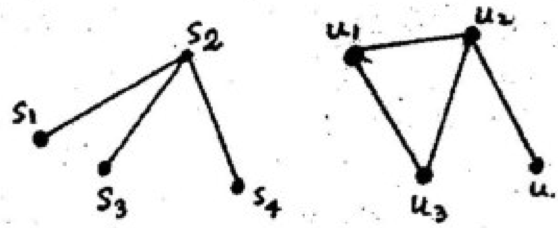


Fig. 1.5.3

1.6. Degrees :

The **Degree** of a point v_i in graph G, denoted d_i or deg v_i, is the number of **lines incident** with v_i. A point is called **odd or even** depending on whether its degree is odd or even.

Since every line is incident with two points, it contributes 2 to the sum of the degrees of the points. We thus have the following theorem.

Theorem 1.2.1: the sum of the degrees of the points of a graph G is twice the number of lines

$$\Sigma \deg v_i = 2q.$$

Corollary 1.2.1(a): In any graph, the number of points of odd degree is even.

Proof: We have by theorem 1.2.1 the sum of the degrees of the points of a graph is even. Let us consider this number as the sum of two parts: one of them is the sum of the even degree, and the other is that of the odd degrees. The former is obviously even; so that latter must also be even since their sum is even. But the sum of odd numbers is even only if there are an even number of them. Therefore the number of points of odd degree is even.

In a (p, q) graph $0 \leq \deg v \leq p - 1$ for every point v. The minimum degree among the points G is denoted min deg G or $\delta(G)$ while $\Delta(G) = \max \deg G$ is the largest such number. If $\delta(G) = \Delta(G) = r$ then all the points have the same degree and G is called **regular of degree r**. We then speak of the degree of G and write $\deg G = r$.

A regular graph of degree 0 has no lines at all. If G is regular of degree I then every component contains exactly one line; if it is regular of degree 2 every component is a cycle, and conversely of course. The first interesting regular graphs are those of degree 3; such graphs are called **cubic**.

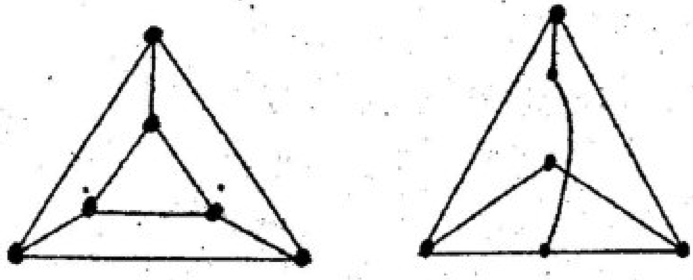


Fig.

Corollary 1.1. (b): Every cubic graph has an even number of points.

The **complete graph** K_p has every pair of its p points adjacent. Thus K_p has p_{c_2} lines and is regular of degree $p - 1$. The graphs $\overline{K_p}$ are **totally-disconnected**, and are regular of degree 0.

Theorem 1.2.2. (Problem of Ramsey): For any graph G with six points, G or $\overline{G}$ contains a triangle.

Proof: Let v be a point of a graph G with six points. Since v is adjacent either in G or in $\overline{G}$ to the other five points of G, we can assume without loss of generality that there are three points u_1, u_2, u_3 adjacent to v in G. If any two of these points are adjacent, then they are two points of a triangle whose third point is v. If no two of them are adjacent in G, then u_1, u_2, u_3 are the points of a triangle in $\overline{G}$.

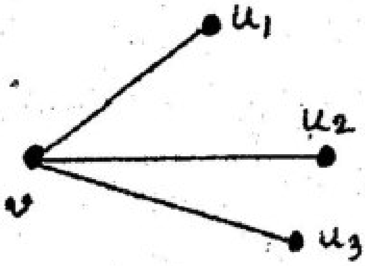


Fig.

The result of theorem 1.2.2 suggests the general question: what is the smallest integer r (m, n) such that every graph with r (m, n) points contains K_m or $\overline{K}_n$?

The values r (m, n) are called **Ramsey numbers**. Of course r(m, n)=r(n, m).

Theorem 1.2.3.: The maximum number of lines among all p points graphs with no triangle is $\left[\frac{p^2}{4}\right]$.

[As usual, let [r] be the greatest integer not exceeding the real number r, and {r} = - [-r], the smallest integer not less than r.]

Proof: The statement is obvious for small value of p. An inductive proof may be given separately for odd p and even p; suppose the statement is true for all even $p \le 2n$. We then prove it for $p = 2n+2$. Let G be a graph with $p = 2n+2$ points and no triangles. Since G is not totally disconnected, there are adjacent points u and v. The subgraph $G' = G - \{u, v\}$ contains $(2n+2)-2 = 2n$ points and no triangles, and by our assumption G′ has atmost $\left[\frac{4n^2}{4}\right] = n^2$ lines.

How many more lines can G have? There is no point w such that both u and v are adjacent to w because then u, v, w will form a triangle in G. Thus if u is adjacent to k points of G′, v can be adjacent to at most $2n-k$ points. Then G has at most

$$n^2 + k + (2n - k) + 1 \text{ lines}$$

i.e., $$n^2 + 2n + 1 \text{ lines}$$

i.e., $$\frac{4}{4}(n^2 + 2n + 1) \text{ lines}$$

i.e., $$\frac{\{2(n+1)\}^2}{4} \text{ lines} = \frac{p^2}{4} = \left[\frac{p^2}{4}\right] \text{ lines}$$

$$p = (2n+2)$$

To complete the proof, we must show that for all even p, there exists a $\left(p, \frac{p^2}{4}\right)$ graph with no triangles. Take two graphs V_1 and V_2 of $\frac{p}{2}$ points each and join each point of V_1 with each pt of V_2. For p = 6, this graph is of Fig-19.

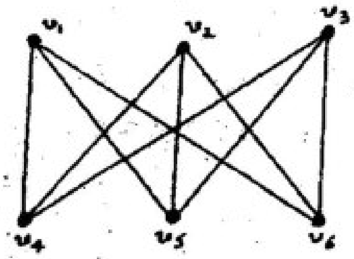


Fig. 19.

The proof is similar for p odd.

A bigraph (or bipartite graph) G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 s.t. every line of G joins V_1 with V_2 or in other words a bigraph is a graph whose points are divided into non- overlapping sets so that points in the same set are not connected by lines.

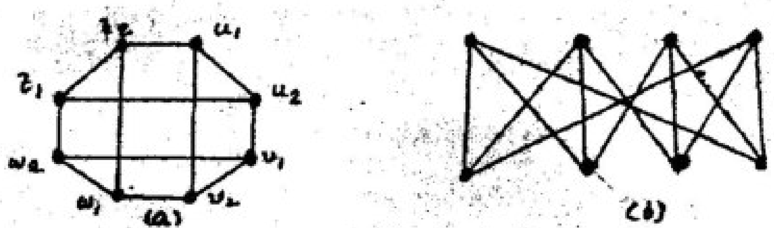


Fig. 19.

For example the graph of Fig-20 (a) can be redrawn in the form of Fig-20 (b) to display the fact that it is a bigraph.

If G contains every line joining V_1 and V_2 then G is a **complete bigraph**. If V_1 and V_2 have m and n points, we write $G = K_{m,n} = k(m, n)$. Clearly $K_{m,n}$ has mn lines. Thus if p is even $K\left(\frac{p}{2}, \frac{p}{2}\right)$ has $\frac{p^2}{4}$ lines. While if p is odd $K\left(\left[\frac{p}{2}\right], \left\{\frac{p}{2}\right\}\right)$ has $\left[\frac{p}{2}\right]\left[\frac{p}{2}\right] = \left[\frac{p^2}{4}\right]$ lines.

Ex. 1. Let G be a (p, q) graph all of whose points have degrees k or k + 1. If G has $p_k > 0$ points of degree k and p_{k+1} points of degree k + 1, then

$$P_k = (k+1)p - 2q$$

Solution: We have

$$\sum_{i=1}^{p} \deg v_i = 2q.$$

There are p_k points of degree k and p_{k+1} $(= p - p_k)$ points of degree k + 1, so

$$P_k(k) + (p - p_k)(k+1) = 2q$$

or $\quad p(k + 1) - p_k = 2q.$

or $\quad p_k = (k + 1)p - 2q.$

Theorem 1.2.4.: A graph is bipartite if and only if all its cycles are even.

Proof: If G is a bigraph then its point set V can be partitioned into two sets V_1 & V_2 so that every line of G joins a point of V_1 with a point of V_2. Thus every cycle $v_1v_2......v_nv_1$ in G necessarily has its oddly subscripted points in V_1, say and the others in V_2, so that the length n is even.

Conversely we assume without loss of generality, that G is connected (for otherwise we can consider the components of G separately). Take any point $v_1 \in V$, and let V_1 consist of v_1 and all points at even distance from v_1, while $V_2 = V - V_1$. Since all the cycles of G are even, every line of G joins a point of V_1 with a point of V_2. For suppose there is a line uv joining two points u & v of G. Then the union of geodesics, from v_1 to v and from v_1 to u together with the line uv contains an odd cycle, a contradiction).

[Since, the points of V_1 are at even distances, so the geodesic from v_1 to v (which is the shortest $v_1 - v$ path) and v_1 to u are even and hence their union is also even].

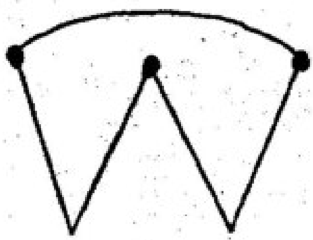


Fig.

1.7 Degree Sequence :

If d_i, $1 \le i \le n$ are the degrees of the vertices of a graph, in any order, then the sequence $(d_i)_1^n$ is called a degree sequence of the graph. An N-sequence $(d_i)_1^n$ is called a degre sequence if it is a degree sequence of some graphs. The graph is said to realize the sequence. The set of distinct non negative integers occuring in a degree sequence of a graph is called its degree set. A set of non negative inteqers is called a gegree set if it is the degree set of some graph. The graph is said to realize the degree set. Two graphs with the same degree sequence are said to the degree equivalent.

It is customary to denote an integer sequence by the elements of its set raised to appropriate powers. This is called the power notation. Thus a gegree sequence of the graph of Figure 1.7.1 is 2,2,3,3,4,4 and this may be represented in power notation as $2^2, 3^2, 4^2$, its degree set being {2, 3, 4}.

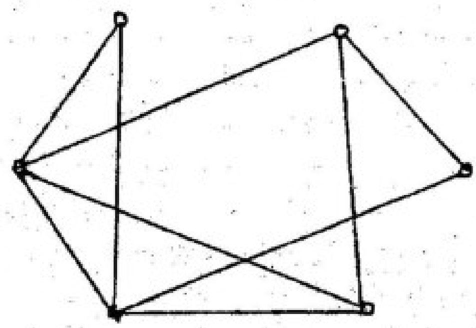


Fig. 1.7.1

If the degree sequence is arranged as an non decreasing positive sequence $d_1^{n_1} d_2^{n_2} .... d_k^{n_k}$, $(d_1 > d_2 >.....> d_k)$ then the sequence $n_1, n_2, ..., n_k$ is called the frequency sequence of the graph.

1.7. Let $D = (d_i)_1^n$ be an N sequence and k be any interger $1 \le k \le n$. Let $D' = (d_i')_1^n$ be the sequence obtained from D by setting $d_k = 0$ and $d_i' = d_i - 1$ for the d_k lergest elements of D other than d_k. Let H_k be the graph obtained on the vertex set $V = (v_1 v_2 ....., v_n)$ by joining v_k to the d_k vertices corresponding to the d_k elements used to obtain D' and H_k is called laying off d_k and D' is called the residual sequence and H_k the subgraph obtained by laying off d_k.

Theorem 1.7.1 (Wang and Kleitman)

An N sequence is degree sequence if the residual sequence obtained by lying off say non zero elements of the sequence is a degree sequence.

Proof : (i) Sufficiency Suppose d_k is the non zero elements laid off and the residual sequence $(d_i')_1^n$ is a degree sequence. Then ther exists a graph G' realizing $(d_i')_1^n$ in which v_k has degree zero and some v_k vertices, say v_{ij}, $1 \le j \le dk$ have degree $d_{ij} - 1$. By joining v_k to these vertices we get a graph G with $(d_i)_1^n$ as its degree sequence. (Obeseve that the subgraph obtained by such joining is precisely the subgraph H_k obtained by lying off d_k).

(ii) necessity We are given that there is a graph realizing $D = (d_j)_1^n$. Let d_k be the element to be laid off. First we claim that there is a graph lealizing D in which v_k is adjacent to all the vertices in the set S of d_k largest elements of $D - (d_k)$. If not G be a graph lealizing

D and such that v_k is adjacent to the maximum possible number of vertices in S. Then ther is a vertex v_i in S to which v_k is not adjacent and hence a vertex v_j outside S to which v_k is adjacent (since $d(v_k) = |S|$. By the definition of S, $d_j \le d_i$. Therefore, there is a vertex v_h in $V-(v_k)$ adjacent to v_i, but not adjacent to v_j (See Fig 1.7.2) Note that v_h may be in S.

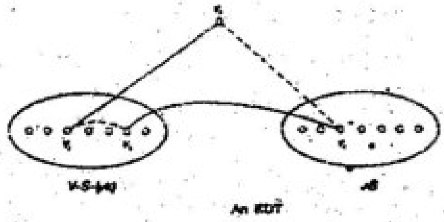


Fig. 1.7.2 An EDT

Now contruct a graph H from G by deleting the edges v_jv_k and v_hv_i and adding the edges v_jv_h and v_iv_k. This operation does not change the degree sequence. Thus H is a graph realizing the sequence, in which one more vertex, namely v_i of S. is adjacent to v_k than in G This contradicts the choice of G and established the claim.

To complete the proof, if G is a graph realizing the given sequence and in which v_k is adjacemnt to all vertices of S, let $G' = G - v_k$.Then G' has the residual degree sequence obtained by laying off d_k.

Suppose the subgraph H on the vertuces (v_i, v_j, v_r, v_s) ofa multi graph G contains the edges v_iv_j and v_rv_s. Then the operation of deleting these edges and introducing a pair of new deges $(v_iv_s \text{ and } v_jv_r)$ or $(v_iv_r \text{ and } v_jv_s)$ is called an elementary degree preserving transformation (and EDT for short)

Remarks (i) The result of an EDT is clearly a degree equivalent multigraph.

(ii) If an EDT is applied to a graph, the result will be a graph only if the latter pair of edges $(v_iv_r \text{ and } v_jv_r)$ or $(v_iv_r \text{ and } v_jv_s)$ does not exist in G

Theorem 1.7.2. (Hakimi)

If G_1 and G_2 are degree equivalent graphs then one can be obtained from the other by a finite sequence of EDTs.

Proof Superpose G_1 and G_2 such that each vertex of G_2 coincides with a vertex of G_1 with the same degree. Imagine the edges of G_1 are colourd blue and the edges of G_2 are coloured red. Then in the surperposed multigraph H, at every vertex, the number of blue deges incident equals the number of red edges incident. We refer to this as blue red degree parity. If there is a blue dege v_iv_j and a red dege v_iv_j in H, we call it a blue red parallel pair. Let K be the graph obtained from H by deleting all such paralled pairs. Then K is the null graph iff G_1 and G_2 are label isomorphic in H and hence originally isomorphic. If this is not the case, we shall show that, by a sequence of EDT's we can create more parallel pairs and delete them, till the final resultant graph is null. This would establish the theorem.

Let B and R denote the sets of blue and red deges in K. If $v_iv_j \in B$ we want to show that we can produce a parallel pair at v_iv_j so that the pair can be deleted. This would establish the claim made above. Now by construction, there is blue red degree parity at every vertex of K. So there are red edges v_iv_k, v_jv_r in K. If $v_k \neq v_r$ (see Fig 1.7.3(a) and EDT in G_2 switching the red deges v_iv_k, v_jv_r to positions v_iv_j, v_kv_r produces a blue red parallel pair at v_iv_j.

If $v_k = v_r$ again by degree parity at v_i there are at least two blue edges. Let v_kv_s be one such blue dege. Then v_s is distinct from both v_i and v_j, for otherwise there would be blue red parallel pair v_iv_k or v_jv_r. But then there is another red edge v_sv_t v_t distinct from v_i or v_j. Suppose $v_t \neq v_i$. The two subcases, $v_t = v_j$ and $v_t \neq v_j$ are illurstrated in Figs 1.7.3(b) and 1.7.3(c) respectively. In the former case, and EDT of G_2 switching v_iv_k and v_sv_t to positions v_iv_j, v_sv_k products a blue red pair at v_iv_j and v_kv_s.

In the latter case one EDT of G_2 switching v_iv_k and v_sv_t to positions v_sv_k, v_tv_i produces a blue red parallel pair at v_kv_s (which can be deleted). Another EDT of G_2 switching the blue red pair v_tv_i, v_jv_k to positions v_iv_j, v_tv_k will produce a blue red pair at v_iv_j (see Fig 1.7.3(d))

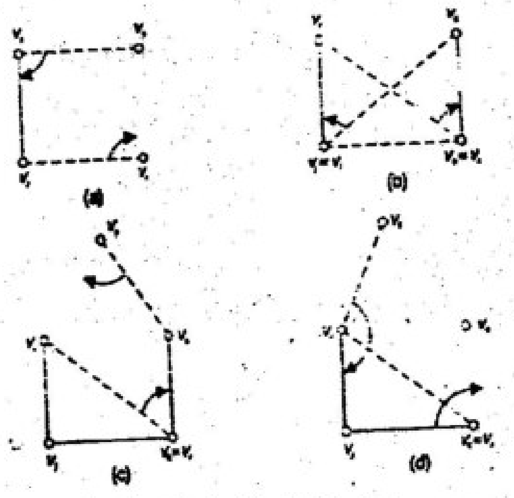


Fig. 17.3 Illustration for Hakimi's theorem

Since in both cases we get a blue red pair at v_iv_j position our claim is established and the proof of the therorem is complete.

1.8 Trees :

There is one simple and important kind of graph called trees which are important for thier applications to many different fields.

A graph is **acyclic** if it has no cycles.

A **tree** is a connected acylic graph

Any graph without cycle is a **forest**. Thus the components of a forest are trees.

Theorem 1.8.1: The following statements are equivalent for a graph G

(1) G is a tree

(2) Every two points of G are joined by a unique path.

(3) G is connected and $p = q + 1$

(4) G is acyclic and $p = q + 1$

(5) G is acylic and if any two nonadjacent points of G are joined by a line x, then G + x has exactly one cycle

(6) G is connected, is not k_p for $p \geq 3$, and if any two nonadjacent points of G are joined by a line x, then G + x has exactly one cycle.

(7) G is not $K_3 \cup K_1$ or $K_3 \cup K_2$, $p = q + 1$, and if two nonadjacent points of G are joined by a line x, then G + x has exactly one cycle.

Proof: $(1) \Rightarrow (2)$

Since G is connected every two points of G are joined by a path. If possible, let us assume that there are two distinct paths p_1 and p_2 joining the points u and v of G. Let w_1 be the first point of the path p_1 (as we traverse p_1 from u to v) which lies on p_2.

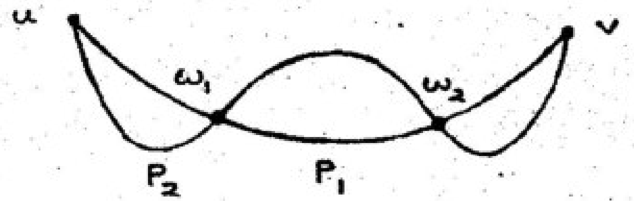


Fig. 1.8.1

Let w_2 be the point next to w_1 on path P_1 which lie on P_2. From the above selection, we see that the portion of P_1 which joins w_1 and w_2 along with the portion of P_2 which joins w_2 and

w_1 froms a cycle. This contradicts the fact that G is acyclic. Therefore every two points of G are joined by a unique path.

(2) ⇒ (3)

Since every two points of G are joined by a unique path, clearly G is connected, we prove $p = q + 1$ by induction. It is obvious for connected graphs of one or two points. Assume it is true for connected graphs with fewer than p points. If G has p points, the removal of any line from G disconnects G, because of the uniquness of paths, and in fact this new graph will have exactly two components, each components consisting of fewer than p points. If n (< p) is the number of points in one component of G - x, then p - n will be the number of points in other component of G - x. Then the corresponding number of lines are n - 1 and p - n - 1 respectively. Now for graph G,

$$q = n - 1 + p - n - 1 + 1$$

or $$q = p - 1 \Rightarrow p = q + 1.$$

Hence the result is true for all values of p.

(3) ⇒ (4)

Assume G has a cycle of length n. Then there are n points and n lines on the cycle and for each of the p - n points not on the cycle, there is a distinct line on the shortest path connecting these points to a point of the cycle. Hence any point of G corresponds to a distinct line of G. So $q \geq p$, which is a contradiction to the fact that $p = q + 1$. Therefore G must be acyclic.

(4) ⇒ (5)

Since G is acyclic, each component of G is a tree. Let there be k components of G. Since each component is a tree, the number of points in it is one more than the number of lines. Then considering the totality of number of points and number of lines in all the components we see that $p = q + k$. But from the hypothesis we have $p = q + 1$, therefore $k = 1$. Hence G is connected and is a tree. So there is exactly one path connecting any two points of G. Let the line x joins two nonadjacent points u and v of G. Then the single path connecting u and v in G together with the line x froms exactly one cycle in G.

(5) ⇒ (6)

Given that by joining two non-adjacent points u and v by a line x, we get a cycle in G + x. So the two non-adjacent points u and v must be connected by a path in G, other than x. Hence G is connected. Now if G is K_p, for $p \geq 3$, then G must contain a cycle. but G is acyclic, so G is not K_p, for $p \geq 3$.

(6) ⇒ (7)

We shall prove that any two points of G are joined by a unique path. Since by joining two non-adjacent points u and v by a line x, we get a cycle, so u and v must be connected by a path other than x. Hence G is connected. If the points u and v are connected by two paths then G has a cycle. If a cycle contains 4 or more points then we get two cycles out of it by joining a line x to it. This is not the case by hypothesis. Hence a cycle of G is K_3. Since $G \neq K_p$, for $p \geq 3$, therefore K_3(if it exists) must be a proper subgraph of G. Since G is connected we may assume that there is another point in G which is joined to a point of this K_3. Then it is clear that a line x may be added so as to form at least two cycles in G + x. This is not the case by hypothesis. Therefore K_3 cannot be a proper subgraph of G and also any twopoints of G are connected by a unique path. Then this implies that G is connected and $p = q + 1$.

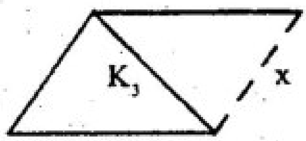


Fig. 1.8.2.

Finally K_3 is not a proper subgraph implies that

$$G \neq K_3 \cup K_1 \text{ or } K_3 \cup K_2$$

(7) ⇒ (1)

If G has a cycle that cycle must be a triangle which is a component of G by an argument in the preceeding paragraph. This component has 3 points and 3 lines. All other components of G must be trees and in order to make $p = q + 1$, there can be only one other component. If this tree contains a path of length two it will be possible to add a line x to G and obtain two cycles in G + x. Thus this tree must be either K_1 or K_2. So G must be $K_3 \cup K_1$ or $K_3 \cup K_2$ which are the graphs which have been excluded. Then G is acyclic. But if G is acyclic and $p = q + 1$ then G is connected since (4) ⇒ (5) ⇒ (6). So G is a tree.

Ex.1: Every non-trivial tree has at least two end points.

Proof: Let u and v be two points of a tree at maximum distance. If possible let v be not an end point. We know that a point which is not an end point in a tree is a cutpoint. So v is a cutpoint and there is a point w in a different component of G - v than u. Hence d(u, w) > d(u, v) which contradicts the fact that u and v are at maximum distance. So v must be an end point of the tree. Similarly it can be shown that u is also an end point of the tree.

The **ecentricity** e(v) of a point v in a connected graph G is maximum d(u, v) for all u in G.

The **radius** r(G) of a graph G is the minimum ecentricity of the points.

The **diameter** d(G) of a graph G is the maximum ecentricity of the points.

A point v is called a **central point** of a graph G if e(v) = r(G).

The set of all central points of a graph G is called the **centre** of G.

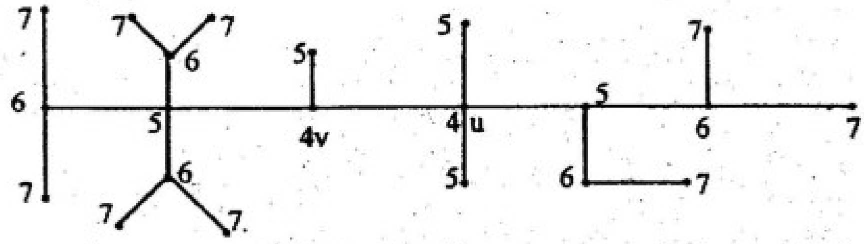


Fig. 1.8.3. The ecentricities of the points of a tree
r(G) = 4, d(G) = 7. Central points are u, v. Centre is {u, v}.

Theorem 1.8.2: Every tree has a centre consisting of either one point or two adjacent points.

Proof: The result is obvious for K_1 and K_2. We show that any other tree T has the same central points as the tree T′ obtained by removing all the end points of T.

We see that the maximum of the distances from a given point u to any other point v of T will occur only when v is an end point of T. Thus the ecentricity of each point in T′ will be exactly one less than the ecentricity of that point in T. Therefore the points of T which possess minimum ecentricity in T are the same points having minimum ecentricity in T′. Hence T and T′ have the same central points. If the process of removing end points is repeated we obtain successive trees having the same centre as T. Since T is finite we eventually obtain a tree which is either K_1 or K_2. In either case all points of this ultimate tree constitute the centre of G; which consists of either the point of K_1 or the two points of K_2.

A **branch** at a point u of a tree is a maximal subtree containing u as an end point.

The **weight** at a point u of T is the maximum number of lines in any branch at u.

Centroid point of a tree is a point v if v has minimum weight. The **centroid** of T is the set of all centroid points.

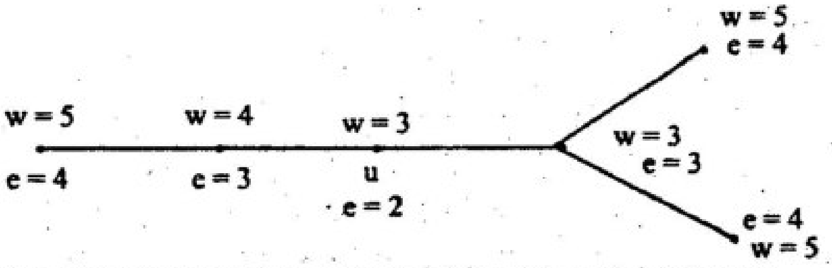


Fig. 1.8.4. The weights and ecentricities of the points of a tree.

Central point u, centre {u}.

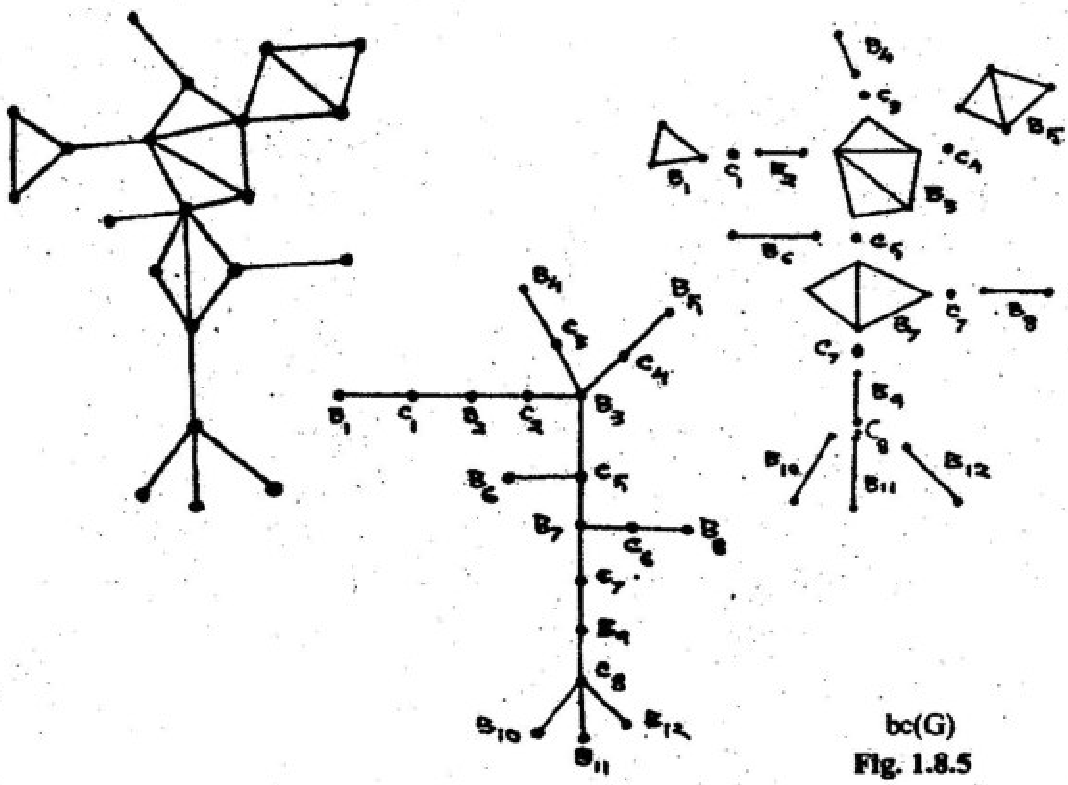
Centroid points u, v, Centroid {u, v}.

Theorem 1.8.3: Every tree has a centroid consisting of either one-point or two adjacent points.

Block-cutpoint graph:

For a connected graph G with blocks $\{B_i\}$ and cutpoints $\{c_j\}$, the **block-cutpoint graph** of G denoted by **bc(G)** is defined as the graph having point set $\{B_i\} \cup \{c_j\}$ with two points adjacent if one corresponds to a block B_i and the other to a cutpoint c_j contained in B_i.

Remark: Every bc(G) is a bi-graph with $v_1 = \{B_i\}, v_2 = \{c_j\}$.



bc(G)

Fig. 1.8.5

Theorem 1.8.4: A graph G is a block-cutpoint graph of some graph H if and only if it is a tree in which the distance between any two end points is even.

In view of this theorem, we will speak of the **block-cutpoint tree** of a graph.

1.9 Spanning Tree : We shall now consider trees at subgraphs of lager graphs. consider a connected graph $G = (V, E)$, which contains a subgraph which is the tree $T = (V', E')$. The deges of T are called branches and the deges of G which are not in T are called chords (both relative to T). If $V = V'$, then T is said to be a spanning tree of the graph G. A collection of spanning trees of graphs G, one for each component of G, is termed as spanning forest.

Proposition 1.9.1

Every connected graph has a spanning tree.

Proof : If the connected graph G is not a tree it has atleast one cycle C. If e = uv is any edge of C, G - e (where P.....is the symmedtric difference). Continuing this process of removing cyclic edges we end up with an acyclic connected subgraph T of G, which is also spanning, since no vertices have been removed.

Proposition 1.9.2

Let v be any vertex of a connected graph G. Then G has a spanning tree T preserving the distances from v.

Proof Required to find a spanning tree T of G such that for each $u \in V = V(G) = V(T)$, $d_G(v, u) = d_T(v, u)$.

Consider the neighbourhoods $N_i(v) = \{u \in V \mid d_G(v, u) = i\}$ of v, $1 \leq i \leq e$ where $e = e(v)$. Let H be the graph obtained from G by removing all edges in each $\{N_i(v)\}$. Clearly H is connected. Recalling the definition 2.13, let $\{B_i(v)\}_H$ does not contain any cycle. If $\{B_2(v)\}_H$ contains cycles, remove edges from $[N_1(v), N_2(v)]$ sequentially, one edge from each cycle, till it becomes acyclic. Proceeding successively by removing edges from $[N_i(v), N_{i+1}(v)]$ to make $\{B_{i+1}(v)_H$ acyclic, for $1 \leq i \leq e - 1$ we get a spanning tree of H and hence of G. Since, in this procedure one distance path from v to each of the other vertics remains intact, we have $d_G(v,u) = d_T(v, u)$ for each $u \in V$, as desired.

1.10. Cycles, Cocyles. Cycle Space and Cocycle Space :

We describe two vector spaces associated with a graph G: its **'cycle space'** and **'cocycle**

space'. For convenience these two vector spaces will be taken over the two element field $F_2=\{0, 1\}$ in which $1 + 1 \equiv 0 \pmod 2$ (even though the theory can be modified to hold for an arbitrary field).

In particular the E_i which occur repeatedly in the following definitions are always either 0 or 1.

As usual let G be a graph with $v_1, v_2, ..., v_p$ points and $x_1, x_2, ..., x_q$ lines.

A **0-chain** of a graph G is a formal linear combinations $\sum \varepsilon_i v_i$ of points of G

A **1-chain** of a graph G is a formal linear combination $\sum \varepsilon_i x_i$ of lines of G

The **Boundary operator** ∂ maps every 1-chain to a 0-chain according to the following rules:

(i) ∂ is linear

(ii) if $x = uv$ then $\partial x = u + v$.

On the otherhand the **Coboundary operator** δ maps every 0-cahin to a 1-chain according to the following rules:

(i) δ is linear

(ii) $\delta(v) = \sum \varepsilon_i v_i$ where $\varepsilon_i = 1$ whenever x_i is incident with v.

1-chain

$$\sigma_1 = x_1 + x_2 + x_4 + x_9$$
$$\partial\sigma_1 = \partial x_1 + \partial x_2 + \partial x_4 + \partial x_9$$
$$= (v_1 + v_2) + (v_1 + v_3) + (v_2 + v_4) + (v_5 + v_6)$$
$$= (1+1)v_1 + (1+1)v_2 + v_3 + v_4 + v_5 + v_6$$
$$= v_3 + v_4 + v_5 + v_6$$

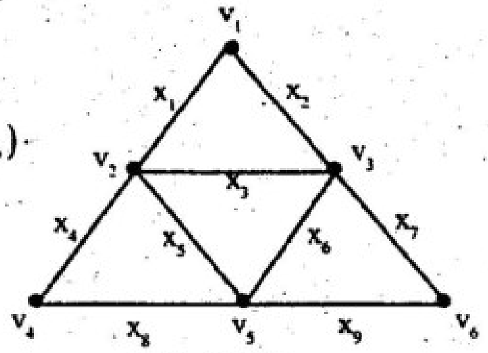


Fig. 1.10.1

0-chain

$$\sigma_0 = v_3 + v_4 + v_5 + v_6$$
$$\delta\sigma_0 = \delta v_3 + \delta v_4 + \delta v_5 + \delta v_6$$
$$= (x_2 + x_3 + x_6 + x_7) + (x_4 + x_8) + (x_5 + x_6 + x_8 + x_9) + (x_7 + x_9)$$
$$= x_2 + x_3 + x_5 + x_5 + (1+1)x_6 + (1+1)x_7 + (1+1)x_8 + (1+1)x_9$$
$$= x_2 + x_3 + x_4 + x_5$$

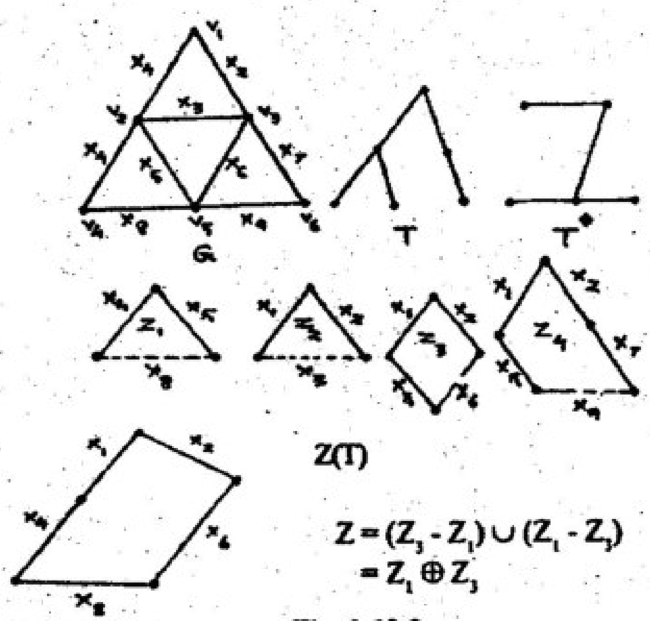
A 1-chain with boundary zero is known as a **cycle vector** of G(and can be regarded as a set of line-disjoint cycles).

The collection of all cycle vectors of a graph G forms a vector space over $F_2 = \{0,1\}$, called the **cycle space** of G. A **cycle basis** of G is defined as a basis for the cycle space of G which consists entirely of cycles. We say that the cycle vector Z depends on the cycles $z_1, z_2, \ldots, z_K$ and it can be written as $Z = \sum_{i=1}^{K} \varepsilon_i Z_i$

A **cutset** of a connected graph G is a collection of lines whose removal results in a disconnected graph.

A **cocycle** of a graph is a **minimal cutset**.

A **Coboundary** of G is the **Coboundary** of some 0-chain in G. The Coboundary of a collection U of points is just the set of all lines joining a point in U to a point not in U. Thus every coboundary is a cutset. Since we define a co-cycle as a minimal cutset of G and any minimal cutset is a coboundary, we see that a cocycle is just a minimal non-zero coboundary. the collection of all coboundaries of G is called the **cocycle space** of G, and a basis for this space which consists entirely of cocycles is called a **cocycle basis** for G.



$$Z = (Z_3 - Z_1) \cup (Z_1 - Z_3)$$
$$= Z_1 \oplus Z_3$$

Fig. 1.10.2

We proceed to construct for the cycle space of G a basis which corresponds to a spanning tree T. In a connected graph G, a **chord** of a spanning tree T is a line of G which is not in T.

The subgraph of G consisting of a spanning tree T and a chord of T has exactly one cycle and that cycle is formed by the unique path joining u and v and the chord uv.

Hence given a spanning tree we get a unique cycle corresponding to a chord of T. Let

Z(T) denote the set of all cycles obtained by the chords of T as mentioned above. Z(T) is linearly independent because a chord belonging to one cocycle is not in any other cycle in Z(T). Also every cycle Z depends on the set Z(T), for Z is the symmetric difference of the cycles determined by the chords of T which lie in Z.

A cycle vector can be regarded as a set of cycles having no common sides between any two. Therefore any cycle vector can be expressed as a formal sum of cycles Z(T). Hence given a spanning tree T of a graph G we get a cycle basis Z(T) of G.

Thus if we define m(G), the **cycle rank**, to be the number of cycles in a basis for the cycle space of G, we have the following result.

Theorem 1.10.1: the cycle rank of a connected graph G is equal to the number of chords of any spanning tree in G.

Corollary 1.10.1(a): If G is a connected (p, q) graph then $m(G) = q - p + 1$.

Corollary 1.10.1(b): If G is a (p, q) graph with k components, then

$$m(G) = q - p + k.$$

Similar results are true for the cocycle space. The **cotree** T^* of a spanning tree T in a connected graph G is the spanning subgraph of G containing exactly those lines of G which are not in T. A cotree of G is the cotree of some spanning tree T. In Fig-3.8, a spanning tree T and its cotree T^* are displayed for the same graph G. The lines of G which are not in T^* are called its **twigs**. The subgraph of G consisting of T^* and any one of its twigs contain exactly one cocycle. The collection of cocycles obtained by adding twigs to T^*, one at a time, is seen to be a basis for the cocycle space of G. It is illustrated in Fig-3.9 for the graph G and cotree T^* of Fig.3.8 for G.

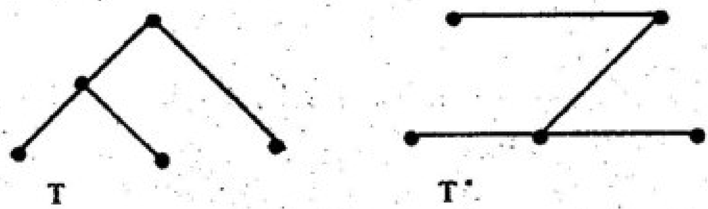
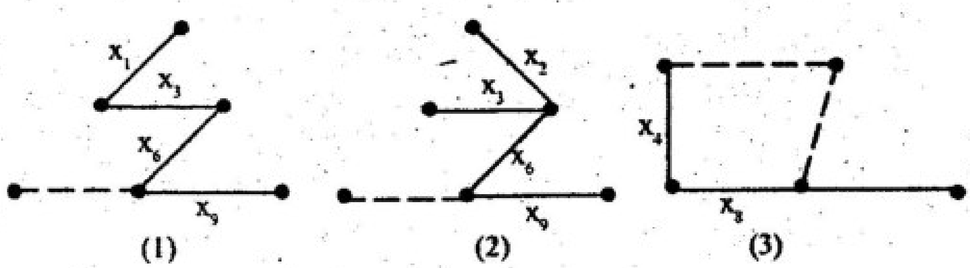
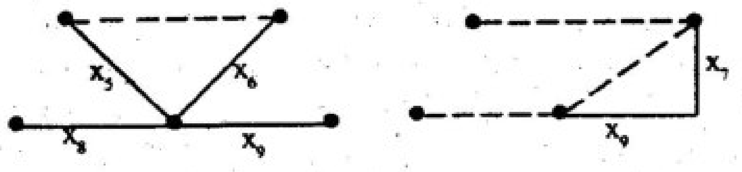


Fig. 1.10.3





(4) (5)

Fig. 1.10.4

Co-cycle basis for G

Coboundary $x_2 + x_3 + x_4 + x_5 = (2) + (3) + (4)$.

The cocycle rank $m^*(G)$ **is the number of cocycles in a basis for the cocycle space of G**

Theorem 1.10.2: The cocycle rank of a connected graph G is the number of twigs in any spanning tree of T.

Corollary 1.10.2(a): If G is a connected (p, q) graph, then $m^*(G) = p - 1$.

Corollary 1.10.2(b): If G is a (p, q) graph with k components, then $m^*(G) = p - k$.

Ex.1. Determine the cycle ranks of

(a) K_p

(a) $K_{m,m}$

(c) a connected cubic graph with p points.

Ex.2. a cotree of a connected graph is a maximal subgraph containing no cocycles.

1.11 Connectivity :

The connectivity $\mathcal{E}$ of a graph G is the minimum number of points whose removal results in a disconnected or trival graph. Thus the connectivity of a disconnected graph is zero. On the other hand the connectivity of a connected graph with a cutpoint is 1.

By removing any number of points, a $\mathcal{E}_p$ graph cannot be made disconnected but by …moving p-1 points it can be reduced to the trivial graph; therefore the connectivity $\mathcal{E}(K_p) = p - 1$.

The line connectivity λ of a graph G is the minimum number of lines whose removal results in a disconnected graph or a trivial graph. Thus $\lambda(K_1) = 0$ and line connectivity of a disconnected graph is zero while that of a connected graph with a bridge is 1.

Theorem 1.11.1: For any graph G,

$$K(G) \leq \lambda(G) \leq \delta(G).$$

Proof: If G has no lines then $\lambda(G) = 0 = \delta(G)$. When all the lines which are incident with a point are removed the graph is disconnected because the particular point remains isolated, therefore removal of the lines which are incident with a point of minimum degree makes the graph disconnected. Therefore $\lambda(G) \leq$ number of lines incident with the point of minimum degree $= \delta(G)$ i.e.

$$\lambda(G) \leq \delta(G) \qquad \text{......(1)}$$

Now we show $\mathcal{K}(G) \leq \lambda(G)$.

Case 1: Let the graph G be disconnected or trivial then $\mathcal{K} = 0 = \lambda$.

Case 1: Let G be connected and has a bridge x.

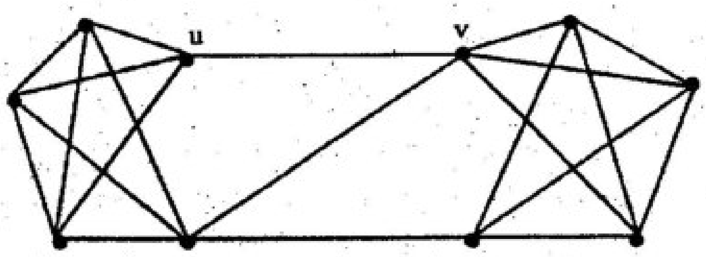


Fig. 1.11.1.

Then $\lambda = 1$. In this case either the graph has a cutpoint incident with the bridge x or G is K_2. In either case $\mathcal{K} = 1$.

Case 3: Suppose that $\lambda \geq 2$. In this case removal of λ -1 lines out of the lines associated to the number λ, produces a subgraph with a bridge uv. For each of these λ -1 lines select an incident point different from u or v. The removal of these points also removes these λ -1 lines(possibly more). Again the above mentioned subgraph with the bridge uv can be made disconnected by removing either u or v. Therefore by removing λ points from the graph G as mentioned above either G can be made disconnected or G is reduced to a trivial graph. Therefore

$$\mathcal{K}(G) \leq \lambda(G) \qquad \text{......(2)}$$

combining (1) and (2) we get

$$\mathcal{K}(G) \leq \lambda(G) \leq \delta(G).$$

Remark: For all integers a, b, c such that $0 < a \leq b \leq c$ there exists a graph G with $\mathcal{K}(G) = a$, $\lambda(G) = b$, $\delta(G) = c$.

Theorem 1.11.2: If G has p points and $\delta(G) \geq \left[\frac{p}{2}\right]$ then $\lambda(G) = \delta(G)$.

Proof: Since $\lambda(G) \leq \delta(G)$, the proof will follow if we show that $\lambda(G) \geq \delta(G)$. Suppose $\lambda(G) < \delta(G)$. Then there is a cutset S such that $\lambda(G) = |s| < \delta(G)$. Let the lines of S be incident with q points in V_1 and p points in $\overline{V}_1$ [V_1 and $\overline{V}_1$ are the points sets in the two components].

Suppose $|V_1| = q$. Then each point of V_1 is an end point of at least one line in S.

If we denote by G_1 the induced subgraph of G on the point set V_1, then G_1 has at least

$$m_1 = \frac{1}{2}[q\delta(G) - \lambda(G)] \text{ lines.}$$

[because the q points has minimum degree $\delta(G)$ and $\lambda(G)$ lines are removed].

Since

$$\lambda(G) < \delta(G), \text{ we get}$$

$$m_1 > \frac{1}{2}[q\delta(G) - \delta(G)]$$

$$= \frac{1}{2}\delta(G)(q-1)$$

or $m_1 > \frac{1}{2}q(q-1)$ as $\delta(G) > \lambda(G) \geq q$. This is a contradiction since in a simple graph there cannot be more than $q(q-1)/2$ lines connecting q points. Therfore $|V_1| > q$. In a similar way we can prove that $|\overline{V}_1| > \overline{q}$.

If $|V_1| > q$ and $|\overline{V}_1| > \overline{q}$, then there are points in both V_1 and $\overline{V}_1$ which are adjacent to only points in V_1 and $\overline{V}_1$ respectively. Thus each of V_1 and $\overline{V}_1$ contains at least $\delta(G)+1$ [The point in V_1 or $\overline{V}_1$ which is adjacent to only points in V_1 or $\overline{V}_1$ should have minimum degre $\delta(G)$ i.e. the point must be adjacent to at least $\delta(G)$ number of points in V_1 or $\overline{V}_1$] points. Therefore G has at least $2\delta(G)+2$ points i.e. $2(\delta + 1) = p$ points. But

$$p = 2\delta(G) + 2 \geq 2\left[\frac{p}{2}\right] + 2 > p.$$

leading to a contradiction. Therefore there is no cutset S with $|S| < \delta(G)$. Hence

$$\lambda(G) \geq \delta(G).$$

We conclude that $\lambda(G) = \delta(G)$

Theorem 1.11.3: Among all graphs with p points and q lines the maximum connectivity is zero when q<p-1 and is $\left[\frac{2q}{p}\right]$ when $q \geq p-1$

Proof: Case, when q < p - 1:

The number of lines in a connected graph with cycles may be more than the number of lines in a connected graph without cycle, both the graphs consisting same number of points. Therefore the number of lines in a connected graph is minimum when it has no cycle. Again a connected graph without cycle is a tree and in a tree q = p-1. So if q < p-1 the graph must be disconnected and hance the connectivity of the graph in this case is zero.

Case when q ≥ p -1

Since the sum of the degrees of any(p, q) graph is 2q, the mean degree is $\frac{2q}{p}$. Therefore $\delta(G) \leq \left[\frac{2q}{p}\right]$, so $\therefore\ \mathcal{E}(G) \leq \delta(G) \leq \left[\frac{2q}{p}\right]$

To show that the value can actually be attained an appropriate family of graphs can be constracted. The same constraction also gives those(p, q) graphs with minimum line connectivity.

For example, consider K_p. Here

$$q = \frac{p(p-1)}{2}$$

$$\left[\frac{2q}{p}\right] = p-1 = \mathcal{E}(G).$$

Connectivity pair: A connectivity pair of a graph G is an ordered pair(a, b) of non-negative integers such that there is some set of 'a' point and 'b' lines whose removal disconnectects the graph, and there is no set of a -1 points and b lines or of a points and b-1 lines with this property. $(\mathcal{E}, 0)$ (θ, λ) are examples of connectivity points where $\mathcal{E}$ is the point connectivity and λ is the line connectivity. It is easy to see that G has $\mathcal{E}+1$ connectivity pairs (a, b_a) where $0 \leq a \leq \mathcal{E}$.

Connectivity function: The connectivity pairs of a graph G determine a function f from the set $0,1,2,\ldots,\mathcal{E}$ into the non-negative integers such that f is strictly decreasing and $f(\mathcal{E}) = 0$. The function f is called connectivity function. f is strictly decreasing because if (a, b) is a connectivity pair b>0, then (a+1, b-1) is also a connectivity pair.

A graph G is said to be **n-connected** if the connectivity of the graph G is greater than or equal to n i.e. $\mathcal{E}(G) \geq n$.

A graph G is called **n-line connected** if $\lambda(G) \geq n$

If A non-trivial graph is connected than obviously $\chi(G) \geq 1$, i.e. a connected graph is 1-connected.

Conversely if $\mathcal{E}(G) \geq 1$ implies that the minimum number of points to be removed from the graph to get a disconnected graph is one and hence the graph must be connected.

If a graph G is a block with more than one line, then G is non-separable i.e. G has got no cutpoints. So in this case $\mathcal{E}(G) \geq 2$, i.e. G is 2-connected.

Conversely if $\mathcal{E}(G) \geq 2$ implies G has no cutpoints and of course G must not be K_2(because in this case $\mathcal{E}(G) \geq 2$). So $\mathcal{E}(G) \geq 2$, implies G is connected having more than one line and has no cutpoints. So G is a block.

Theorem 1.11.4: If G is n-connected and n≥2, then every set of n points of G lie on a cycle.

Proof: We use induction on n. We know that any two points of a connected graph lie on a cycle(because in this case the graph is a block). Suppose any n-1 points of a n-1 connected graph lie on a cycle. Let G be a n-connected graph and $v = \{v_1, \ldots\ldots, v_n\}$ be any set of n points of G, by removing one of the n-points we get a n-1 connected graph. Hence by the induction hypothesis there is a cycle C passing through $v_1, \ldots\ldots, v_{n-1}$. If v_n also lies on C, there is nothing to prove. If not we have to consider two cases:

(i) There are n v_n-c paths $p = \{p_1, p_2, \ldots, p_n\}$ in G which have only v_n in common. Then the points $v_1, v_2, \ldots, v_{n-1}$ divide C into n-1 segments and one of these (including the end points) should contain the end points of two of the paths in P. Adding these two paths to C and removing the segment of C between thier end points we get a cycle c′ of G containg all the points of U.

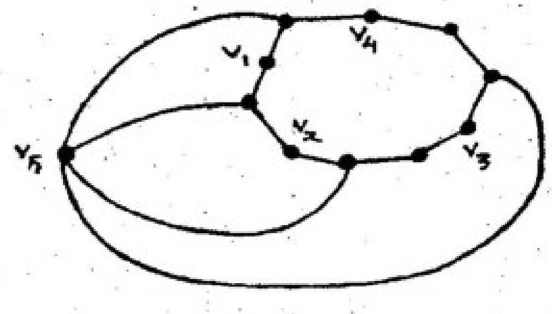


Fig. 1.11.2.

(ii) If there are no such n-paths as in (i), then there is a set S of n-1 points not containing v_n, which meets all the v_n-c paths. Now G being n connected, G-S is connected and hence every point of C can be joined in G-S by a path to v_n, contradicting the nature of S(i.e. S meets all the v_n-c paths), except when $S = V(c) = \{v_1,...,v_{n-1}\}$. But in this exceptional case, since $d(v_n) \geq n$ [as $\mathcal{E} \geq n$ and $\mathcal{E} \leq \delta(G)$] we can find n-1 v_n-c paths having only the point v_n in common. Adding two of these paths which connect neighbouring points of C to v_n and deleting the line of C between these neighbouring points we get from C a cycle c′ containing all points of U.

Theorem 1.11.5: The minimum number of points separating two non-adjacent points s and t is the maximum number of disjoint s-t paths.

Proof: If k points separate s and t then there can be no more than k disjoint paths joining s and t.

It remains to show that if it takes k points to separate s and t in G, there are k disjoint s-t paths in G. This is certainly true if k = 1. Assume that it is not true for some k>1. Let h be the smallest such k, and let F be a graph with the minimum number of points for which the theorem fails for h. We remove lines from F until we obtain a graph G such that h points are required to separate s and t in G but for any line x of G, only h-1 points are required to separate s and t in G-x. We first investigate the properties of this graph G and complete the proof of the theorem.

By the definition of G, for any lime x of G there exists a set S(x) of h-1 points which separates s and t in G-x. Now G-S(x) contains at least on s-t path, since it takes h points to b separate s and t in G. Each such s - t path must contain the line x = uv. Since it is not a path in G-x. So $u, v \notin s(x)$ and if $u \neq s$, t then $s(x) \cup \{u\}$ separates s and t in G.

If there is a point w adjacent to both s and t in G the G-w requires h-1 point to separate s and t and so it has h-1 disjoint s-t paths. Replacing w, we have h disjoint s-t paths in G. So we have shown:

Case I : No point is adjacent to both s and t in G.

Let W be any collection of h points separating s and t in G. An s-W path is a path joining s with some $w_i \in W$ and containing no other points of W. Call the collections of all s-W paths and w-t paths p_s and p_t respectivily. Then each s-t path begins with a member of p_s and ends with a member of p_t, because every such path contains a point of W. Moreover the paths in p_s and p_t have the points of W and no others in common, since it is clear that each w_i is in at least one path in each collection and if some other point were in both an s-w and a w-t path, then there would be an s-t path containing no point of W. Finally either p_s - w = {s} or p_t - w = {t}, since, if not then both p_s plus the lines {w_1t, w_2t, ...} and p_t plus the lines {sw_1, sw_2, ...} are graphs with fewer points than G in which s and t are non-adjacent and h connected, and therefore in each there are h disjoint s-t paths. Combining the s-w and w-t portions of these paths, we can construct h disjoint s-t paths in G, and thus have a contradiction. Therefore we have proved.

Case I I: Any collection W of h points separating s and t is adjacent either to 's' or to 't'.

Now we can complete the proof of the theorem. Let P = {s, u_1, u_2, ..., t} be a shortest s-t path in G and let $u_1u_2 = \mathcal{E}$. Note that by (I) $u_2 \neq t$ (because then u_1 will be adjacent to both s and t). Form $s(x) = \{v_1, v_2, ..., v_{h-1}\}$ as above separating s and t in G-x. By(I) $u_1t \notin G$ (then u_1 will be adjacent to both s and t), so by (II), with $W = s(x) \cup \{u_1\}, sv_i \in G$, for all i. Thus by (I) $v_it \notin G$, for all i. However if we pick $W = s(x) \cup \{u_2\}$ instead, we have by (II) that $su_2 \in G$, contradicting our choice of P as a shortest s-t path and completing the proof of the theorem.

Theorem 1.11.6: A graph is n-connected if and only if every pair of points are joined by at least n point disjoint paths.

Proof: Let G be a graph with at least (n+1) points. Obviously the theorem is true if n = 1. So we need to prove the theorem for $n \geq 2$.

Necessary part:

If s and t are non-adjacent then the necessity of the theorem follows from **theorem 1.11.5.**

Suppose that s and t are adjacent and that there are at most n-1 point disjoint s-t paths in G. Let e = (s, t). Consider now the graph G′ = G -e. Since there are at most n - 1 point disjoint s-t paths in G, there cannot be more than n-2 point disjoint s-t paths in G′. Thus there exists a set $A \subset v - \{s, t\}$ of points with

$$|A| \leq n - 2$$

whose removal disconnects s and t in G′. Then

$$|V - A| \geq |V| - |A| \geq (n+1) - (n-2) = 3$$

and therfore, there is a point u in V-A different from s and t (as the number of points in V-A ≥ 3).

Now we show that there exists a s-u path in G′ which does not contain any point of A.

Clearly this is true if s and u are adjacent. If s and u are not adjacent then there are n point disjoint s-u paths in G(as G is n-connected) and hence there are n-1 point disjoint s-u paths in G'. Since

$$|A| \leq n-2,$$

at least one of these n-1 paths will not contain any point of A.

In a similar way we can show that in G' there exists a u-t path which does not contain any point of A.

Thus there exists in G' a s-t path which does not contain any point of A. This, however, contradicts that A is a s-t disconnecting set in G'. Hence the necessity.

Sufficiency:

G is connected because there are n point disjoint paths between any two distinct points of G. Further, not more than one of these paths can be of length 1, since there are no parallel lines in G. The union of the remaining n-1 paths must contain at least(n-1) distinct points other than s and t. Hence

$$|v| \geq (n-1)+2 > n.$$

Suppose in G there is a disconnecting set A with $|A| < n$. Then consider the subgraph G' of G on the point set V-A. This graph contains at least two distinct components. If we select two points s and t from any two different components of G', then there are at most $|A| < n$ point disjoint s-t paths in G. This contradicts that any two point are connected by n point-disjoint paths in G.

Hence the suffenciency.

Theorem 1.11.7: For any two points of a graph the maximum number of line disjoint paths joining them equals the minimum number of lines which separate them.

Theorem 1.11.7: A graph with at least 2n points is n-connected if and only if for any two disjoint sets v_1 and v_2 of n points each, there exists n disjoint paths joining these two sets of points.

Note: That in this Theorem these n disjoint paths do not have any points at all in common, not even thier end points.

Proof: To show the sufficiency of the condition, we form the graph G' from G by adding two new points w_1 and w_2 adjacent to exactly the points of v_i, $i = 1, 2$. Since G is n connected, so is G' and hence by **theorem 1.11.5** there are n disjoint paths joining w_1 and w_2. The restrictions of these paths to G are clearly the n disjoint v_1 - v_2 paths we need.

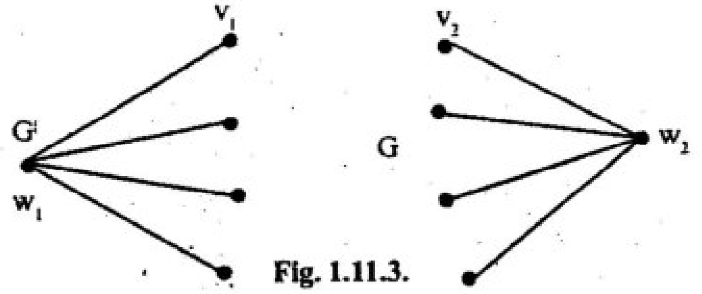


Fig. 1.11.3.

To prove the 'other half', let S be a set of at least n-1 points which separates G_1 and G_2 with point sets v_1' and v_2' respectively. then since

$$|v_1'| \geq 1, |v_2'| \geq 1 \text{ and}$$

$$|v_1'| + |v_2'| + |s| = |v| \geq 2n,$$

there is a partition of S into two disjoint subsets s_1 and s_2 such that

$$|v_1' \cup s_1| \geq n,$$

$$|v_2' \cup s_2| \geq n.$$

Picking any n subsets v_1 of $v_1' \cup s_1$ and v_2 of $v_3' \cup s_2$, we have two disjoint sets of n point each. Every path joining v_1 and v_2 must contain a point of S, and since we know there are n disjoint v_1- v_1 paths we see that $|s| \geq n$ and hence G is n connected.

Hall's marraige problem:

Given a set of boys and a set of girls where each girl know some of the boys. Under what condition can all girls get married, each to a boy she knows? In this context we are going toy prove our next **theorem 1.11.9** which may be reformulated to produce what is often referred to as **Hall's marriage problem:** if there are n girls, then the marriage problem has a solution if and only if every subset of k girls $(1 \leq k \leq n)$ collectively know at least k boys.

Theorem 1.11.8: There exists a system of distinct representative for a family of sets $S_1, S_2, ..., S_m$ if and only if the union of any k of these sets contain at least k elements, for all k from 1 to m.

Proof: The necessity is immediate. For the sufficiency we first prove that if the collection $\{s_i\}$ satisfies the stated conditions and $|s_m| \geq 2$ then there is an element e in s_m such that the collection of sets $s_1, s_2, ..., s_{m-1}, s_m - \{e\}$ also satisfies the condition. Suppose this is not the case. Then there are elements e and f in s_m and subsets J and K of $\{1, 2, ..., m-1\}$, such that

$$\left|\left(\bigcup_{i\in J} s_i\right)\cup\left(s_m-\{e\}\right)\right| < J+1$$

and

$$\left|\left(\bigcup_{i\in K} s_i\right)\cup\left(\delta_m-\{f\}\right)\right| < |K|+1.$$

But then

$$|J|+|K| \geq \left|\left(\bigcup_{J} s_i\right)\cup\left(s_m-\{e\}\right)+\left(\bigcup_{K} s_i\right)\cup\left(s_m-\{f\}\right)\right|$$

$$\geq \left|\left(\bigcup_{J\cup K} s_i\right)\cup s_m\right| + \left|\bigcup_{J\cap K} s_i\right|$$

$$\geq |J\cup K|+1+|J\cap K| > |J|+|K$$

$$[\ n(A\cup B)=n(A)+n(B)-n(A\cap B)]$$

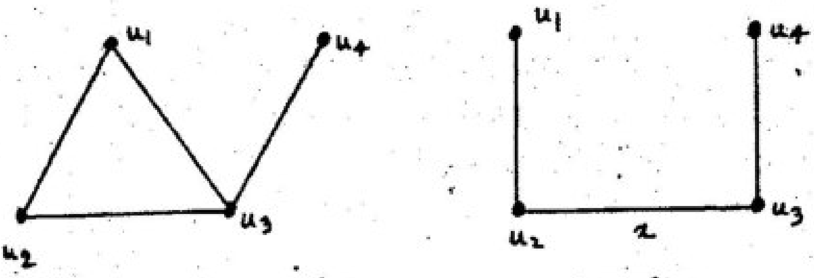
which is a contradiction. The sufficiency now follows by induction on the maximum of the number $|s_i|$. If each set is a singleton, there is noting to prove. The induction is made by application(repeated if necessary) of the above result to the sets of largest order.

1.12. Cut verties, Cut deges and Blocks :

Some connected graphs can be disconnected by the removal of a single point called cutpoint or by single line called bridge. The fragments of a graph held together by its cutpoints are its blocks. These concepts are developed in this chapter.

A **cutpoint or cut vertices** of a graph is a point whose removal increases the number of components.

Example: Here u_3 is a cutpoint, others are not.



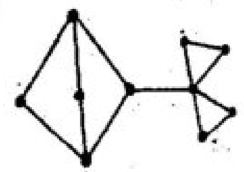
x is a bridge

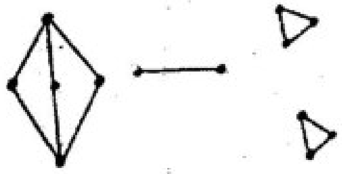
Fig.1.12.1

A **bridge** of a graph is a line whose removal increases the number of components.

A **non-separable graph** is connected, nontrivial and has no cutpoints.

G:





Blocks of G

Fig. 1.12.2. **Fig. 1.12.3.**

A **block** of a graph is a maximum non-separable subgraph.

Theorem 1.12.1: Let v be a point of a connected graph G, then the following statements are equivalent:

(1) v is a cutpoint of G

(2) there exist points u and w distinct from v such that v is on every $u-w$ path.

(3) there exist a partition of the set of points $V-\{v\}$ into subsets U and W such that for every points $u \in U$, $w \in W$, the point v is on any $u-w$ path.

Proof: $(1) \Rightarrow (3)$

Since v is a cutpoint of G, $G-v$ is disconnected and has at least two components. Form a partition of points $V-\{v\}$ by considering two subsets U and W such that U consists of the points of one components and W the points of the others. Then the two points $u \in U$ and $w \in W$ lie in different components of $G-v$. Therefore every $u-w$ path in G must contain the point v.

$(3) \Rightarrow (2)$

This is obvious because (2) is a particular case of (3). Here U may be constructed as the set of points which are connected to $u \in G-v$ and W as the set of points which are connected to w in $G-v$.

$(2) \Rightarrow (1)$

If v lies on every path in G joining two points u and w distinct from v then by the removal of the point v every path connecting u and w in G is disconnected and u and w will be disconnected in $G-v$. Therefore $G-v$ must have at least two components. However, G has a single component because it is connected. Therefore v must be a cutpoint of G.

Theorem 1.12.2: Let x be a line of a connected graph G. The following statements are equivalent

(1) x is a bridge of G

(2) x is not on any cycle of G

(3) there exist points u and v of G such that the line x is on every path joining u and v

(4) there exists a partition of V into subsets U and W such that for every point $u \in U$ and $w \in W$, the line x is on every path joining u and w.

Proof: $(3) \Rightarrow (2)$

Let v_1, v_2 denote the end points of the line x. Then v_1 and v_2 must be two points on every path joining u and v. If x lies on a cycle then v_1 and v_2 are connected in $G - x$, therefore u and v are connected in $G - x$ via the path joining v_1 and v_2 in $G - x$; this contradicts that x lies on every path joining u and v.

$(2) \Rightarrow (1)$

Let v_1 and v_2 be the end points of x. Since x is not on any cycle of G, then v_1 and v_2 are connected by x only. Then in $G - x$, v_1 and v_2 are not connected and hence they lie in two components of $G - x$. Therfore x is a bridge of G.

$(1) \Rightarrow (4)$

Since x is a bridge of G, $G - x$ is disconnected and has at least two components. Form a partition of V by letting U consist of the points of one of these components and W the points of the other. Then any two points $u \in U$ and $w \in W$ lie in different components of $G - x$. Therefore every $u - w$ path in G contains x.

$(4) \Rightarrow (3)$

(3) is a particular case of (4). So by the chain $(1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$, the equivalence is established.

Theorem 1.12.3: Let G be a connected graph with at least three points. The following statements are equivalent.

(1) G is a block.

(2) Every two points of G lie on a common cycle.

(3) Every point and a line of G lie on a common cycle.

(4) Every two lines of G lie on a common cycle.

(5) Given two points and one line of G, there is a path joining the points, which contains

the line.

(6) For every three distinct points of G there is a path joining any two of them which contains the third.

(7) For every three distinct points of G, there is a path joining any two of them which does not contain the third.

Proof:

(1) ⇒ (2)

Let u and v be distinct points of G, and let U be the set of points different from u which lie on a cycle containing u. Since G has at least three points and no cutpoints, it has no bridge; therefore, every point adjacent to u is in U. So U is not empty.

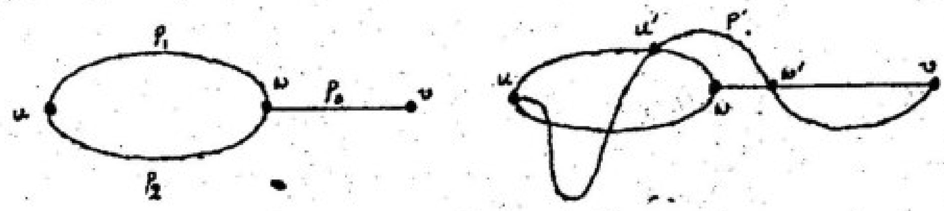


Fig. 1.12.4

Suppose v is not in U. Let w be a point in U for which the distance d(w, v) is minimum. Let P_0 be a shortest w-v path and let P_1 and P_2 be the two u-w paths of a cycle containing u and w. Since w is not a cut point, there is a u – v path P′ not containing w. Let w′ be the point nearest u in P′ which is also in P_0 and let u′ be the last point of the u - w′ subpath of P′ in either P_1 or P_2. Without loss of generality we assume u′ is in P_1.

Let Q_1 be the u - w′ path consisting of the u - u′ sub-path of P_1 and the u′ - w′ sub-path of P′. Let Q_2 be the u - w′ path consisting of P_2 followed by the w - w′ sub-path of P_0. Then Q_1 and Q_2 are disjoint u - w′ paths together they form a cycle, so w′ is in U. Since w′ is on a shortest w – v path, d(w′, v)<d(w,v). This contradicts our choice of w, proving that u and v do lie on a cycle.

(2) ⇒ (3)

Let u be a point and vw be a line of G. Let Z be a cycle containing u and v. A cycle Z′ containing u and vw can be formed as follows. If w is on Z, then Z′ consists of vw together with the v – w path of Z containing u. If w is not on Z there is a w – u path P not containing v, since otherwise v would be a cutpoint. Let u′ be the first point of P in Z. Then Z′ consists of vw followed by the w - u′ sub-path of P and the u′ - v path on Z containing u.

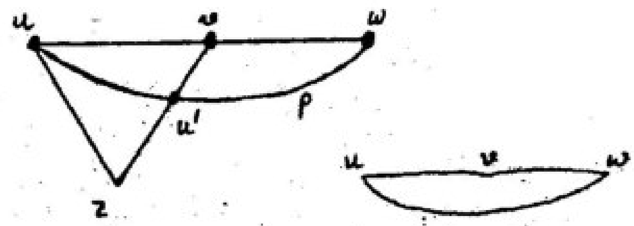


Fig. 1.12.5

(3) ⇒ (4)

Consider the lines u_1v_1 and u_2v_2 in G. Let Z_1 denote the cycle containing u_1v_1 and u_2. Z_2 denote the cycle containing u_1 and u_2v_2.

If v_2 lies on Z_1 or v_1 lies on Z_2 there is nothing to prove. Suppose v_2 does not lie Z_1 and v_1 does not lie on Z_2.

Let u'_1 be the nearest point to v_1 on Z_2 and $v_1 - u_2$ path of Z_1. Let u'_2 be the nearest point to v_2 which lie on Z_1 and $u_1 - v_2$ path of Z_2.

Let us take the cycle consisting of $u_2 - u'_1$ path of Z_2 followed by $u'_1 - v_1$ path of Z_1, followed by v_1u_1 line followed by $u_1u'_2$ path of Z_1 followed by $u'_2\ v_2$ path of Z_2 then followed by v_2u_2 line.

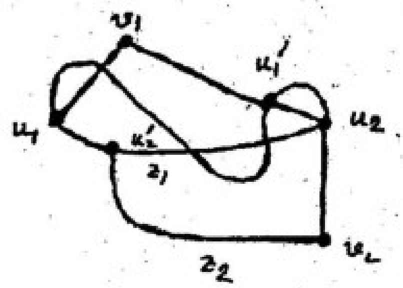


Fig. 1.12.6

(4) ⇒ (5)

[Any two points of G are incident with one line each, which lie on a cycle by (4). Hence any two points of G lie on a cycle and we have (2) so also (3)]

Let u and v be distinct points and x is a line. Then there is a cycle Z_1 containing u and x (by 3) and there is another cycle Z_2 which contain v and x. If v lies on Z_1 or u lies on Z_2 there is nothing to prove. Thus we consider $v \notin Z_1$ and $u \notin Z_2$.

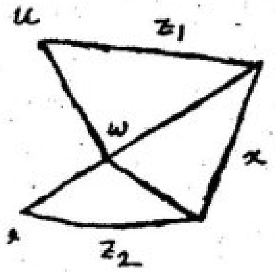


Fig. 1.12.7

Begin with u and proceed along Z_1 until reaching the first point w of Z_2, then take the path on Z_2 joining w and v which contain x. This walk constitutes a path joining u and v that contains x.

$(5) \Rightarrow (6)$

Let u, v, w be distinct points of G, and let x be any line incident with w. By (5) there is a path joining u and v which contains x and hence must contain w.

$(6) \Rightarrow (7)$

Let u, v, w be distinct points of G. By (6) there is a u – w path P containing v. The u – v sub-path of P does not contain w.

$(7) \Rightarrow (1)$

By (7) for any two points u and v no point lies on every u – v path. Hence, G must be a block.

Theorem 1.12.4: Every non-trivial connected graph has at least two points which are not cutpoints.

Proof: Let u and v be points at maximum distance in G and assume v is a cutpoint. Then there is a point w in a different component of G – v than u. Hence v is in every path joining u and w, so $d(u, w) > d(u, v)$, which is impossible. Therefore v and similarly u are not cutpoints of G.

Block-graph: Let the blocks of G be denoted by $G_1, G_2, \ldots\ldots G_k$. Let S be the set of points and lines of G, S_i be the set of points and lines of G_i and $F=\{S_1,S_2, \ldots\ldots S_k\}$. Then the intersection graph $\Omega(F)$ is called the Block-graph of G and is denoted by B(G).

Remark: Every block of G corresponds to a point of B(G) and such two points in B(G) are adjacent whenever the corresponding blocks contain a common cut point of G

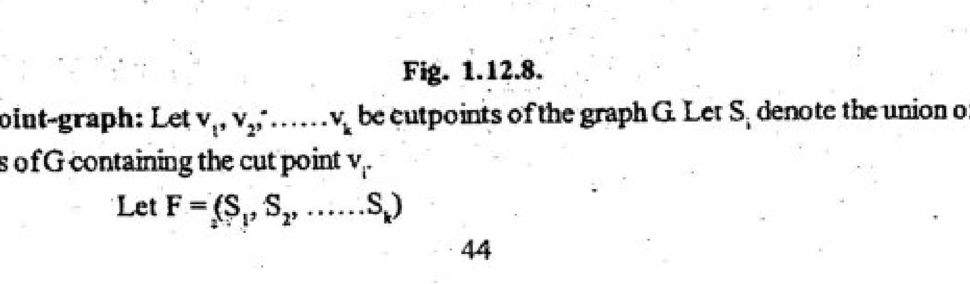


Fig. 1.12.8.

Cutpoint-graph: Let $v_1, v_2, \ldots\ldots v_k$ be cutpoints of the graph G. Let S_i denote the union of all blocks of G containing the cut point v_i.

Let $F = (S_1, S_2, \ldots\ldots S_k)$

Then the intersection graph $\Omega(F)$ is called the cutpoint-graph of G and is denoted by C(G).

Remark: Every point of C(G) corresponds to a collection of blocks of G which contain a common cutpoint. Two such points of C(G) are adjacent if the cutpoints of G to which they correspond lie on a common block (corresponding collection of blocks contain a common point).

$C_1 = \{ S_1, S_2, S_3, S_4\}$, $C_2 = \{ S_4, S_6\}$

$C_3 = \{ S_5, S_6\}$

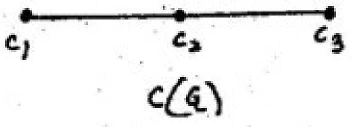


Fig. 1.12.9

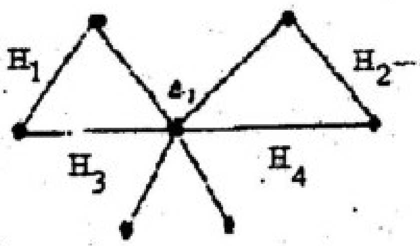
Theorem 1.12.5: A graph H is a block-graph of some graph if and only if every block of H is complete.

Proof: Let us first assume that H is a block-graph. Then we get a graph G such that H = B(G). Let us assume that there is a block H_1 of H which is not complete. Then there are at least two points in H_1 which are non-adjacent and lie on a common cycle J of length at least 4.

Let $v_1, v_2, \ldots v_k$, $k \geq 4$ be the points of H_1 which lie on J. Let $B_1, B_2, \ldots\ldots B_k$ be the blocks of G which correspond to the points $v_1, v_2, \ldots\ldots v_k$ respectively. Then $B_1 \cup B_2 \cup B_3 \cup B_4 \cup \ldots\ldots \cup B_k$ in G is connected and has no cut point [Because H_1 is a block and $B = B_1 \cup B_2 \cup \ldots\ldots \cup B_k$ corresponds to H_1 and so B has got no cutpoint]. Therefore $B_1 \cup B_2 \cup \ldots\ldots \cup B_k$ is contained in a block B (Now each $B_i \subset B$ leads to a contradiction to the fact that every B_i is a block). Therefore every block B_i is a proper subgraph of block B. This contradicts the maximality of a block. Therefore H_1 must be complete.

Conversely let H be a given graph in which every block is complete. Let $H_1, H_2, \ldots\ldots, H_k$ be the blocks of H. Then every point of B(H) corresponds to same block H_i. Then we may denote the points of B(H) by $H_1, H_2, \ldots\ldots, H_k$.

Now to each point H_i of B(H) add end lines equal to the number of points of the block H_i which are not cut point of H. Then we get a new graph G from B(H) by adjoining such end lines. We see that every end line is a block of G and corresponds to a point of H which is not a cutpoint. Again if two blocks H_i and H_j are connected in H through a cutpoint v then the corresponding points H_i and H_j of $B(H) \subset G$ are adjacent.



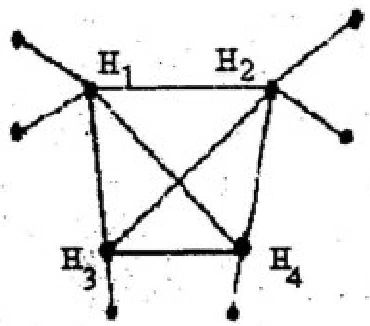


Fig.1.12.10

Then the line H_iH_j of G is contained in a block B of G. Now we see that the block B of G corresponds to the cutpoint v of H. In this wary we get a one-one correspondence between the blocks of G and the points of H. Hence H is a block graph of G.

1.13. Connectivity Parameters :

In this section we introduce the golbal connectivity parameters κ and λ and their local variants.

If a graph cannot be disconnected by the removal of a single vertex (that is if it has no cut vertex) or a single edge (that is it has no cut edge) we ask for the minimum number of vertices (edges) whose removal results in a disconnected or trivial graph. These are precisely the parameters κ and λ. However, to clarity their relation are start from a more general points of view.

A subset S of $V \cup E$ is a disconnecting set (or separating set of the graph $G = (V, E)$ if $k(G-S) \rangle k(G)$ or $G-S$ is the trivial graph. If a disconnecting set S is a subset of V it is called a vertex cut of G. if it is a subset of E it is called and dege cut of G. If a disconnecting set S contains vertices and deges it is called mixed cut. Recall that a disconnecting set/vertex cut/edge cut is minimal if no proper subset of it has same property and that it is minimum if it has least cardinality among all such minimal sets.

(i) A minimal vertex cut is called a knot and a minimum vertex cut a clot. The cardianality of clot called the vertex connectivity number of clot number of the graph G and is denoted by $\kappa(G)$

(ii) A mimimal dege cut is called a bond and a minimum edge cut a band. The cardinality of band is called the edge connectivity number of band number of the graph G andis denoted by $\lambda(G)$.

(iii) The minimum cardinality of a mixed disconnecting set is denoted by $\sigma(G)$.

If S is a disconnecting set of the grph $G=(V,E)$ and vertices s and t are in the same component of G but in different components of $G-S$, then S is called and $s-t$ separating set in G

(ii) Minimal $s-t$ separating vertex cuts (edge cuts) are called $s-t$ knots (bonds) and minimum $s-t$ separating vertex cuts (edge cuts) are called $s-t$ clots (bands).

(iii) The cardinality of an $s-t$ clot is called the $s-t$ clot number and is denoted by $\kappa(s,t)$ and the cardinality of an $(s-t)$ band the $(s-t)$ band number and is denoted by $\lambda(s,t)$. The cardinality of minimum $s-t$ separating mixed cut is denoted by $\sigma(s,t)$

The following results are obvious.

Proposition 1.13.1

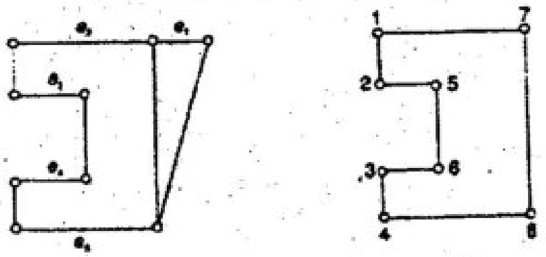
$\kappa(k_n)=n-1$. If G is incomplete $\kappa(G)\le n-2$

Proposition 1.13.2

$$\kappa(G)=\min_{\substack{s,t\in V\\ st\notin E}}\kappa(s,t) \qquad \lambda(G)=\min_{s,t\in V}\lambda(s,t) \quad \sigma(G)=\min_{s,t\in V}\lambda(s,t)$$

If A is any non empty subset of the vertex set V of a graph $G=(V,E)$ the set $[A,\overline{A}]$ of all deges of G with ont end in A and other end in $\overline{A}=V-A$ is called a cut of G

This is a concept intermediate between that of and dege cut and a bond. Every cut is clearly and dege cut but the converse is not true as seen from Fig. 1.13.1 (a). Also every bond is a cut (see Proposition 5.10 below) while every cut is not bond as seen from Fig 1.13.1 (b)



(a) (b)

$F=\{e_1, e_2, e_3, e_4, e_5\}$ is an edge cut which is not a cut (a)

$A=\{1, 2, 3, 4\}$ $[A,\overline{A}]$ is a cut which is not a bond (b)

Fig. 1.13.1.

Proposition 1.13.3

Every minimal cut is a bond and every bond is minimal cut

Proof (i) If $C=[A,\overline{A}]$ is a minimal cut, no subset of the edges of C is a cut, and this means that $G-C$ has only the two components $\langle A\rangle$ and $\langle\overline{A}\rangle$ and C is a bond.

(ii) If F is a bond, $G-F$ has only two components C_1 and C_2 and if V_1, V_2 are their vertex sets, then $F=[V_1,V_2]$ with $V_2\ V_2=\overline{V}_1$. Thus F is a cut and a minimal cut.

Proposition 1.13.4.

Every cut is a disjoint union of minimal cuts.

Proof

Let $C=[A,\overline{A}]$ be a cut of G. If is not a minimal cut, $\langle A\rangle$ will have components say $C_1,C_2\ C_1,C_2,\ldots\ldots\ldots\ldots C_s$ with at least one of r and s being greater than one. Consider the simple coalescence H with C_i coalesced to vertices $c_i 1\le i\le r$ and the C_i' coalesced to vertices c_i', $1-\le i\le s$. Then H is a dipartite graph. If we can partition the dege set of H into a disjoint union of bonds of H, the deges of G. Tc achieve such a partition of $E(H)$ we first take the cut deges of H as members of the partition. If F is the set of such cut deges, for the remaining members of the partition we take the stars at the remaining (non isolated) vertices c_i $(or\ c_i')$. It is easy to see that these meet the requirements.

Lemma 1.13.1 If s, t $\notin$ E then $\kappa(s,t)=\min\ \{\kappa(s,t)=\lambda(s,t)\}$.

Proof:

Since an dege cut is a special case of a mixed cut it is enough to prove that $\kappa(s,t)\le\sigma(s,t)$. To this end we prove that from any mixed $s-t$ separating set we can get and $s-t$ separating vertex cut with no more elements. Let S be a minimum mixed $s-t$ separating set. If ij is an dege in S both i and j cnnot coincide with s and t, since $st\notin E$. It $i=s$, add i to S and remove from S alldeges with i as an end vertex. If $i\ne s$, add j to S and remove from S all edges with j as an end vertex. The resulting possible mixed set is clearly and $s-t$ separating set with no move elements than S. We can repeat this process and remove all edges from S and obtain a vertex cut S' with at most $|S|$ elements. Since $\kappa(s,t)\le|S'|\le|S|=\sigma(s,t)$ the proof is complete.

Corollary If $st\notin E$, then $\kappa(s,t)\le\lambda(s,t)$. Observe also that if $st\in E$ then $\kappa(s,t)$ is not defined.

Proposition 1.13.5

For any graph $G, \sigma(G) = \kappa(G)$

Proof Case (i) $G = K_n$. Then $\kappa(G) = n-1$ and $\lambda(G) = n-1$. Let S be a minimum mixed disconnecting set of G. Let $S = T \cup F$ with $T \subseteq V$ and $F \subseteq E$ and $|T| = n_1$ and $|F| = m_1$. Then $G-T$ is a K_{n_1-n} and so $|F| \geq \lambda(K_{n-n}) = n - n_1 - 1$.

Then $\sigma = |S| = m_i + n_1 \geq n-1 = \kappa$. Since always $\sigma \leq \kappa$ we have $\sigma = \kappa$.

CASE (ii) G is incomplete. If possible let there be a minimum $s-t$ separating mixed set $S = M \cup (st)$ with $\sigma = |S| \langle \kappa$. As shown in the proof of Lemma 5.4, M can be replaced by a set of vertices T (a subset of the vertex set of the induced subgraph $\langle M \rangle$ to provide a vertex cut of $G^* = G - st$ with cardinality at most $|M|$. Let C_1 and C_2 be the components of $G-S$ to which s and t, respectively, belong. Suppose there is another component C_3 of $G-S$ ane let v be a vertex of C_3. (see Fig. 1.13.2). Then $T \cup (s)$ is a $v-t$ separating vertex cut of G. But then $|T \cup (s)| \leq |S| < \kappa$, a contradiction. Thus C_1 and C_2 are the only components of $G-S$. Also if $u \in V(C_1)$ and $u \neq s$, then $T \cup (s)$ is a $u-t$ separating vertex cut of G again leading to a contradiction. Thus $C_1 = (s)$ and similarly $C_2 = (t)$ and G has $|V(M)| + 2$ vertices and if incomplete. Hence $\kappa(G) \leq n-2 = |V(M)| < \kappa$, c contradiction. Thus $\sigma \not< \kappa$ and hence $\sigma = \kappa$.

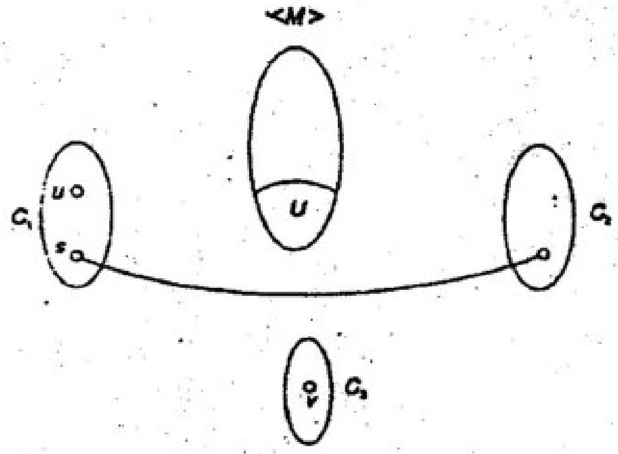


Fig. 1.13.2

Theorem 1.13.1 (Whitney)

For any graph, $G, \kappa \leq \lambda \leq \delta$

Proof Let G be a connected graph.

(i) If v is a vertex of minimum degree δ, the set of deges incident with v is an edge cut of G. Hence $\lambda \leq \delta$

(ii) To prove $\lambda \leq \delta$ we observe that if $G=K_n$ then $\kappa = \lambda = n-1$. Let G be an incompelte graph.

If $\lambda(G)=0$, G is disconnected and $\kappa(G)=0$ as well. If $\kappa(G)=1, G$ has a cut edge and hence also a cut vertex, so that $\kappa(G)=1$

So let $\kappa(G) \geq 2$ and F be a set of λ deges disconnecting G. Removing any set of $\lambda - 1$ deges from this set leaves a connected graph with a cut edge (the remaining deges of (F) e=uv say, For each of the $\lambda - 1$ deges mentioned above choose one end vertex which is neither u nor v. Let H be the graph obtained be removing from the original graph G, set S of vertices so chosen (at most $\lambda - 1$ in number). If H is disconnected, $\kappa(G) \leq |S| \leq \lambda - 1$. If not $e = vu$ is a cut edge of H and hence u is a cut vertex of H. But then $S \cup (u)$ is a vertex cut of G of cardinality at most λ, so that $\kappa(G) = \lambda(G)$. Thus in any case $\kappa(G) \leq \lambda(G)$

1.14 A thread between vertices s and t of a graph G is a set of paths between s and t which have pairwise no vertices in common except s and t. The number of paths in the thread is the ply number of the thread. A thread with ply number q will be called a q thread. The maximum ply number of a thread between s and t is called the $s-t$ number and is denoted by $p(s,t)$. The minimum value of $p(s,t)$ for all pairs $s,t \in V$ such that $st \notin E$ is denoted by $p(G)$ and is called the ply number of G.

A lace between vertices s and t of a graph G is a set of paths between s and t which are dege disjoint but need not be vertex disjoint. the maximum number of paths in a lace between s and t is the $s-t$ lace number and is denoted by $l(s,t)$. A lace with lace number q will be called a *q lace*. The minimum value of $l(s,t)$ for $s,i \in V$ is denoted by $l(G)$ and is called the lace numer of G

Note :

If the graph G to which the numbers ralate has to be specified in a context we will use notations like $p(s,t|G), l(s,t|G)$, etc. It is intuitively clear that vertex pairs (s,t) with higher ply or lace number are more strongly connected than other vertex pairs of graph and that it should be more difficult to separate them by removal of vertices of edges. There should therefore be some relation between these parameters and the separation parameters $s-t$ clot number and $s-t$ band number introduced earlier. The celebrated theorems of Menger to be proved below simply assert the equality of these (respective) parameters.

Theorem 1.14.1 Menger's Therorem : Vertex From

For any pair of non adjacent vertices s and t of a graph, the clot number equals the ply number. That is $\kappa(s,t) = p(s,t)$ for every pair $s,t \in V$ such that $st \notin E$

Proof : We prove the statement by induction on the number of deges in G. The statement is obvious for a graph with $m=1$ of $m=2$. Let us then assume that the statement is true for all graphs with less than m déges and let G be a grph with m edges. Suppose the statement is false for G. Then we should have

$p(s,t|G) < \kappa(s,t|G) = q(say)$ (1)

since for any graph we obviously have $p(s,t) \le \kappa(s,t)$

Let $e = uv$ be an dege of G. since the deletion graph $G_1 = G - e$ and the elementary contraction graph $G_2 = G|e$ have less number of deges than G, the induction hypothesis applies to them and we have the following equations.

$p(s,t|G_1) = \kappa(s,t|G_1)$ and $p(s,t|G_2) = \kappa(s,t|G_2)$ (2)

if I is an (s,t) clot in G_1 and $\hat{J}$ is an (s,t) clot in G_2 we have

$|I| = \kappa(s,t|G_1) = p(s,t|G_1) \le (s,t|G) < q$

and $|\hat{J}| = \kappa(s,t|G_2) = p(s,t|G_2) \le p(s,t|G) < q$ using (2) and (1)

Therefore $|\hat{J}| \le q-1$. To $\hat{J}$ there corresponds and (s,t) vertex cut J of G such ahta $|J| \le |\hat{J}|+1$ since by an elementary contraction $\kappa(s,t)$ can be decreased by at most one and this decrease actually occurs when $e \in E(\langle J \rangle)$

Thus $|J| \le |\hat{J}|+1 \le q-1+1 = q$ (3)

But then $q \le |J| \le q$ since J is an (s,t) vertex cut in G.

Thus $|J| < q$ *and* $|J| = q$

and $u,v \in J$ by (3)

Now let $H_s = \{w \in I \cup J\}$ there exist and $s-w$ path in G. vertex disjoint from $I \cup J - \{w\}$ and $H_t = \{w \in I \cup J|\}$ there exists $t-w$ path in G, vertex disjoint from $I \cup J - \{w\}\}$

Clearly H_s and H_t are (s,t) separating vertex cuts in G.

Hence $|H_s| \geq q$ and $|H_t| \geq q$

Obviously $H_s \cup H_t \subseteq I \cup J$. We now verify that $H_s \cap H_t \leq I \cap J$. For this let $w \in H_s \cap H_t$. Then there exists an $s-w$ path P_1 and $w-t$ path P_2 in G vertex disjoint from $I \cup J - \{w\}$, $P_1 \cup P_2$ then contains a path P. If $e \notin P$ we would have $u, v \in V(P) \cap J \subseteq \{w\}$ which is impossible. Therefore $e \notin P$ and so $P \subseteq G - e$. since I is an $(s-t)$ separator in $G-e$ and J is an $(s-t)$ separator in G, P has a vertex common with I and also with J. Therefore $w \in I \cap J$ thus verifying the claim.

Combining (4), (5) and the above observation we have

$$2q \leq |H_s| + |H_t| = |H_s \cup H_t| + |H_s \cap H_t| \leq |I \cup J| + |I \cap J| = |I| + |J| < q + q = 2q$$

giving a contradiction.

Thus (1) is false and we have $\kappa(s,t|G) = p(s,t|G)$ completing the induction.

Theorem 1.14.2 Menger's Theorem : Edge form

For any pair of vertices s and t of a graph G the band number equals the lace number

Thst is $\lambda(s,t) = l(s,t)$ for very pair $s,t \in V$

Proof

We use induction on the numbe of deges of G. For $m = 1$ or 2, the theorem is obious. We assume that the theorem is true for all graphs with less than m deges and let G be a graph with m edeges. Let $\lambda(s,t|G) = k$. We distinguish two cases.

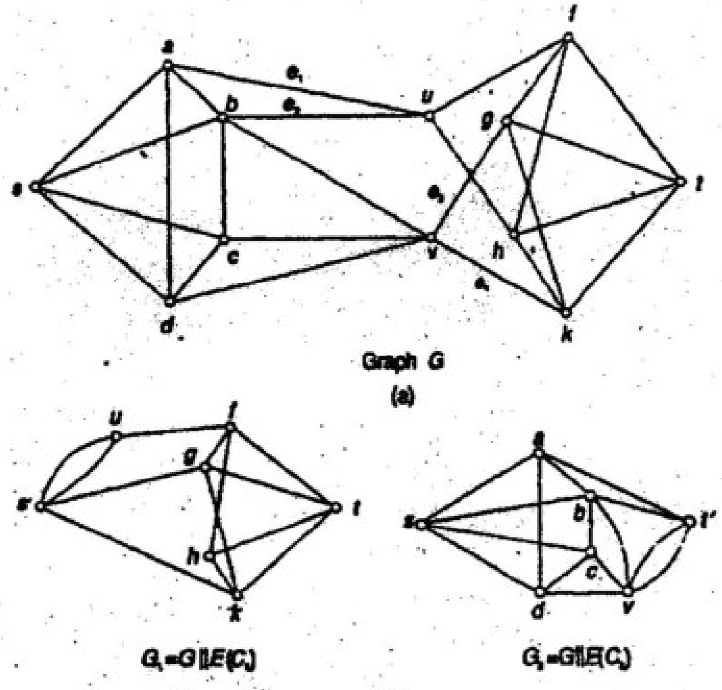
Case (i) Suppose G has an $(s-t)$ band F such that not all deges of F are incident wiht s, nor all deges of F are incident with t. Then $G-F$ consists of two non trivial components C_1 and C_2 with $s \in C_1$ and $t \in C_2$. Let $G_1 = G||E(C_1)$ and $G_2 = G||E(C_2)$. That is G_i is obtained from G by contracting the edges of C_i, $i = 1,2$ (see Fig. 1.14.1) Since G_1 and G_2 have less edges than G, the induction hypothesis applies to them. Also the edges corresponding to F provides an (s-t) band. in G_1 and G_2, so that $\lambda(s,t|G_1) = \lambda(s,t|G_2) = k$. Hence by induction hypothesis, there is k lace between s and t in G_1 and so also in G_2. The section of the paths of the k-lace in G_2 which are in C_1 and the section of the paths of the k lace in G_1 which are in C_2 can now be combined to get a k lace between s and t in G

Case (ii) Every $(s-t)$ band of G is such that either all its deges are incident with

s or all its edges are incident with t.

If G has an edge e which is not in any $s-t$ band of G. then $\lambda(s,t|G-e)=\lambda(s,t|G)=k$ and since the induction hypothesis is applicable to $G-e$, there is a k lace between s and t in $G-e$ and hence in G .

Thus we may assume that every edge of G is in at least one $(s-t)$ band of G. This implies that every $s-t$ path P of G is either a single edge or a pair of edges. Any such path P can therefore contain at most one edge of any $(s-t)$ band. Then



$\kappa = 2,\ \lambda = 3,\ \delta = 4.\ F = \{e_1, e_2, e_3, e_4\},\ C_1 = (s, a, b, c, d, u),$
$C_2 = (t, f, g, h, k, u)$

Fig. 1.14.1 Graph G

$G-E(P)=G_1$ is a grph with $\lambda(s,t|G_1)=k-1$ and to which the induction hypothesis applies. Hence there is a $(k-1)$ lace between s and t in G_1. Together with P, this gives a k lace between s and t in G

1.15 Exercises :

1. Draw all graph with five points.
2. A closed walk of add length contains a cycle.

3. **Verify whether each of the following is a degree sequence and construct an graph realizing** it if it is degree sequence.

(i) $1^3 234567$ (ii) $1^3 2^3$ (iii) $1^3 2^3 3^5 579$ (iv) 4^7 (v) 4^6 (vi 5^8 (viii) $2^9 3^2$

4. Give an example of a graph which cannot be generated by the wang kleitman alogorithm.

5. For what values of k is $3^k k$ a degree sequence $(k \in n)$?

6.Prove that the following four statements are equivalent

(i) G is unicyclic (ii) G is connected and $p=q$ (iii) for some line x of G, the graph $G-x$ in true.(iv) G is connedcted and the set of lines of G which are not bridges from an cycle

7. The intersection of cycle and cocycle contains and even number of lines.

8. G is a block if and only if every two lines lie and a common cocycle, Prove

9. show that $\kappa = \lambda$ for any connected cubic graph.

References :

1. Graph theory, Frank Harary

2. Graph Theroy, Ronald Gould

3. Basic Graph Theory, K. R. Parthasarathy

●●●

UNIT 2

Introduction 2.1

One feature of graph theory that has helped to popularize the subject lies in its application to the area of puzzles and games. Often a puzzle can be converted into a graphical problem; to determine the existence or non-existence of an 'eulerian trail' or a 'hamiltonian cycle', within a graph. The concept of an eularian graph was formulated when Euler studied the problem of the 'Konigsberg bridges.

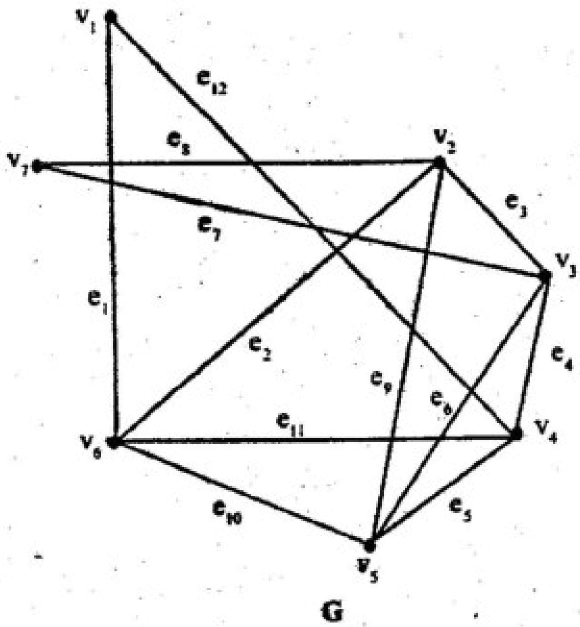
In this unit characterization of eularian graphs are presented. Some necessary conditions and some sufficient conditions for graphs to be Hamiltonian are also given.

2.2. Eulerian graph :

2.2.1 Definition : Eulerian graph

A graph G is called Eulerian if there exists walk which traverses each lines exactly once and goes through all the points in and ends at the starting point

A closed trail containing all points and lines in an Eulerian graph is called an Eulerian trail.



G

Fig. 2.1.

2.2.2 Theorem :

The following statements are equivalent in a connected graph G:

(i) G is Eulerian.

(ii) Every point of G has even degree.

(iii) The set of lines of G can be partitioned into cycles.

Proof: We proced to show the results in the following order:

(i) ⇒ (ii)

Let G be any Eulerian graph.

To show that every point of G has even degree let T be an Eulerian trail in G. Suppose, we traverse T starting from v_1(say) in G and let T be $v_1 = x_1 e_1 x_2 e_2 x_3 \ldots e_{r-1} x_r e_r x_{r+1} = v_1$, where all the lines e_i are distincts while the points $x_1, \ldots, x_n$ may not be all distinct. It is clear that any two succesive lines e_i and e_{i+1}, $i = 1, 2, \ldots, r - 1$ contributes 2 to the degree of the point x_{i+1}. Also, in addition to this, the point v_1 gets a contribution of 2 to its degree by the initial and the final lines e_1 and e_r. thus, all the points are of even degrees.

(ii) ⇒ (iii)

Let all the points of G have even degrees.

To show that G can be partitioned into cycles.

We have G is connected and non trivial and let all points have even degree each. So all the points have atleast degree 2 each. Thus, degree of each point of G is greater than 1. We also know that, in a non trivial tree there are atleast two points with degree 1. Therefoer, G cannot be a tree. So, G must contain atleast one cycle. let z_1 be a cycle of G. Then the removal of the lines of Z_1, results a spanning subgraph G_1 of G in which every point have still even degree. If G_1 has no lines, then $G = z_1$ and the result(iii) holds immediately. Otherwise, G_1 has atleast one cycle z_2(say). Then we repeat the argument applied to G_1 to get a graph G_2 in which again each points have even degree. Now, proceeding in this way, when a totally disconnected graph G_n is obtained, we have a partition of the lines of G into n cycles say,

$$X = Z_1 \cup Z_2 \cup \ldots \cup Z_n$$

This shows that the lines of G can be partitioned into cycles.

(iii) ⇒ (iv)

Let the lines of G be partitioned into cycles.

To show that G is Eulerian.

Let z_1 be one of the cycles of the partition of the lines of G. If G consists only of this cycles, than G is obiviously Eulerian. Otherwise, there is another cycle z_2 with a point v(say) in common with z_1(since G is connected). Then the walk begining at v(say) and consisting of the cycle z_1 and z_2 in succession is a closed trail T_{12} containing all the lines of these two cycles once. If G has only these two cycles, then T_{12} is the required Eulerian trail that traverses all the lines of G exactly once showing thereby that G in Eulerian. If it is not so, there exists another cycle z_3 such that there is a point v '(say) common to T_{12} and z_3 where z_3 is different from both z_1 and z_2. Then the closed trail begins at v' and traversing T_{12} and Z_3 is succession which contains all the lines of the cycles z_1, z_2, z_3. By continuing this process, we can ultimately construct a closed trail that contains all the lines of G and contains all the points of G

This implies that G is an Eulerian graph.

Thus, all the three statements are equivalent to each other.

Corollary 2.2.2.(a) : Let G be a connected graph with exactly 2n odd points, $n \geq 1$. Then the set of lines of G can be partitiones into n open trails.

Proof : Let G be any connected graph with exactly 2n number of odd points, $n \geq 1$.

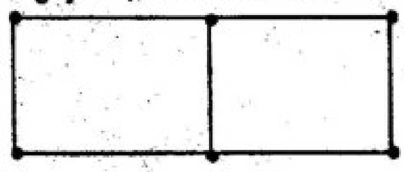
let r_i and s_i, $1 \leq i \leq n$ be the 2n odd degree points of G. Now to G add n-new points say $w_1, w_2, \ldots, w_n$ together with lines (r_i, w_i) and (s_i, w_i), for $1 \leq i \leq n$. In the resulting graph G′ every point is of even degree and so G′ is Eulerian. It may be noted that, any Euler trail of G′, the lines (r_i, w_i) and (s_i, w_i) for all $1 \leq i \leq n$, appear consecutively. The removal of 2n lines will then result in 'n' line disjoint open trail of G such that each line of G is present in precisily one of these trails. These open trails give the required partition of the lines of G.

Corollary 2.2.2.(b) : Let G be a connected graph with exactly two odd points. Then G has an open trail containing all the points and lines of G.

This corollary is a special case of the above corollary 2.3.(a)

2.2.3 Definition : Hamiltonian graph and Hamiltonian cycle.

If a graph G has a spanning cycle z, then G is called **Hamiltonian graph.**

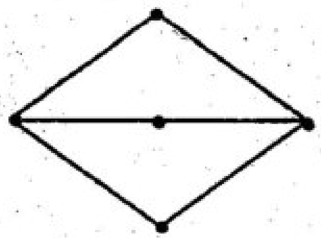


Hamiltonian graph

Fig. 2.2

The spanning cycle z is called the **Hamiltonian cycle.**

A path containing all the points of Hamiltonian graph is called a **Hamiltonian path.**



Graph containing Hamiltonian path but not containing any Hamiltonian cycle

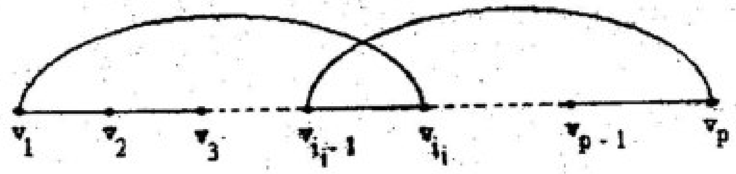
Fig. 2.3

Theorem 2.2.4: Let G have $p \geq \beta$ points. If, for every $n, 1 \leq n < \frac{p-1}{2}$, the number of points of degree not exceeding n is less than n and if for odd p, the number of points of degree atmost $\frac{p-1}{2}$ does not exceed $\frac{p-1}{2}$, then G is Hamiltonian.

Proof: Assume that the theorem does not hold and let G be a maximal non-Hamiltonian graph with p points satisfying the hypothesis of the theorem. Since G is maximal non-Hamiltonian, so the addition of any line to a graph satisfying the conditions of the theorem results in a Hamiltonian graph, and as such any two non-adjacent points must be joined by a spanning path.

We first show that every point of degree at least $\frac{p-1}{2}$ is adjacent to every point of degree greater than $\frac{p-1}{2}$. Assume without loss of generelity, degree $v_p \geq \frac{p-1}{2}$, and degree $v_2 \geq \frac{p}{2}$, but v_1 and v_p are non-adjacent. Then, there is a spanning path $v_1v_2...v_p$ connecting v_1 and v_p in G. Let the point adjacent to v_1 be $v_{i_1}, v_{i_2}, ..., v_{i_n}$, where n = degree v_1 and $2 = i_1 < i_2 < ... < i_n$. Clearly v_p cannot be adjacent to any point of G of the form v_{i_j-1}, for otherwise there would be a hamiltonian cycle,

$$v_1v_2...v_{i_j-1}...v_pv_{p-1}...v_{i_j-1}v_{i_j}v_1 \text{ in G}$$



but, since $n \geq \frac{p-1}{2}$, we must have,

$$\frac{p}{2} \leq \text{degree } v_p \leq (p-1) - n < \frac{p}{2}$$

which is impossible.

So, v_1 must be adjacent to v_p.

It follows that if the degree $v \geq \frac{p}{2}$, for all points v then G is Hamiltonian. For the above agrument implies that every pair of points in G are adjacent and so G is complete. But it is a contradiction since a complete graph k_p is Hamiltonian for all $p \geq 3$. Therefore there is a point v in G with degree $< \frac{p}{2}$. Let m be the maximam degree among all such points, and choose v_1 so that deg $v_1 = m$. By hypothesis the number of points of degree not exceeding m is atmost $m < \frac{p}{2}$. Thus, there must be more than 'm' points having degree greater than m, and hence atleast $\frac{p}{2}$. therefore there exists some points sayv_p of degree atleast $\frac{p}{2}$ not adjacent to v_1 (since degree $v_1 < \frac{p}{2}$). Since v_1 and v_p are not adjacent; there is a spanning path $v_1 v_2 \ldots v_p$. Also, as above we write the points $v_{i_1}, v_{i_2}, \ldots, v_{i_m}$ as the points of G adjacent to v_1 and note that v_p cannot be adjacent to any of these m points v_{i_j-1}, for $1 \leq j \leq m$.

Since, v_1 and v_p are not adjacent and degree $v_p \geq \frac{p}{2}$, m must be less than $\frac{p-1}{2}$ (by the first part of the proof).

[degree $v_p = (p-1) - m \geq \frac{p}{2} \Rightarrow m \leq (p-1) - \frac{p}{2} < \frac{p-1}{2}$]

thus by hypothesis, the number of points of degree atmost m is less than m and so atleast one of the m points v_{i_j-1}, say v' must have degree atleast $\frac{p}{2}$. We have degree $v' \geq \frac{p}{2}$, for otherwise, the number of points with degree $\leq m$ will be greater than m.

We have thus exhibited two non-adjacent points v_p and v' each having degree atleast $\frac{p}{2}$. But this is a contradiction to the fact that any two points with degree atleast $\frac{p}{2}$ are adjacent.

This completes the proof.

Corollory 2.2.5.(a) : If $p \geq 3$, and u and v are non-adjacent points of the graph G such that

$\deg u + \deg v \geq p$, then G is Hamiltonian.

Proof: Let G be the maximal non-Hamiltonian graph. Therefore addition of the lines uv makes the graph Hamiltonian and hence between u and v, there is a spanning path. Let u be adjacent to $u_{i1}, u_{i2}, \ldots, u_{in}$ and v cannot be adjacent to any point of the form $u_{i_j-1} (2 \leq i_1 < i_2 < \ldots < i_n)$, otherwise we should get a Hamiltonian cycle.

$uv_{i1} \ldots u_{ij-1} v \ldots u_{ij} u.$

$\therefore$ $\deg u + \deg v \leq m + (p-1) - m = p - 1 < p$ contradicting the hypothesis.

Hence, G must be Hamiltonian.

Corollary 2.2.5.(b) : If for all points v of G, $\deg v \geq \frac{p}{2}$, where $p \geq 3$, then G is Hamiltonian.

Proof : Case I : If $G \cong K_p$, then obviously G is hamiltonian.

Case II : Suppose G is not complete.

Let u and v be two non-adjacent points of G. Then

$\deg u + v \geq p/2 + p/2 = P.$

So, G is Hamiltonian by corollory 2.2.5 (a),

Theorem : 2.2.6 Every cubic Hamiltonian graph has atleast three spanning cycles.

2.3 Factorization :

A factor of a graph G is a spanning subgraph of G which is not totally disconnected. We say that G is the sum of factors G_i if it is their line disjoint union and such a union is called a factorisation of G.

If $G_1, G_2, ..., G_n, (n \geq 2)$ are line disjoint factors of a graph G such that $\bigcup_{i=1}^{n} x(G_i) = x(G)$, then we write

$$G = G_1 \oplus G_2 \oplus ... \oplus G_n.$$

An n-factor of a graph G is defined to be a spanning subgraph of the graph with degree of each of its points being equal to n i.e. an n-factor is a regular graph of degree n.

If G is the sum of n-factors, their union is called an n-factorisation and G itself is n-factorable.

Examples :

(i)

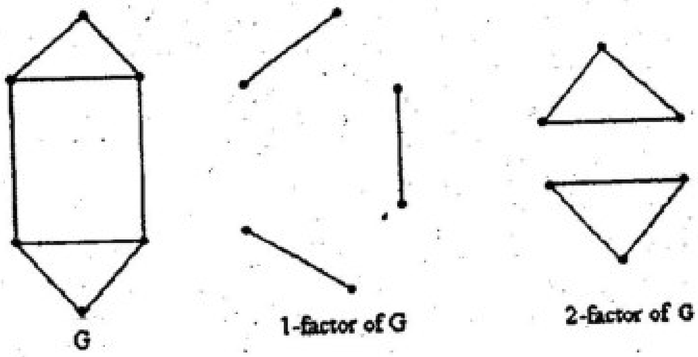
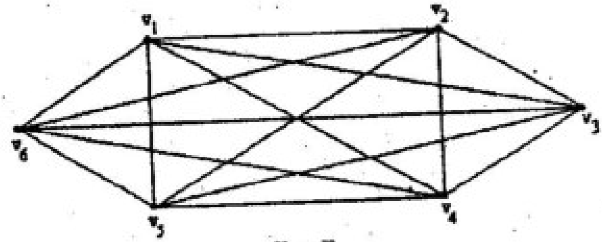


Fig. 2.4.

(ii) A 1-factorisation of K_6 is given below :



$K_6 = K_{2,4}$

Fig. 2.5.

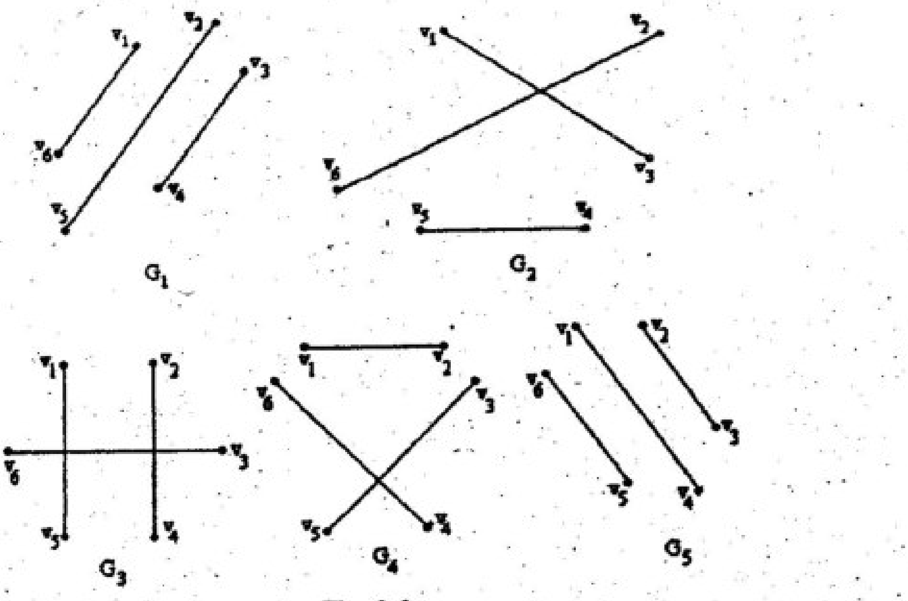


Fig. 2.6.

$$X(K_6) = X(G_1) \cup X(G_2) \cup X(G_3) \cup X(G_4) \cup X(G_5)$$

$G \rightarrow$ 1-factor of G_i.

G_i is a spanning subgraph of G and hence

$$V(G_i) = V(G) = p(\text{say}).$$

Let the number of lines in G_i is q. Degree of each point of G_i is 1. So the equation $\sum_{i=1}^{p} d(v_i) = 2q$

gives

$$p = 2q.$$

$\therefore$ G has even number of points.

Example : display a 1-factorisation of K_8.

Solution : The following is a display of a 1-factorisation of K_8 :

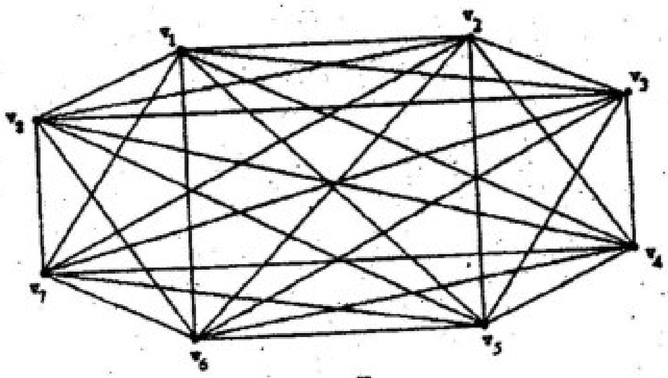


Fig. 2.7.

$$X_1 = \{v_1v_8\} \cup \{v_7v_2, v_6v_3, v_5v_4\}$$
$$X_2 = \{v_2v_8\} \cup \{v_1v_3, v_7v_4, v_6v_5\}$$
$$X_3 = \{v_3v_8\} \cup \{v_2v_4, v_1v_5, v_7v_8\}$$
$$X_4 = \{v_4v_8\} \cup \{v_3v_5, v_2v_6, v_1v_7\}$$
$$X_5 = \{v_5v_8\} \cup \{v_4v_6, v_3v_7, v_2v_1\}$$
$$X_6 = \{v_6v_8\} \cup \{v_5v_7, v_4v_1, v_3v_2\}$$
$$X_7 = \{v_7v_8\} \cup \{v_6v_1, v_5v_2, v_4v_3\}$$

Theorem 2.3.2. : The complete graph K_{2n} is 1-factorable.

Proof : We need only display a partition of the set of lines of K_{2n} into (2n - 1) 1-factors. For this purpose, we denote the points of G by $v_1, v_2, v_3, ..., v_{2n}$ and for each i = 1, 2, 3, ..., 2n - 1, we define the set of lines :

$$X_i = \{v_iv_{2n}\} \cup \{v_{i-j}v_{i+j}; j = 1, 2, ..., n - 1\}$$

where each of the subscripts i - j and i + j is expressed as one of the numbers 1, 2, 3, ..., 2n - 1 modulo(2n - 1). [e.g. if i = 1, j = 2 then $v_{i-j} = v_{-1}$ as $-1 \equiv (2n - 2) \bmod (2n - 1)$].

The collection $\{X_i : i = 1, 2, ..., 2n - i\}$ is easily seen to give an appropriate partition of X and the sum of the subgraphs induced by X_i is a 1-factorisation of K_{2n}.

[For K_6 we have,

$X_1 = \{v_1v_6, v_2v_5, v_3v_4\}$ $X_2 = \{v_2v_6, v_1v_3, v_5v_4\}$ $X_3 = \{v_3v_6, v_2v_4, v_1v_5)$

$X_4 = \{v_4v_6, v_3v_5, v_1v_2\}$ $X_5 = \{v_5v_6, v_4v_1, v_2v_3\}$]

2.3.3 Definition : A set of mutually non-adjacent lines is called independent.

Here $\{v_1v_2, v_3v_4\}$, $\{v_1v_4, v_2v_3\}$ are non-adjacent. By an odd component of G we mean one with an odd number of points.

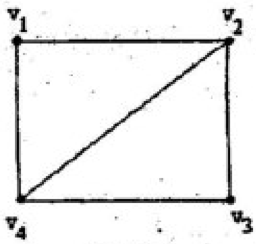


Fig. 2.6.

Theorem 2.3.4. : A graph G has a one factor if and only if p is even and there is no set S of points such that number of odd components of G - S exceeds | S |.

Proof : Let S be any set of points of G and let H be a component of G - S. In any one factor of G, each point of H must be paired with either to another point of H or a point of S. But if H has odd number of points, then at least one point of H is matched with a point of S. let k_0 be the number of odd components of G - S. If G has a 1-factor, then $|S| \geq k_0$, since in a 1-factor each point of S can be matched with at most one point of G - S and therefore can take care of atmost one odd component. That is, the number odd components can't exceed | S |. This proves the necessity of the theorem.

Conversely, for a subset D of V(G), denote the number of odd components of G - S by K_0(G - S). Hence the hypothesis for G can be restated as $k_0(G\text{-}S) \leq |S|$, for every proper subset S of V(G). In particular,

$$k_0(G - \phi) \leq |\phi| = 0$$

$$\Rightarrow k_0(G - \phi) = k_0(G) = 0$$

This means that G has only even components and therefore has even order p.

Furthermore we note that for each proper subset S of V(G), the number of odd components of G - S and | S | are of same parity, since p is even. We proceed by induction on even positive integers p. If G is a graph of order p = 2 such that $k_0(G - S) \leq |S|$ for every proper subset S of V(G), then G $\cong K_2$ and G has a 1-factor.

Assume for all graphs H of even order less then p(where $p \geq 4$, p is even) that if

$$k_0(H - W) \leq |W|$$

for every proper subset W of V(H), then H has a 1-factor.

Let G be a graph of order p(even). Assume that

$$k_0(G - S) \leq |S|$$

for every proper S of V(G). We consider two cases :

Case I : Suppose $k_0(G - S) < |S|$ for every proper subset S of V(G) with $2 \leq |S| < p$ since $k_0(G - S)$ and | S | are of the same parity, so

$$k_0(G - S) \leq S - 2$$

for all subset S of V(G) with $2 \leq |S| < p$.

Let e = uv be a line of G and consider G - u - v. Let T be a proper subset of V(G - u - v). It follows that

$$k_0(G - u - v - T) \leq |T|$$

For suppose, to the contrary that

$$k_0(G - u - v - T) > |T| = |T \cup \{u, v\}| - 2$$

so that $$k_0(G - (T \cup \{u, v\})) \geq |T \cup \{u, v\}|$$

which contradicts the hypothesis of our assumption. Thus by the induction hypothesis G - u - v has a 1-factor and hence so does G.

Case II : Suppose there exists a subset R of V(G) such that $k_0(G - R) = |R|$ where $2 \leq |R| < p$. Among all such sets R, let S be one of maximal cardinality where $k_0(G - S) = |S| = n$. Further let $G_1, G_2, ..., G_n$ denote the odd components of G - S. These are the only components of G - S, for if G_0 were an even components of G - S and $u_0 \in V(G_0)$, then we have,

$$k_0(G - (S \cup \{u_0\})) \geq n + 1 = |S \cup \{u_0\}|$$

implying necessarily that

$$k_0(G - (S \cup \{u_0\})) = n + 1$$

which contrdicts the maximality of S.

For i = 1, 2, 3, ...,n, let S_i denote the set of those points of S adjacent to one or more points of G_i. Each set S_i is non-empty; otherwise some G_i would be an odd components of G. The union of k of the sets $S_0, S_1, ..., S_n$ contains at least k points for each k with $1 \leq k < n$. For otherwise there exists $k(1 \leq k \leq n)$ such that the union of T of some k sets contains less than k points. This would however imply that $k_0(G - T) > |T|$, which is impossible. Thus we may apply the theorem(which states that, there exists a system of distinct representative S for a family of sets $S_1, S_2, ..., s_m$ if and only if the union of any k of these contains at least k elements, for all k from 1 to m) to produce a system of distinct representatives for $S_1, S_2, ..., s_n$. This implies that S contains points $v_1, v_2, ..., v_n$ and each G_i contains a point $u_i \in G_i$, $1 \leq i \leq n$ such that $u_iv_i \in X(G)$, for i = 1, 2, 3, ..., n. Let w be a proper subset of $V(G_i - u_i)$, $1 \leq i \leq n$. we show that

$$k_0(G_i - u_i - W) \leq |W|.$$

For, if possible let $k_0(G_i - u_i - W) > |W|$, sine $G_i - u_i$ is of even order for each i, $k_0(G_i - u_i - W)$ and $|W|$ are of same parity and so

$$k_0(G_i - u_i - W) \geq |W| + 2$$

Thus,

$$\begin{aligned} k_0(G_i - (S \cup W \cup \{u_i\})) &= k_0(G_i - u - w) + k_0(G - S) - 1 \\ &\geq |S| + |W| + 1 \\ &= |S \cup W \cup \{u_i\}| \end{aligned}$$

This however contradicts the maximal property of S. Therefore, we must have that,

$$k_0(G_i - u_i - W) \leq |W|.$$

as claimed implying by the inductive hypothesis that for i = 1, 2, ..., n the subgraph $G_i - u_i$ has a 1-factor. This fact together with the existence of the line $u_iv_i(1 \leq i \leq n)$ produces a 1-factor in G.

This completes the proof.

2.3.5 Definition : 2-Factorable Graphs : Since the degree of all points in a 2-factorable graph is even, so the complete graph K_{2n} is not 2-factorable because the degree of each point of K_{2n} is (2n - 1). the odd complete graph are 2-factorable.

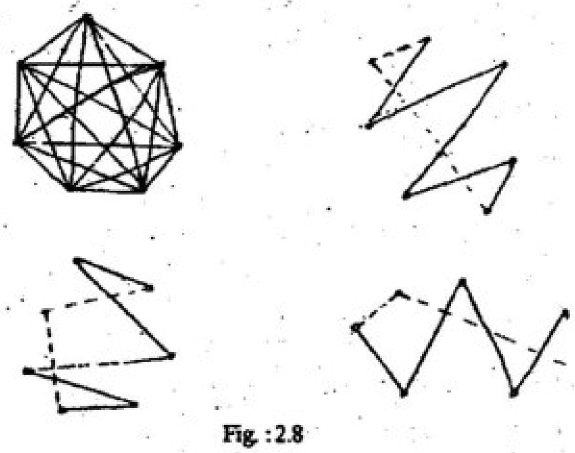


Fig. : 2.8

Theorem 2.3.6. : The graph k_{2n+1} is the sum of n spanning cycles.

Proof : In order to construct n line disjoint spanning cycles in K_{2n+1}, first label its points by $v_1, v_2, ..., v_{2n}$,

v_{2n+1}. Then we construct n-paths on the points $v_1, v_2, \ldots, v_{2n}$ as follows:

$$p_i = v_i v_{i-1} v_{i+1} v_{i-2} \ldots v_{i+n-1} v_{i-n},\ i = 1, 2, 3, \ldots, n.$$

Thus the j^{th} point of p_i is v_k where

$$k = i + (-1)^{j+1}\left[\frac{j}{2}\right]$$

and all subscripts are taken as the integers 1, 2, 3, ..., 2n(modulo 2n). The spanning cycles Z_i is then constructed by joining v_{2n+1} to the end points of p_i.

We explain the technique by taking K_7 as an example :

$P_1 = v_1v_6\ v_2v_5\ v_3v_4$

$P_2 = v_2v_1\ v_3v_6\ v_4v_5$

$P_3 = v_3v_2\ v_4v_1\ v_5v_6$

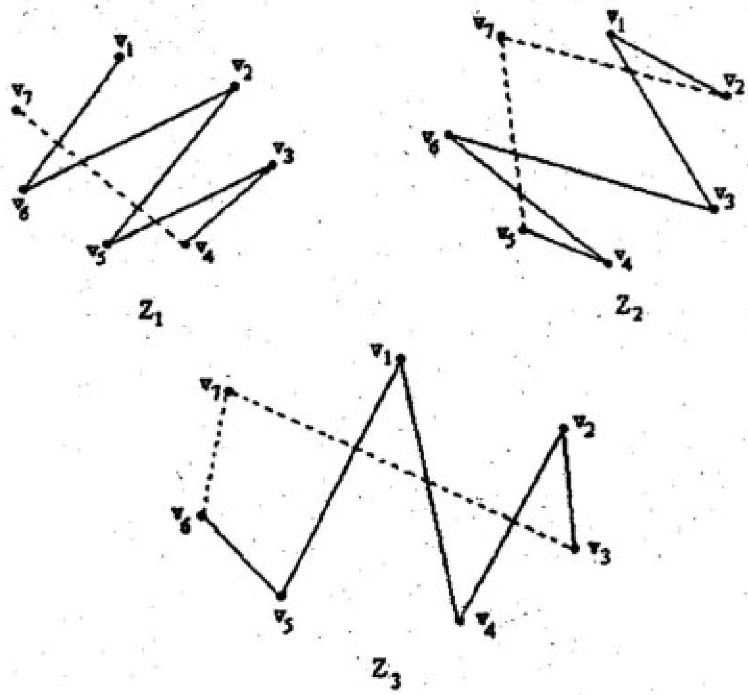


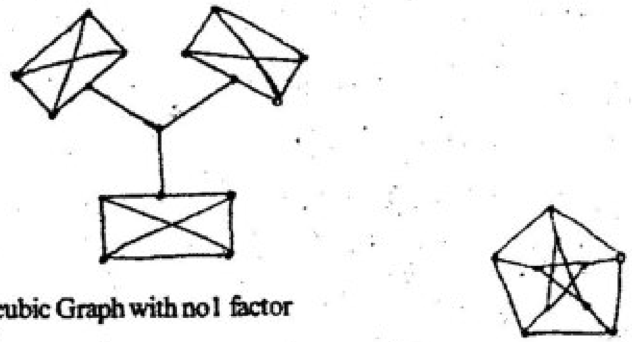
Fig. 2.9.

2.3.7. Theorem :

The complete graph K_{2n} is the sum of a 1-factor and n-1 spanning cycles.

Ofcourse, every regular graph of degree 1 is itself a 1-factor and every regular graph of degree 2 is a 2-factor. If every component of a regular graph G of degree 2 is an even cycle, then G is also 1-factorable since it can be expressed as the sum of two 1-factors. If a cubic graph contains a 1-factor, it must also have a 2-factor, but there are many cubic graphs which donot have 1-factors.

The graph of Fig. 2.9 has three bridges.



A cubic Graph with no 1 factor

Peterson Graph

Fig. : 2.10

Theorem 2.3.8. : Every bridgeless cubic graph can be expressed as the sum of a 1-factor and 2-factor.

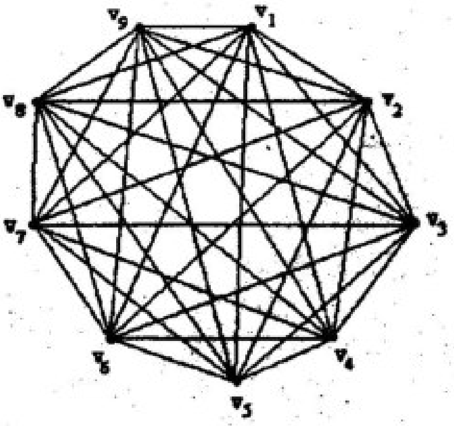
Proof : Peterson proved that every cubic graph that fails to cintain a 1-factor possesses bridges. It is sufficient to show that every bridges cubic graph has a one-factor(since the remaining lines form a 2-factor).

Assume to the contrary that G has no 1-factor. Then by theorem 8.4., V(G) has a proper subset s such that the number of odd components of G - S exceeds | S |. Let k = | S | and let $G_1, G_2, \ldots, G_n (n > k)$ be the odd components of G - S. Then there must exists at least one line joining a point of G_i to a point of S for each i = 1, 2,3, ..., n, for otherwise G_i is a cubic graph of odd order. On the otherhand, since G contains no bridges, there can't be exactly one such line i.e. there are a least two lines joining G_i to S, for each i = 1, 2, 3, ..., n.

Suppose that for some i, $1 \le i \le n$, there are excatly two lines joining G_i and S. Then there are an odd number of odd points in the components G_i of G - S which can't happen. Hence for each i, $1 \le i \le n$, there are at least three lines joining G_i and S. Therefore the total number of lines joining $\bigcup_{i=1}^{n} V(G_i)$ and S is at least 3n. However, since each of the k points of S has degree 3, the total number of lines joining $\bigcup_{i=1}^{n} V(G_i)$ and S is at most 3k. Therefore $3k \ge 3n$ i.e. $k \ge n$ which is a contradiction to $k < n$. Hence no such set S exists and so by theorem 8.4., we conclude that G has a 1-factor.

This completes the proof.

Exercise : Express K_9 as the sum of 4 spanning cycles?



K_9

$P_1 = v_1v_8v_2v_7\ v_3v_6v_4v_5$

$P_2 = v_2v_1\ v_3v_8v_4v_7v_5v_6$

$P_3 = v_3v_2\ v_4v_1\ v_5v_8v_6v_7$

$P_3 = v_4v_3\ v_5v_2\ v_6v_1v_7v_8$

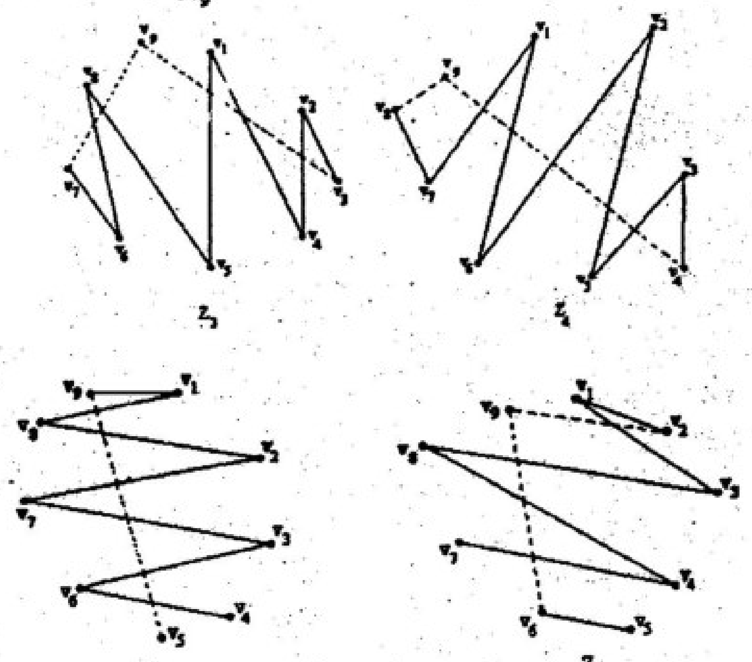


Fig. 2.11

2.4. Coverings :

2.4.1 Definition : A point and a line are said to cover each other if they are incident for example $x = uv$, then x covers u and v and u and v cover x.

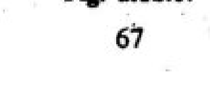


Fig. 2.12.1.

A set of points which covers all the lines of a graph G is called a point cover, while a set of lines which covers all the points is a line cover.

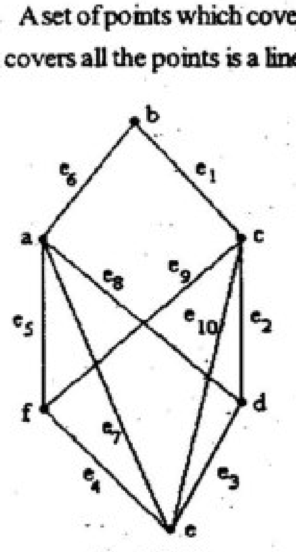


Fig. 2.12.2

{a, c, e, f} - point cover
{a, c, d, e} - point cover
{b, d, e, f} - point cover
{a, c, e} - minimum point cover
{b, d, f} - independent set of points
$\{e_1, e_3, e_5\}$ - independent set of lines
$\{e_1, e_3, e_5\}$ - minimium line cover

The smallest number of points in any point cover for G is called a point number and is denoted by $\alpha_0(G)$ or α_0. Similarly α_1 or $\alpha_1(G)$ is the smallest number of lines in any line cover of G and is called a line covering number.

A point cover(line cover) is minimum if it contains $\alpha_0(\alpha_1)$ elements.

A set of points in G is independent if no two of them are adjacent.

An independent set S of G is maximum if G has no independent set S′ with $|S'| \geq |S|$. The number of points in a maximum independent set of G is called the point indepence number of G and is denoted by $\beta_0(G)$ or β_0.

Analogously, an independent set of lines of G has no two of its lines adjacent, and the maximum cardinality such a set is the line independence number $\beta_1(G)$ or β_1.

For example, for the complete graph k_p

$$\beta_0(k_p) = 1, \beta_1(k_p) = \left[\frac{p}{2}\right]$$

$$\beta_0(\bar{k}_p) = p, \beta_0(c_n) = \left[\frac{n}{2}\right]$$

$$\beta_1(\bar{k}_p) = 0, \beta_1(c_n) = \left[\frac{n}{2}\right]$$

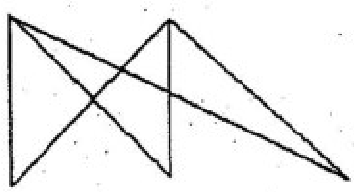


Fig. 2.12.3.

$$\beta_0(k_{m,n}) = \max(m,n), \beta_1(k_{m,n}) = \min(m,n)$$

For $k_{2,3}$, $\beta_1(k_{m,n}) = 2$, $\beta_0(k_{2,3}) = 3$.

Theorem 2.4.2. : For any non trivial connected graph G

$$\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1.$$

Proof : Let M_0 be any maximum independence set of points. So $|M_0| = \beta_0$. Since no two points M_0 are adjacent so the remaining $(p - \beta_0)$ points constitute a point cover for G so that

$$\alpha_0 \leq p - \beta_0 \qquad \text{......(i)}$$

On the other hand if N_0 is minimum point cover for G then no lines can join any two of the remaining of the $p - \beta_0$ points of G. So that the set $V - N_0$ is independent and hence

$$\beta_0 \leq p - \alpha_0 \qquad \text{......(ii)}$$

From (i) and (ii) we have $\quad \alpha_0 + \beta_0 = p$

To obtain the second inequality, let us consider an independent set M_1 of β_1 lines(i.e., M is the maximum independent set).

A line cover Y is then produced by taking $Y = M_1 \cup P$ where P is the set of $p - 2\beta_1$ lines.

Let x is the set of points of G which are not in M_1, then $|X| = p - 2\beta_1$.

Now for each point v in X. Select a point which is incident on v and adjacent to a line in M_1. Let p be the set of $p - 2\beta_1$ lines so selected.

$$|Y| = \beta_1 + p - 2\beta_1 = p - \beta_1$$

$$|Y| \geq \alpha_1 \Rightarrow p - \beta_1 \geq \alpha_1$$

$$\Rightarrow \alpha_1 + \beta_1 \leq p \qquad \text{......(iii)}$$

In order to show inequality in the other direction let us consider a minimum line cover N_1 of G Clearly N_1 cannot contain a line both whose end points are incident with lines also in N_1. This implies that N_1 is the sum of stars of G. If one line is selected from each of these stars we obtain a independent set of lines W.

Now $\quad |N_1| + |W| \geq p$

$$\Rightarrow \alpha_1 + \beta_1 \geq p \qquad \text{......(iv)}$$

From (iii) and (iv) we have

$$\alpha_1 + \beta_1 = p$$

Hence

$$\alpha_1 + \beta_1 = p = \alpha_1 + \beta_1.$$

A collection of independent lines is some time called a matching of G

Theorem 2.4.3. : If G is a bepartite graph then the number of lines in a maximum matching equals to the point covering number i.e. $\beta_1 = \alpha_0$.

Proof : Let M^* be a maximum matching and α_0 a point covering number and let $|k^*| = \alpha_0$ in a bipartite graph G(X, Y, E) where $V = X \cup Y$ and E in the set of lines in G.

Consider any subset A of X. each line e of G is incident on either a point in A or a point in X - A.

Further any line incident on a point in A is also incident to a point in $\lceil \overline{A}$, the set of points adjacent to those in A. Thus the set $(X - A) \cup \lceil \overline{A}$ is a point cover of G and hence

$$|X - A| + |\lceil \overline{A}| \geq \alpha_0$$

But $|M^*| = \min_{A \subseteq X} \{|X - A| + |\overline{A}|\} \geq \alpha_0$ ......(1)

Again at least β_1 points are required to cover the lines of M^*, the point cover k^* must contain at least β_1 points. Therefore we have

$\beta_1 \leq \alpha_0$ ......(2)

Frome (1) and (2) we have

$\alpha_0 = \beta_1$. Hence proved.

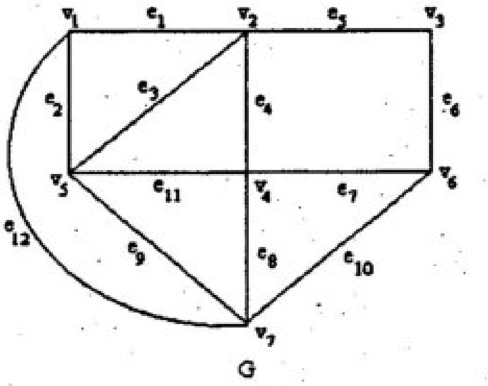


Fig. 2.13

The problem of finding a maximum matching, the so called matching problem, is closely related to that of finding a minimum point cover.

Let $M \subseteq X(G)$ be a matching. In an alternative M-trail exactly one of any two consecutive lines is in M

$M = \{e_4, e_2\}$. [From Fig. 2.13]

An augmenting M-trail is an alternating M-trail whose end points are not incident with any line of M.

An augmenting M-trail $\{e_5, e_4, e_{11}, e_2, e_{12}\}$ which begins at u_3 and ends at v_7 which are not points of $M = \{e_4, e_2\}$.

Consider $M = \{e_5, e_1, e_{12}\}$.

M has no augmenting M-trail. the matching is **unaugmentable** and it is maximum.

Thus every maximum matching is unaugmentable.

Theorem 2.4.4 : Every unaugmentable matching is maximum.

Proof : Let M be unaugmentable. Let M′ be a maximum matching for which | M - M′ |, the number of lines which are in M but not in M′ is minimum. If this number is zero, then M = M′. Thus the result is obvious. Otherwise let us construct a trail w of maximum length whose lines alternate in M - M′ and M′ being maximum is unaugmentable and hence trail w cannot begin and end with lines of M - M′ (otherwise M - M′ will be unaugmentable) and has equally many lines in M - M′ and M′. Now we form a maximum matching N from M′ by replacing these lines of w which are in M′ by the lines of W in M - M′

Then | M - N′ | < | M - M′ | contradicting the choice of M′. Hence M must be maximum.

2.5 Critical Points and lines :

2.5.1 Definition : If H is a subgraph of G, then $\alpha_0(H) \le \alpha_0(G)$. In particular this inequality holds when H is G - v or H = G - x for any point v or line x. If $\alpha_0(G - v) < \alpha_0(G)$ then v is called critical point. If $\alpha_0(G - x) < \alpha_0(G)$ then x is called critical line of G.

If v and x are critical, it follows that

$$\alpha_0(G - v) = \alpha_0(G - x) = \alpha_0 - 1.$$

Theorem 2.5.2 : A point v is critical in a graph G if and only if some minimum point cover of G contain v.

Proof : If M is a minimum point cover of a graph G which contain v, then m - {v} covers G - {v}. Hence

$$\alpha_0(G - v) \le \alpha_0 \,|\, M - \{v\} | = | M | - 1 = \alpha_0(G) - 1.$$

So v is a critical point of G.

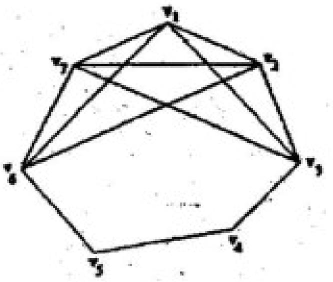
Conversely suppose that v is a critical point of G. Consider a minimum point cover M′ for G - {v} then the set M′ ∪ {v} is a point cover for G and since it contains one more element than M′ so it is minimum.

If the removal of a line x = uv from G decreases the point covering number, then the removal of u or v must also results in a graph with smaller point covering number. Thus if a line is critical both of its end point are critical.

But if a graph has critical points, it need not have critical lines. For example, every point of C_4 is critical but no lines of C_4 is critical.

A graph in which every point is critical is called point critical while one having all lines critical called line critical. Thus a graph G is a point critical iff each point of G is lies in some minimum point cover of G.

Every line critical graph without isolated point is point critical.



is a line critical graph and so is a point critical graph

Fig. 2.14

2.6. Exercise :

1. The graph K_4 has a unique 1-factorization. Find the the number of 1-factorizations of $K_{3,3}$ and of K_6.

2. The number of 1-factors in K_{2n} is

$$\frac{(2n)!}{\left(2^n n!\right)}$$

3. If an n-connected graph G with P even is regular of degree n, then G has a 1-factor.

4. Is the Peterson graph hamiltonian?

5. Prove or disprove. Every point cover of a graph G contains a minimum point cover.

6. For any graph G, $\alpha_0(G) \geq \beta_1(G)$ and $\alpha_1(G) \geq \beta_0(G)$.

7. Find a necessary and sufficient condition that $\alpha_1(G) = \beta_1(G)$.

●●●

Unit-3

3.1 Introduction :

The Concept of planarity revolves around the possibility of drawing a given graph in a plane such that its edges do not intersect. It is very important that we have an efficient way of establishing whether or not a given graphs is planar. In administering a test of planarity for any graph, we can consider its components one at a time.

3.2. Subdivision of graph :

A subdivision of the edge e = uv of a graph G is the replacement of the edge e by a new vertex w and two new edges uw and wv. This operation is also called an elementary subdivision of G.

A graph H obtained by a sequence of elementary subdivisions from a graph G is said to be a subdivision graph of G or to be homeomorphic from (or a homeomorph of) G. Two graphs H_1 and H_2 which are homeomorphics of the same graph G are said to be homeomorphic to each other; they are also said to be homeomorphically reducible to G. A graph G is homeomorphically irreducible if whenever a graph H is homeomorphic to G then H is homeomorphic from G.

The graphs H_1 and H_2 of Fig. 3.1 are homeomorphs of K_4;K_4 itself is homeomorphically irreducible.

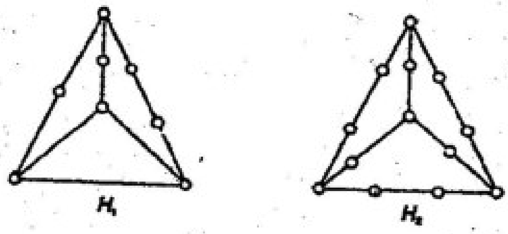


Fig. 3.2.1. A pair of homeomorphic graphs

It is eassy to see that 'being homeomorphic to each other' is an equivalence relation in the set of all graphs. It can be further proved that each equivalence class contains a unique homeomorphically irreducible graph which may therefore be taken as the representative of the class.

3.3. Identification of vertices :

Let U be a subset of the vertex set V of the general graph G = (V,E). Let H be the general graph obtained from G by the following operations:

(i) Replace the set of vertices U by a single new vertes u.

(ii) Replace each edge e = ab with $a \in U$ and $b \in V - U$ by a corresponding edge e' = ub.

(iii) Replace each edge e = ab with a, b $\in$ U (b possible being the same as a with a loop at u in H.

Let K be the multigraph obtained from H by dropping all loops and L be the graph obtained from K by replacing each multiple edge by a single edge. Then H,K,L are respectively said to be obtained from G by a general/multiple/simple identification of the vertices of U. We adopt the notation H = G : U,K = G:U, L = G.U.

Remark : (a) The operation in (ii) may result in the generation of multiple edges, and that in (iii) in the generation of loops even when G is an ordinary graph.

(b) For b ≠ u in H, $qH[u,b] = \{qG[a,b] \mid a \in U\}$.

qH[a,b] = number of edges between a and b in H.

3.4. Plane and Planar Graphs :

A graph is said to be embedded in a surface S when it is drawn on S so that no two edges intersect. We shall use "points and lines" for abstract graphs, "vertices and edges" for geometric graphs (embedded in some surface). A graph is planar if it can be embedded in the plane;

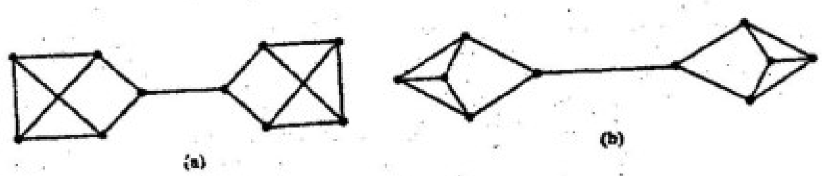


Fig. 3.4.1 A planar graph and an embessing.

A plane graph has already been embedded in the plane. For example, the cubic graph of fig. 3.4.1 (a) is planar since it is isomorphic to the plane graph in Fig.3.4.1 (b)

We will refer to the regions defined by a plane graph as its faces, the unbounded region being called the exterior face. When the boundary of a face of a plane graph is a cycle, we will sometimes refer to the cycle as a face. The plane graph of Fig. 3.4.2 has three faces f_1, f_2 and the exterior face f_3. Of these, only f_2 is bounded by a cycle.

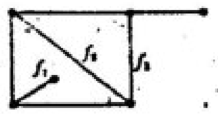


Fig. 3.4.2 A plane graph

The subject of planar graphs was discovered by Euler in his investigation of polyhedra. With every polyhedron there is associated a graph consisting only of its ver.'ces and edges, called its 1-skeleton. For example, the graph Euler formula for polyhedra is one of the classical results of mathematics.

Theorem 3.4.1 : A graph is planar if and only if each of its block is planar.

Proof : Clearly a graph G is planar if and only if each of its component is planar. So, we may assume G to the connected.

If G is planar, then each block of G is planar. For the converse, we apply induction on the number of G. If G has only one block and this block is planar then of courv G is planar. Assume every graph with fewer than n (≥ 2) blocks each of which is planar, is planar, is a planar graph and suppose G has n blocks all of which are planar.

Let B be an end block of G and denote by v the cutpoint of G common to B. Delete from G all vertices of B different from v calling the resulting graph G'. By the induction hypothesis G' is a planar graph.

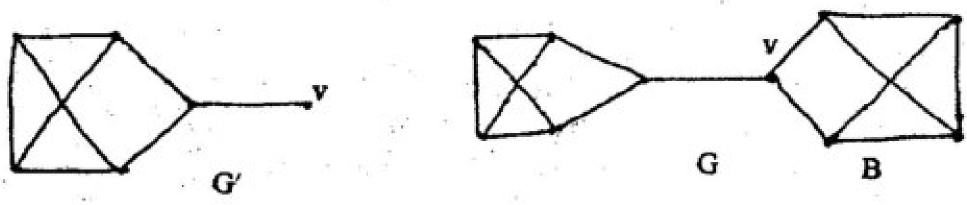


Fig. 3.4.3

Since the block B is planar it may be embedding in the plane so that v lies on the exterior region. In any region of a plane embedded of G' containing v, the plane block B may now be suitably placed so that the two vertices of G' and B labeled v are identified. The results is a plane graph of G

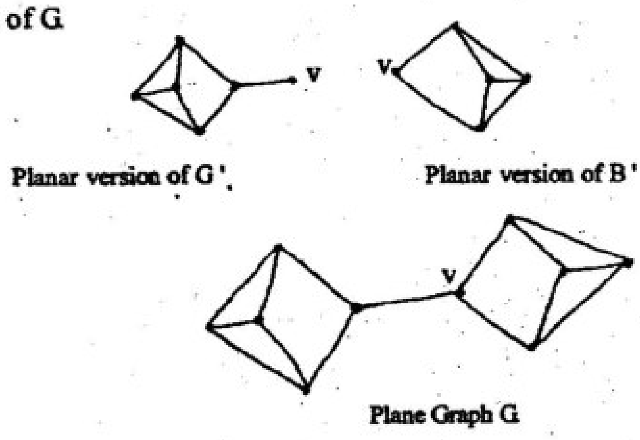


Fig. : 3.4.4

Hence G is planar.

Theorem. 3.4.2 Every 2-connected plane graph can be embedded in the plane so that any specified face is the exterior.

Proof : Let f be a nasexterior face of a plane block G. Embed G on a sphere and call some point interior to f the "North Pole". Consider a plane tangent to the sphere at the South Pole and project G onto that plane from the North Pole. The result is a plane graphs isomorphic to G in which f is the exterior face.

Corollary. 3.4.2(a) Every planar graphs can be embedded in the plane so that a prescribed line is an edge of the exterior regions.

3.5. Outerplanar Graph :

A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face; we usually choose this face to be exterior. Figure 3.5.1. Shows an outerplanar graph (a) and two outerplane embedding (b) and (c). In (c) all vertices lie on the exterior face

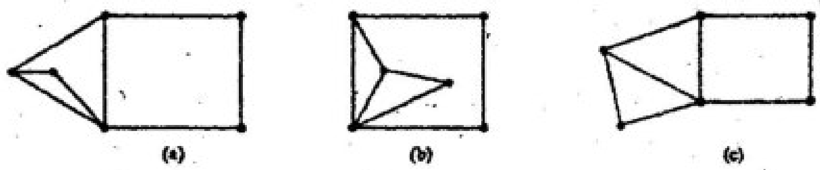


Fig. 3.5.1. An outerplanar graph and two outerplane embeddings.

In this section we develop theorems for outerplanar graphs parallel with those for planar graphs.

Theorem 3.5.1 A graph G is maximal outerplanar if no line can be added without losing outerplanarity. An outerplanar graph G is maximal outerplanar if no line can be added without losing outerplanarity. Clearly, every maximal outerplane graph is a triangulation of a polygon, while every maximal plane graph is a triangulation of the sphere. The three maximal outerplane graphs with 6 vertices are shown in Fig.3.5.2.

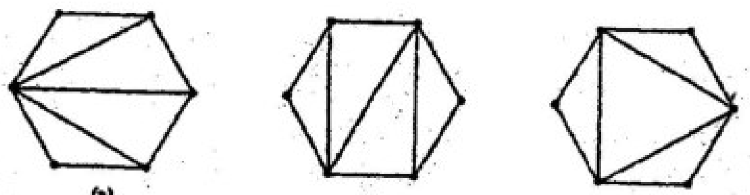


Fig. 3.5.2. Three maximal outerplanar graphs

Theorem 3.5.2. Let G be a maximal outerplanar graph with $p \geq 3$ vertices all lying on the exterior face. Then G has p–2 interior faces.

Proof. Obviously the result holds for p = 3. Suppose it is true for p = n and let G have p = n + 1 vertices and m interior faces. Clearly G must have a vertex v of degree 2 on its exterior face. In forming G – v we reduce the number of interior faces by 1 so that m – 1 = n – 1 = p – 2, the number of interior faces of G. Thus by inductions, the result is true.

Theorem 3.5.3. Every maximal outerplanar graph G with P points has

(a) 2P – 3 lines,

(b) at least three points of degree not exceeding 3,

(c) at least two points of degree 2,

(d) k(G) = 2.

Theorem 3.5.4. A graph is outerplanar if and only if it has no subgraph homeomorphic K_4 or $K_{2,3}$ except $K_4 - x$.

Proof. Two graphs are homeomorphic if both can be obtained from the same graph by a sequence of subdivisions of lines. For example, any two cycles are homeomorphic, and Fig. 3.5.3. shows a homeomorph of K_4.

(a) G be a maximal outplanar graph with P points.

∴ The no. of interior faces of G is P – 2.

By Euler's Pohyhedron formula, $V - E + F = 2$

$F = P - 2 + 1$

$V = P$

$\therefore E = V + F - 2$

$= P + P - 1 - 2$

$= 2P - 3.$

i.e. $2P - 3$ lines.

(b) Σ deg $V_i = 2q = 2\ (2P - 3)$

$= 4P - 6$

$= 4(P - 3) + 6$

Thus G has 2 points of deg 3 or 3 points of deg 2.

(c) Also G has atleast two points of degree2.

(d) As G has a point of degree2, say, v be a point of deg 2. So the removing of the point incident with the line which incident with v makes a disconnected graph.

Thus $K(G) \leq 2$.

Also here, δ (G) = 2

$\therefore K(G) = 2.$

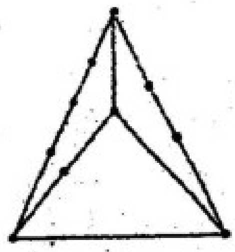


Fig.3.5.3. A homeomorph of K_4

Theorem 3.5.5 Every outerplanar graph with at least seven points has a nonouterplanar complement, and seven is the smallest such number.

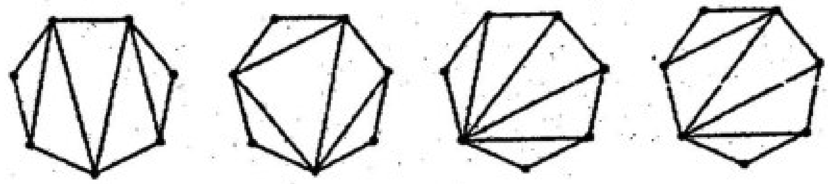


Fig.3.5.4. The four maximal outerplanar graphs with seven points.

Proof. To prove the first part, it is sufficient to verify that the complement of every maximal outerplanar graph with seven points is not outerplanar. This holds because there are exactly four maximal outerplanar graphs with P = 7 (Fig.3.5.4) and the complement of each is readily seen to be non-outerplanar. The minimality follows from the fact that the (maximal) outerplanar graph of Fig. 3.5.2. with six points has an outerplanar complement.

3.6. Euler's Polyhedron Formula :

For any spherical polyhedron with V vertices, E edges and F faces, $V - E + F = 2$.

Before proving this equation, we will recast this in graph theoretic term.

A plane map is a connected plane graph together with all its faces. One can restate theorem 10.1 for a plane map in terms of the number p of vertices, q of edeges and r of faces

$p - q + r = 2$ .... (1)

However this equation has already been proved where it was established that the cycle rank m of a connected graph G is given by,

$m = p - q + 1$.

Since it is easily seen that if (1) holds for the blocks of G separately, then (1) holds for G also. We assume that G is 2-connected.

Thus every face of a plane embedding of G is a cycle.

A graph is 2-connected

$\Leftrightarrow$ it is a block.

[if G is n-connected $n \geq 2$, then every set of n points of G lie on a cycle.]

we have just noted that, p = V, q = E for a plane map. It only remains to link m with F. We now show that the interior faces of a plane graph G constitute a cycle basis for G, so that they are m in number. This holds because the edges of every cycle X of G can be regarded as a symmetric difference of the faces of G contained in Z the exterior face is thus the sum (mod 2) of all the interior faces (regarded as edge set).

[The plane graph of G has three faces, f_1, f_2 and the exterior face f_3

$f_3 = f_1 + f_2$ (mod 2)

= 0,1]

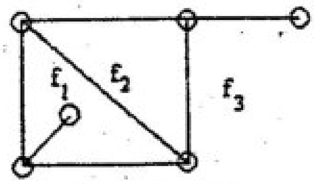


Fig. : 3.6.1

The result will be either 0 or 1. But for any bounded region, the exterior of the region can be regarded as 1.

m = F − 1.

Hence, m = q − p + 1 gives

F − 1 = E − v + 1

⇒ V − E + F = 2.

Corollary 3.6.1. (a). : If G is a (p,q) plane map in which every face is an n-cycle, then

$$q\frac{n(p-2)}{n-2}$$

Proof : Since every face of G is an n-cycle, each line of G is on two faces, has n-edeges. Thus the no. of lines in r faces is nr. But since while counting the line in each face, each line is connected twice, hence

$$nr = 2q$$

$$\Rightarrow r = \frac{2q}{n}$$

$$p - q + r = 2 \text{ gives}$$

$$p - q + \frac{2q}{n} = 2$$

$$\Rightarrow pn - qn + 2q = 2n$$

$$\Rightarrow (p-2)n = (n-2)q$$

$$\Rightarrow q = \frac{n(p-2)}{n-2}.$$

Maximal planar graph : Maximal planar graph is one in which no line can be added without losing planarity.

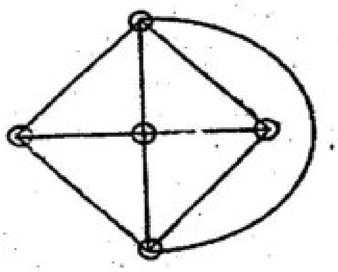


Fig. : 3.6.2

In any embedding of a maximal planar graph G having order $p \geq 3$, the boundary of every region of G is a triangle. For this reason maximal planar graphs are also referred to as triangulated planar graphs.

Corollary 3.6.1. (b). : If G is a (p,q) maximal plane graph with $p \geq 3$, then $q = 3p-6$.

Proof : Let r be the number of region of G, note that the boundary of every face of G is a tringle and that each edge lies on the boundary of two such faces. If we sum the number of edeges on the boundary of a face over all faces, we obtain 3r. We note that the sum also counts each edge twice.

$$\therefore 3r = 2q.$$

Applying Euler's formula.

$$p - q + \frac{2q}{3} = 2$$

$$\Rightarrow 3p - 3q + 2q = 6$$

$$\Rightarrow q = 3p - 6.$$

Corollary 3.6.1. (c) : If G is any planar (p,q) graph with $p \geq 3$, then $q = 3p \leq 6$.

Proof : Let us add edge to G until it is maximal planar. Let G be such maximal planar graph obtain from G. Now let G' be a (p', q') graph, wher $p' = p$ and $q' \geq q$.

$$\therefore q' = 3p' - 6$$

$$\Rightarrow 3p - 6 = q' \geq q$$

$$\Rightarrow 3p - 6 \geq q.$$

$$\therefore \quad q \leq 3p - 6.$$

Corollary 3.6.1. (d) : If G is plane graph in which every face is a 4-cycle then $q = 2p - 4$.

Proof : Let r be the number of regions of G. Here the boundary of every face of G is a 4-cycle and that each edge lies on the boundary of two such faces. If we sum the number of edges on the boundary of a face over all faces, we obtain 4r. We note that the sum also counts each edge twice

$$\therefore 4r = 2q.$$

$$\Rightarrow r = \frac{q}{n}.$$

Now

$$p - q + \frac{q}{2} = 2$$

$$\Rightarrow 2p - 2q + q = 4$$

$$\Rightarrow q = 2p - 4.$$

Corollary 3.6.1. (e) : The graph k_5 and $k_{3,3}$ are non-planar.

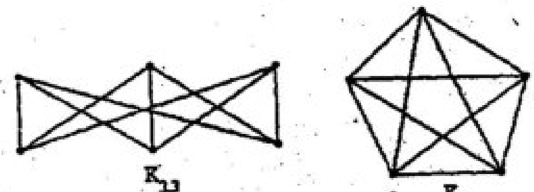


Fig. 10.5.

Proof : Suppose if possible $k_{3,3}$ is planar. Let g be a plane embedding of $k_{3,3}$. Since $k_{3,3}$ has no triangle, (in fact no odd cycle) every region of G must contain at least 4 edges. Then $q \leq 2p - 4$.

Here in $k_{3,3}$ $q = 9$, $p = 6$

$\therefore 2p - 4 = 8$

So, $q \leq 2p - 4$

which is a contradiction. So $k_{3,3}$ is not planar

Again suppose if possible k_5 is planar. Let G be a plane embedding of k_5.

Then $q \leq 3p - 6.$

Here $q = 10$, $p = 5.$

$\therefore$ $3p - 6 = 9$

$\therefore$ $q \leq 3p - 6$

which is a contradiction

$\therefore$ k_5 is not planar.

3.7. Kuratowski's Theorem :

A graph is planar if and only if it has no subgraph homeomorphic to k_5 or $k_{3,3}$.

Proof : Since k_5 and $k_{3,3}$ are non-planar by corollary 3.6.1.(e), it follows that if a graph contains a subgraph homeomorphic to either of these, it is also non-planar.

For the converse, Assume that there is a non-planar graph with no subgraph homeomorphic to either k_5 or $k_{3,3}$. Let G be any such graph having the minimum number of lines. Then G must be a block with $\delta(G) \geq 3$. Let $x_0 = u_0v_0$ be an arlitrary line of G. The graph $F = G - x_0$ is necessarily planar.

[Lemma 3.7.(a) There is a cycle in F containing u_0 and v_0.

Lemma 3.7.(b)– There exists a u_0–v_0 separating outer pices meeting $Z(u_0, v_0)$, say at u_1, and $Z(v_0,u_0)$, say at v_1, such that there is an inner pices which is both u_0–v_0 separating and u_1– v_1 separating.]

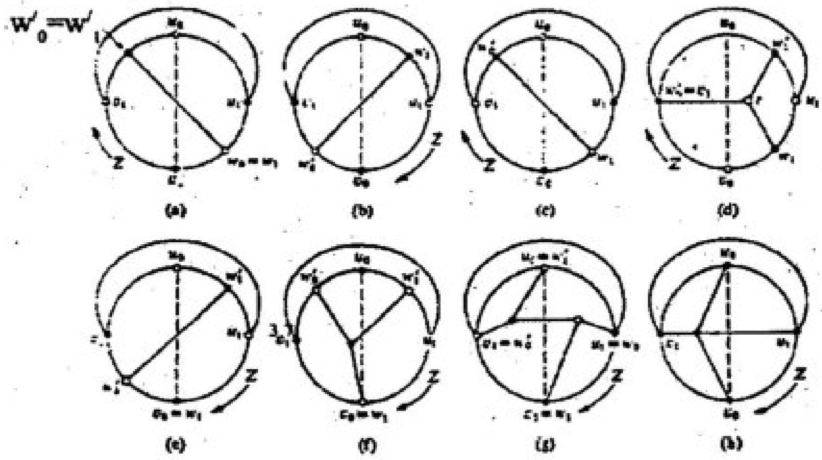


Fig. The possibilities for nonplanar subgraphs.

Let H be the inner piece guaranteed by Lemma 3.7.(b) which is both u_0–v_0 separating and u_1– v_1 separating. In addition, let $w_0, w_0', w_1,$ and w_1' be vertices at which H meets $Z(u_0, v_0)$, $Z(v_0, u_0)$, $Z(u_1, v_1)$ and $Z(v_1, u_1)$, respectively. There are now four cases to consider, depending on the relative position on Z of these four vertices.

Case 1. One of the vertices w_1 and w_1' is on $Z(u_0, v_0)$ and the other is on $Z(v_0, u_0)$. We can then take, say, $w_0 = w_1$ and $w_0' = w_1'$ in which case G contains a subgraph homeomorphic to $k_{3,3}$, as indicated in Fig 3.7.(a), in which the two sets of vertices are indicated by open and closed dots.

Case 2. Both vertices w_1 and w_1' are on either $Z(u_0, v_0)$ or $Z(v_0, u_0)$. Without loss of generality we assume the first situation. There are two possibilities : either $v_1 \neq w_0'$ or

$v_1 = w'_0$. If $v_1 \neq w'_0$, then G contains a subgraph homeomorphic to $k_{3,3}$, as shown in Fig.3.7.(b) or (c), depending on whether w'_0 lies on $Z(u_1, v_1)$ or $Z(v_1, u_1)$, respectively. If $v_1 = w'_0$ (see Fig.3.7.(d), then H contains a vertex r from which there exist disjoint paths to w_1, w'_1 and v_1, all of whose vertices (except w_1, w'_1 and v_1) belong to H. In this case also, G contains a subgraph homeomorphic to $k_{3,3}$.

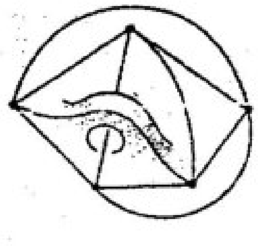
Case 3. $w_1 = v_0$ and $w'_1 = u_0$. Without loss of generality, let w'_1 be on $Z(u_0, v_0)$. Once again G contains a subgraph homeomorphic to $k_{3,3}$. If w'_0 is on (v_0, v_0), then G has a subgraph $k_{3,3}$ as shown in Fig. 3.7.(e). If, on the other hand, w'_0 is on $Z(v_1, u_0)$, there is a $K_{3,3}$ as indicated in Fig.3.7.(f). This figure is easily modified to show G contains $K_{3,3}$ if $w'_0 = v_1$.

Case 4. $w_1 = v_0$ and $w'_1 \neq u_0$. Here we assume $w_0 = u_1$ and $w'_0 \neq v_1$, for otherwise we are in a situation covered by one of the first 3 cases. We distinguish between two subcases. Let P_0 be a shortest path in H from u_0 to v_0, and let P_1 be such a path from u_1 to v_1. The path P_0 and P_1 must intersect. If P_0 and P_1 have more than one vertex in common, then G contains a subgraph homeomorphic to $K_{3,3}$, as shown in Fig.3.7.(g); otherwise, G contains a subgraph homeomorphic to K_5 as in Fig.3.7.(h).

Since these are all the possible cases, the theorem has been proved.

3.8. Genus, Thickness, Coarseness, Crossing Number :

In this section four topological invariants of a graph G are considered. These are genus : the number of handles needed on a sphere in order to embed G, thickness : the number of planar graphs required to from G, coarseness : the maximum number of line-disjoint nonplanar subgraphs in G, and crossing number : the number of crossings there must be when G is drawn in the plane. We will concentrate on three classes of graphs— complete graphs, complete bigraphs, and cubes— and indicate the values of these invariants for them as far as they are known.



K_5

Fig. 3.8.1. Embedding a graph on an orientable surface.

As observed by Koing, every graph is embeddible on some orientable surface. This can easily be seen by drawing an arbitrary graph G in the plane, possible with edges that cross each other, and then attaching a handle to the plane at each crossing and allowing one edge to go over the handle and the other under it. For example, Fig. 3.8.1. shows an embedding of K_5 in a plane to which one handle has been attached. Of course, this method often uses more handles than are actually required. In fact, Konig also showed that any embedding of a graph on an orientable surface with a minimum number of handles has all its faces simply connected.

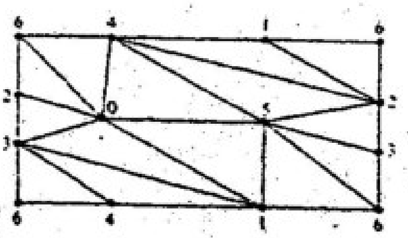


Fig. 3.8.2-An embedding of K on the torous

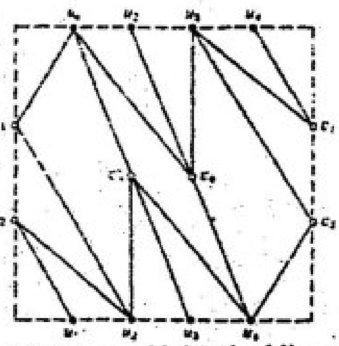


Fig. 3.8.3 A toroidal embedding of $K_{4,4}$

We have already noted that planar graphs can be embedded on a sphere. A toroidal graph can be embedded on a torus. Both K_5 and $K_{3,3}$ are toroidal: in fact Figs. 3.8.2. and 3.8.3. show embeddings of K_7 and $K_{4,4}$ on the torus, represented as the familiar rectangle in which both pairs of opposite sides are identified.

The genus $\gamma(G)$ of a graph G is the minimum number of handles which must be added to a sphere so that G can be embedded on the resulting surface. Of course, $\gamma(G) = 0$ if and only if G is planar, and homeomorphic graphs have the same genus.

The Euler characteristic equation, $V - E + F = 2$, for spherical polyhedra. More generally, the genus of a polyhedrom* is the number of handles needed on the sphere for a surface to contain the polyhedron. Theorem 3.6.1. has been generalized to polyhedra of arbitrary genus, in a result also due to Euler.

Ex.1. Show that if a plane graph of order n and size m has f region and k components, then $n - m + f = k + 1$.

Proof. Suppose the components are G_i with ni vertices and mi edges and f_i faces $(i = 1,2,....,k)$. Then for each i, $n_i - m_i + f_i = 2$, $i = 1,2,....,k$. The extension region is the same for all components. If the extension region is not considered, then $n_i - m_i + f_i = 1$, for each i.

On summation, we get

$$\sum_{i=1}^{k}\left(n_i - m_i + f_i\right) = \sum_{k \text{ summands}} 1$$

$$\Rightarrow \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} f_i = k$$

$$\Rightarrow n - m + f = k$$

So, with the inclusion of the common extension region, we obtain the relation $n - m + f = k + 1$.

Theorem 3.8.1. (Euler's Formula, Generalized) :

Let G be a graph with genus γ, and let n,m and f denote, respectively, the numbers of vertices, edges, and faces in an embedding of G on a surface of genus γ. Then

$n - m + f = 2 - 2\gamma$

Note : $g = 0$ for planar graph, and then the above theorem reduces to the original Eulir's formula. This theorem gives us a lower bound for the genus of a graph.

Corollary. 3.8.1.(a) : Let G be a simple graph with n vertices and m edges. Then the genus $\gamma(G)$ of the graph G satisfies.

$$\gamma(G) \geq \left[\frac{1}{6}(m - 3n) + 1\right],$$

where [x] denotes the smallest inteqer greater than or equal to x.

We can conclude from this corollary that the genus $\gamma(K_n)$ of the complete graph K_n satisfies the inequality

$$\gamma(K_n) \geq \left[\frac{1}{6}\left(\frac{1}{2}n(n-1) - 3n\right) + 1\right]$$

$$= \left[\frac{1}{12}\left(n^2 - 7n + 12\right)\right] = \left[\frac{1}{12}(n-3)(n-4)\right] \quad \text{e.g. } \gamma(K_8) \geq \left[\frac{20}{12}\right] = 2$$

Hence K_8 cannot be embedded on a torus.

Ex.2. The vertices of a certain graph G have degrees 3, 4, 4, 4, 5, 6, 6. Prove that G is non-planar.

Solution : $p = 7,\ q = \frac{1}{2}\sum \deg v = 16$

$\therefore 3p - 6 = 15 < 16 = q$

$\Rightarrow q > 3p - 6$

For a planar graph, $3p - 6 \geq q$.

$\therefore$ G is non-planar.

Ex.3. Is there any non-planar graph of order 4.

Solution : No such graph exists, because, if it so, then it must have a subgraph homeomorphic to $K_{3,3}$ or K_5, which is impossible. Both thickness and coarseness involve constructions which factor a graph into spanning subgraphs (planar and non-planar respectively). $K_{3,3}$ or a homeomorph theroof is a most convenient subgraph for coarseness construction. Figure 3.8.4. shows four line-disjoint homeomorph of $K_{3,3}$ contained in K_{10}

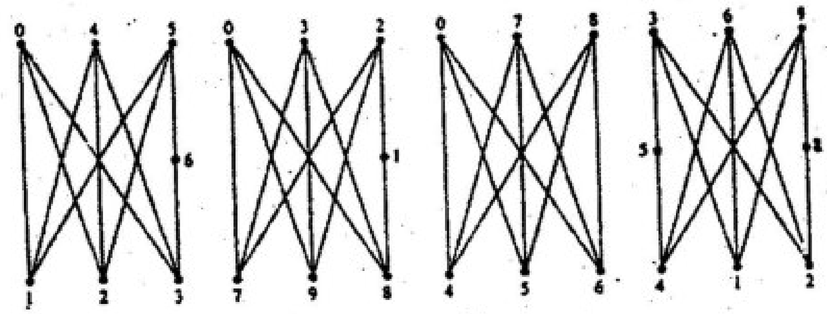
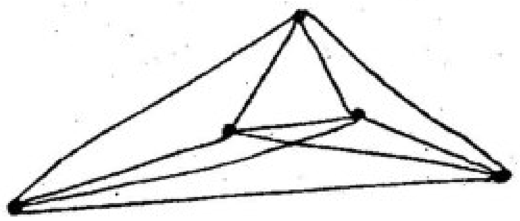


Fig. 3.8.4 Four nonplanar subgraphs of K_{10}

The crossing number $\gamma(G)$ of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Obviously, $\gamma(G) = 0$ iff G is planar.

e.g.



A drawing of K_5 with one crossing.

Ex.4. Show that the crossing number of $K_{2,2,3}$ is $\gamma(K_{2,2,3}) = 2$.

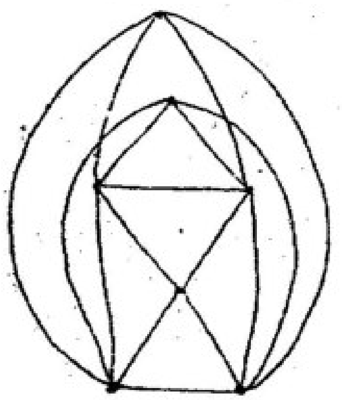


Fig. : 3.8.5

Let $\gamma(K_{2,2,3}) = C$

Since $K_{3,3}$ is non-planar and $K_{3,3} \subset K_{2,2,3}$ it follows that $K_{2,2,3}$ is non-planar so that $C \geq 1$. Let there be given a drawing of $K_{2,2,3}$ in the plane with C crossing. At each crossing we introduce a new vertex, producing a connected plane graph of G of order $p = 7 + C$ and size $q = 16 + 2C$. Since G is planar, so $q \leq 3p - 6$.

Let u_1u_2 and v_1v_2 be two non-adjacent edges of $K_{2,2,3}$ that cross the given drawing giving rise to a new vertex. If G is a triangulation, then $C_1 : u_1v_1u_2v_2u_1$ is a cycle of G implying that the induced subgraph $\langle\{u_1, u_2, v_1, v_2\}\rangle$ in $K_{2,2,3}$ is isomorphic to K_4. However, $K_{2,2,3}$ contains no such subgraph. Thus G is not a triangulation so that $q \leq 3p - 6$.

$\Rightarrow 16 + 2c < 3(7 + c) - 6 = 15 + 36$

$\Rightarrow C > 1$

$\Rightarrow C \geq 2$

The inequality $C \geq 2$ follows from the fact that there exists a drawing of $K_{2,2,3}$ with two crossing. Hence $C = 2 = \gamma(K_{2,2,3})$.

3.9. Exercise :

1. If a (p_1, q_1) graph and a (p_2, q_2) graph are homeomorphic then $p_1 + q_2 = p_2 + q_1$.
2. Find the genus and crossing number of the peterson graph.
3. Prove or disprove : A non planar graph G has $\gamma = 1$, iff $G - x$ is planar for some line x.

4. If G is outplanar with triangles, then

$$q \leq (3p-4)/2.$$

5. Prove or disprove : A graph is planar iff every subgraph with atmost six points of degree at least 3 is homeomorphic to a subgraph of $K_2 + P_4$.

3.10. References :

1. Graph Theory, F. Harary.
2. Basic Graph Theory, K.R. Parthasarathy.
3. Graphs— An Introductory Approach,
   Robin J. Wilson and Jhon J. Watkins.

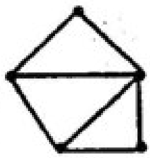
●●●

UNIT-4

4.1 Introduction : Algebraic graph theory is a branch of mathematics in which algebraic methods are applied to problems about graphs. In this unit we begin by defining a matrix which will play an important role in algebraic graph theory. Then we will study the speclriem of the adjacency matrix of a graph. This matrix completely determines the graph and its spacial properties. Also we shall apply spectral techniques to the vertex coloring problem, using inequalities involving the eigenvalues of a graph.

4.2. Adjacency matrix

4.2.1 Definition : The adjacency matrix $A = [A_{ij}]$ of a labeled graph G with p points is the $p \times p$ matrix in which $a_{ij} = 1$ if v_i is adjacent with v_j and $a_{ij} = 0$ atheririse. Thuse there is a one-one correspondence between labeled graphs with P points and $p \times p$ symmetric binary matrics with zero diagonal.

G : 

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Suppose λ is an eigenvalue of a matrix A. If A is real and symmetric, it follows that λ is real and the multiplicity of λ as a root of the equation det $(\lambda I - A) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ.

4.2.2 Definition : The spectecum of a graph G is the set of numbers which are eigenvalues of A(G) together with their multiplicities. If the distinct eigenvalues of A(G) are $\lambda_0 > \lambda_1 > .... > \lambda_{s-1}$ and their multiplicities are $m(\lambda_0)\ m(\lambda_1) .... m(\lambda_{s-1})$, then we shall write

$$\text{Spec } G = \begin{pmatrix} \lambda_0 & \lambda_1 & .... & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & .... & m(\lambda_{s-1}) \end{pmatrix}$$

For example, the complete graph K_p is the graph with P points in which each distinct pair are adjacent. Thus the graph K_4 has adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and an easy calculation shows the spectrum of k_4

$$\text{Spec } K_4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

4.2.3 Theorem : We shall usually refer to the eigenvalues of $A = A(G)$ as the eigenvalues of G Also, the characteristic. Polynomial det $(\lambda I - A)$ will be refeured to as the characteristic polynomial of G and denoted by $X(G;\lambda)$. Let us suppose tha the characteristic polynomial of G is

$$X(G;\lambda) = \lambda^n + C_1\lambda^{n-1} + C_2\lambda^{n-2} + ...+ C_n.$$

In this form we know that— C_1 is the sum of zeros, that is, the sum of the eigenvalues. This is also the trace of A which, as we have already noted, is zero. Thus $C_1 = 0$. More generally, it is proved in the theory of matrices that all the co-efficients can be expressed in terms of the principal minors of A, where a principal minor is the determinant of a submatrix obtained by taking a subset of the rows and the same subset of the columns. This lead to the following simple result.

4.2.3. Theorem : The co-efficeints of the characteristic polynomial of a graph G satisfy.

1. $C_1 = 0$
2. $-C_2$ is the number of edges of G.
3. $-C_3$ is twice the number of triangles in G.

Proof : For each $i \in \{1,2,...,n\}$ the number $(-1)^i C_i$ is the sum of those principal minors of A which have i rows and columns. So we can argue as follows—

1. Since the diagonal elements of A are all zero, $C_1 = 0$.

2. A principal minor with two rows and columns and which has a non-zero entry, must be of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

There is one such minor for each pair of adjacent vertices of G, and each has value -1. Hence $(-1)^2 C_2 = -|E(G)|$, giving the result.

3. There are essentially three possibilities for non-trivial principal minors with three rows and columns :

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix},$$

and of these, the only non-zero one is the last (whose value is 2). This principal minor corresponds to three mutually adjacent vertices in G, and so have the required description of C_3.

These simple results indicate that the characteristic polynomial of a graph is an object of the kind we study in algebraic graph theory; it is an algebraic construction which contains graphical information.

Suppose A is the adjacency matrix of a graph G. Then the set of polynomials in A, with complex co-efficeints forms an algebra under the esual matrix operations.

4.3 Definition : The adjacency algebra of a graph G is the algebra of polynomials in the adjjacency matrix A = A(G). We shall denote the adjacency algebra of G by A(G).

Since every element of the adjacency algebra is a linear combination of powers of A, we can obtain results about A(G) from a study of these powers. We define a walk of length l in G, from v_i to v_j, to be a finite sequence of vertices of G,

$v_i = u_o, v_j, \ldots, v_l = u_j$,

such that u_{t-1} and u_t are adjacent for $1 \leq t \leq e$.

4.4 Theorem : The number of walks of length l from Γ,, is the entry in position (i,j) of the matrix v_i to v_j.

Proof : The result is true for $\ell = 0$ (since $A^0 = I$) and for $\ell = 1$ (since $A^1 = A$ is the adjacency matrix). Suppose that the result is true for $l = \Gamma$. The set of walks of length L+1 from v_i to v_j is in bijective correspondence with the set of walks of length L from v_i to verticis v_h adjacenct to v_j. Thus the number of such walks is

$$\sum\left(A^L\right)_{ih} = \sum_{h=1}^{n}\left(A^L\right)_{ih} a_{hj} = \left(A^{L+1}\right)_{ij}$$

It follows that the number of walks of length L+1 joining v_i to v_j is $\left(A^{L+1}\right)_{ij}$. The general result follows by induction.

4.5 Some Additional Results :

1. A reduction formula for X :

Suppose G is a graph with vertex v_1 of degree 1 and let v_2 be the vertix adjacent to v_1.

Let G_1 be the induced subgraph obtained removing v_1 and G_{12} be the induced subgraph obtained by removing $\{v_1, v_2\}$

Then $x(G;\lambda) = -\lambda X(G_1;\lambda) - x(G_{12},\lambda)$

This formula can be used to calculate the characteristic polynomial of any true, because a tree always has a vertex of degree 1.

2. The characteristic polynomial of a path :

Let Pn be the path graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edges $\{v_i, v_{i+1}\}$ $(1 \leq i \leq n-1)$.

For n ≥ 3, we have

$$x(P_n:\lambda) = \lambda X(P_{n-1};\lambda) - x(P_{n-2}:\lambda)$$

3. The spectrum of a bipartite graph :

A graph is bipartite if its vertex set can be partitioned into two parts v_1 and v_2 such that each edge has one vertex in v_1 and one vertex in v_2. If we order the verticis so that those in v_1 come first, then the adjacency matrix of a bipartite graph takes the form.

$$A = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$$

If x is an eigenvector corresponding to the eigenvalue λ and c is obtained from x by changing the signs of the entries corresponding to vertices in v_2, then x^2 is an eigenvector corresponding to the eigenvalue $-\lambda$. It follows that the spectrum of a bipartite graph is symmetric with respect to the origin.

4. The derivative of X :

For i = 1,2,...,n, let G_i denote the induced subgraph $\langle V(G)/V_i \rangle$. Then

$$x'(G;\lambda) = \sum_{i=1}^{n} x(G_i;\lambda)$$

4.6. Vertex partition and the spectrum :

A color partition of a general graph G is a partition V(G) into subsets called color classes,

$$V(G) = V_1 \cup V_2 \cup .... \cup V_l$$

such that each $V_i(1 \leq i \leq l)$ contains no pair of adjacent vertices.

In other words, the induced subgraph $<V_i>$ have no edges. The chrematic number of G, written V(G), is the least natural number l for which such a partition is possible.

We define a vertex-coloring of G to be the assignment of colors to the vertices, with the property that adjacent vertices have different colors, so clearly a vertex-coloring in which ℓ colors are used gives rise to a color-partition with ℓ color-classes.

If V(G) = 1, then G has no edges.

If V(G) = 2, then G is a bipartite graph.

4.6.1. Theorem : Suppose the bipartite graph G has an eigenvalue λ of multiplicity $m(\lambda)$. Then $-\lambda$ is also an eigenvalue of G, and $m(-\lambda) = m(\lambda)$.

Proof : If G is a bipartite graph, then G has no odd cycles and consequently no elementary subgraph with an odd number of vertices. It follows that the characteristic polynomial of G has the form

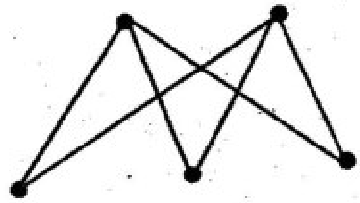
$$X(G;Z) = Z^n + C_2Z^{n-2} + C_4Z^{n-4} + \ldots = Z^\delta P(Z^2)$$

[$\therefore$ there is no odd cycles, so $C_1 = C_3 = C_5 = \ldots = 0$]

where $\delta = 0$ and P is a polynomial function. Thus the eigenvalues, which are the zeros of x, have the required property.

Example : Find the spectrum of $K_{2,3}$.

$$A(G) : \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$



Now the characteristic polynomial is

$$X(K_{2,3}) = Z_5 + C_2Z^{5-2} = 0$$

$$\therefore Z^3 (Z^2+C_2) = 0$$

$$\Rightarrow Z = 0,0,0 \text{ or } Z^2 + C_2 = 0$$

$$\Rightarrow Z^2 = -C_2 = 6$$

$$\Rightarrow Z = \pm\sqrt{6}$$

Spectrum of $K_{2,3}$ is $\begin{pmatrix} +\sqrt{6} & 0 & -\sqrt{6} \\ 1 & 3 & 1 \end{pmatrix}$

Note : The spectrum of the complete bipartite graph $K_{a,b}$ can be found in the following manner, We may suppose that the verticies of $K_{a,b}$ are labelled in such a way that its adjacency matrix is

$$A = \begin{pmatrix} 0 & J \\ J^t & 0 \end{pmatrix}$$

where J is the a $\times$ b matrix having all entries + 1. The matrix A has just two linearly endependent rows and so its rank is 2. Consequently 0 is an eigenvalue of A with multiplicity $a + b - 2$. The characteristic polynomial is thus of the form

$$Z^{a+b} + C_2Z^{a+b-2} = Z^{a+b-2}(Z^2 + C_2)$$

The eigenvalues are obtained by showing the above characteristics polynomial

$$Z^{a+b-2}(Z^2 + C_2) = 0$$

$$\Rightarrow Z = \underbrace{0,0,\ldots,0}_{a+b-2} \quad \text{'or'}$$

$$Z^2 = -C_2 = \text{number of edges of } K_{a,b} = ab$$

$\Rightarrow Z = \pm\sqrt{ab}$

So, the eiginvalues are $\sqrt{ab}$, 0, $-\sqrt{ab}$ with multiplicities 1, a + b − 2, 1.

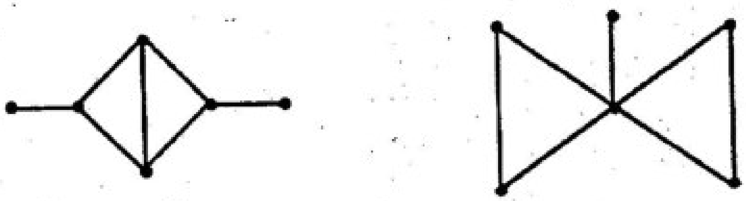
Hence the spectrum of $K_{a,b}$ is

$$\begin{pmatrix} \sqrt{ab} & 0 & -\sqrt{ab} \\ 1 & a+b-2 & 1 \end{pmatrix}$$

This example illecstrate that the spectrun of a bipartite graph is symmelrical with respect to the origin.

4.7. Cospectral fraphs :

Two non-isomerphic graphs are said to be cospectral if they have the same eigenvalues with the same multiplicitis.



Two cosectral graphs

Example : The graphs $K_{1,4}$ and $K_1 \cup C_4$ are cospectral graphs. Since the trace of a squarematrix is equal to the sum of its eigenvalues and the eigenvalues of A^r are the rth poor of the eigenvalues of A, we see that trace of A^r is determined by the spectrum of A.

The spectrum of a disconnected graph is the union of the spectra of its components.

Almost every tree has a cospectral mate .

4.8 Exercise :

1. Find the adjacency matrix of K_5.
2. Find the spectrum of K_3 and $K_{1,4}$.

4.9. References :

1. Graph Teory: F. Harary.
2. Algebraic Graph Theory: N. Biggs.

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