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M.A./M.Sc. in Mathematics

Semester 4

Paper III

Functional Analysis II

Contents:

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Unit 1
Topological Vector Spaces

1.1 Definition :

Let X be a set. A topology on X is a collection T of subsets of X such that

1. both X and ϕ belong to T.

2. for every subcollection of T, the union of the elements of the subcollection also belong to T.

3. for every finite subcollection of T, the intersection of the elements of the subcollection also belongs to T.

The set X with the topology T is called the topological space (X, T).

Directed Set :

A directed set is a non-empty set I with a relation $\leq$ such that

(1) $\alpha \leq \alpha$ whenever $\alpha \in I$.

(2) if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$, $\forall\, \alpha, \beta, \gamma \in I$

(3) for each pair α, β of elements of I, there is a $\gamma_{\alpha,\beta}$, in I such that

$\alpha \leq \gamma_{\alpha,\beta}$ and $\beta \leq \gamma_{\alpha,\beta}$

That is, a directed set is a non-empty preordered set that satisfies (3)

Net :

A net or Moore-Smith sequence in a set X is a function from a directed set I into X. The set I is the index set for the net.

If $f : I \to X$ is a net, then for each α in I. the α^{th} term $f(\alpha)$ of the net is often denoted by x_α, and the entire net is often denoted by $(x_\alpha)_{\alpha \in I}$ or just (x_α). By analogy with sequences, it is said that x_α precedes x_β in a net when,

$\alpha \leq \beta$

Examples :

1. $(N, \leq)$ is directed set, so every sequence in N is a net.

2. $(R, \leq)$ is a directed set. So a function $f : R \to X$ is a net.

Difference between sequence and nets :

1. A sequence has a first term which cannot be preceded by any other terms.

2. But in a net, there is no first term, that is a term can be preceded by infinitely many terms :

Example :

$(R^2, \leq)$ is a directed set defined by

$(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ henever $\alpha_1 \leq \alpha_2$

Then $(x_{(\alpha,\beta)})$ is a net in R^2, where $x_{(\alpha,\beta)} = \alpha + \beta$

$\forall\ (\alpha, \beta) \in R^2$.

This net has no first term.

If possible let $x_{(\alpha_0, \beta)} = \alpha_0 + \beta$ be the first term.

If $\alpha < \alpha_0$, $(\alpha, \beta) < (\alpha_0, \beta)$ and

$x_{(\alpha, \beta)} < x_{(\alpha_0, \beta)}$

i.e. $\alpha + \beta < \alpha_0 + \beta$

$\therefore$ There exists infinitely many α s.f $\alpha < \alpha_0$.

$\Rightarrow$ Infinitely many $x_{(\alpha, \beta)}$ preceds $x_{(\alpha_0, \beta)}$

$\Rightarrow x_{(\alpha_0, \beta)}$ is not the first term.

Note :

In a partially ordered set $\alpha \leq \beta, \beta \leq \alpha$

$\Rightarrow \alpha = \beta$

But in a directed set it does not happen.

In $(R^2, \leq)$, $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ if $\alpha_1 \leq \alpha_2$, $\forall (\alpha_i, \beta_i) \in R^2$

$i = 1, 2$

Then $(1, 2) \leq (1, 3)$ and $(1, 3) \leq (1, 2)$

But $(1, 2) \neq (1, 3)$.

Finite Net :

Let I be a three-element set {u, v, w}. Define $\leq$ on I by letting these be all of the corresponding relations : $\alpha \leq \alpha$ for each α in I; $u \leq w$ and $v \leq w$.

Then $(I, \leq)$ is a directed set. Define a net (x_α) in R with index set I by letting

$x_u = 0$, $x_v = \pi$ and $x_w = -3$.

(a) The index set {u, v, w} is finite.

(b) Nets can have last term.

$u \leq w \Rightarrow x_u = 0$ precedes $x_w = -3$.

$v \leq w \Rightarrow x_v = \pi$ precedes $x_w = -3$.

Thus −3 is the last term.

(c) Nets can have more than one first term.

Here 0 and π are first terms.

(d) The index set for a net need not be a chain.

An important Net :

Let (X, T) be a topological space and that $x \in X$. Let I be the collection of all nbhds of x with the relation $\leq$ given by declaring that $U \leq V$ when $U \supseteq V$.

Then I is a directed set. If $x_u \in U$ for each U in I, then (x_u) is a net in X.

Convergent Net :

Let (x_α) $\alpha \in I$ be a net in a topological space (X, T) and let x be an element of X. Then (x_α) converges to x, and x is called a limit of (x_α), if for each nbhd U of x, there is an α_u in I such that $x_\alpha \in U$ whenever $\alpha_u \leq \alpha$.

This convergence is denoted by $x_\alpha \to x$ or

$$\lim_{\alpha} x_\alpha = x$$

Example 1 :

The net (x_u, x_v, x_w) converges to −3 when I = {u, v, w}, $u \leq u$, $v \leq v$, $w \leq w$, $u \leq w$, $v \leq w$. s.t

$x_u = 0$, $x_v = \pi$, $x_w = -3$.

as $x_\alpha \in (-3 - \varepsilon, -3 + \varepsilon)$ $\forall$ $w \leq \alpha$.

Example 2 :

The net $\{x_u : U \text{ is a nbhd of } x\}$. $U \leq V$ if $U \supseteq V$ and $x_u \in U$. We can show that $x_u \to x$.

Let W be a nbhd of x, so $x_w \in W$

Now, $W \leq U \Rightarrow U \subseteq W \Rightarrow x_u \in U \subseteq W$

$$\Rightarrow x_u \in W$$

$\therefore$ $x_u \in W$, $\forall$ $W \leq U$

$\Rightarrow x_u \rightarrow x.$

1.2 Subbasis :

Let X be a set and let $\mathfrak{S}$ be a collection of subsets of X. Let $B_{\mathfrak{S}}$ be the collection of all sets that are intersections of finitely many members of $\mathfrak{S}$. Then the topology generated by the subbasis $\mathfrak{S}$ is the topology generated by the basis $B_{\mathfrak{S}}$.

Proposition :

Suppose that $\mathfrak{S}$ is a subbasis for the topology of a topological space X, that $(x_\alpha)_{\alpha\in I}$ is a net in X, and that $x \in X$. Then $x_\alpha \rightarrow x$ if and only if the following is true :

For every members U of $\mathfrak{S}$ that contains x, there is an α_U in I such that $x_\alpha \in U$, whenever

$$\alpha_U \leq \alpha.$$

Proof :

First suppose that

$x_\alpha \rightarrow x$, and $U \in \mathfrak{S}$ and $x \in U$.

So U is a nbhd of x, and hence $\exists\ \alpha_u \in I$ such that $x_\alpha \in U$ whenever $\alpha_u \leq \alpha$.

Conversely let, for every members U of $\mathfrak{S}$ that contains x, there is an α_u in I such that $x_\alpha \in U$ whenever $\alpha_u \leq \alpha$.

Let $\mathfrak{F} = \{U_1, U_2, ..., U_k\}$ be a finite subcollection of $\mathfrak{S}$.

$\therefore x_\alpha \in U_i$ when $\alpha_{U_i} \leq \alpha$ $(1 \leq i \leq k)$.

Let $\alpha_{\mathfrak{F}}$ = U.B of $(\alpha_{U_1}, \alpha_{U_2}, .., \alpha_{U_K})$

$\therefore \alpha_{\mathfrak{F}} \leq \alpha \Rightarrow \alpha_{U_i} \leq \alpha$ $(1 \leq i \leq k)$

$\Rightarrow x_\alpha \in U_i$ $(1 \leq i \leq k)$

$$\Rightarrow x_\alpha \in \leq \bigcap_{i=1}^{k} U_i$$

Let U be a nbhd of $x \Rightarrow \exists\ \mathfrak{F} = \{U_1, U_2, ..., U_k\}$

$$\text{s.f } \bigcap_{i=1}^{k} U_i \subseteq U$$

$\Rightarrow x_\alpha \in U$ if $\alpha_{\mathfrak{F}} \leq \alpha$

$\Rightarrow x_\alpha \rightarrow x.$

1.3 Product topology :

Let $\{X_\alpha : \alpha \in I\}$ be a family of topological spaces.

Let $\mathfrak{S}$ be the collection of all subsets of the cartesian product $\prod_{\alpha \in I} X_\alpha$ of the form $\prod_{\alpha \in I} U_\alpha$, where each U_α is open and at most one U_α is not equal to the corresponding X_α. Then the product topology of $\prod_{\alpha \in I} X_\alpha$ is the topology generated by the subbasis $\mathfrak{S}$.

Proposition :

Let $\{X^{(\alpha)} : \alpha \in I\}$ be a family of topological spaces and let X be their topological product. Suppose that $(X_\beta)_{\beta \in J}$ is a net in X, and x is a member of X. Then $x_\beta \to x$ if and only if

$x_\beta^{(\alpha)} \to x^{(\alpha)}$ for each α in I.

Proof :

Let $\mathfrak{S}$ be the usual subbasis for the topology of X, that is, the collection of all subsets of X of the form $\prod_{\alpha \in I} U^{(\alpha)}$ such that each $U^{(\alpha)}$ is open and at most one is not equal to the corresponding $X^{(\alpha)}$.

Then $x_\beta \to x$ in $\prod_{\alpha \in I} U^{(\alpha)}$

$\Leftrightarrow$ For every member U of $\mathfrak{S}$ that contains x, there is a β_U in J such that

$x_\beta \in U$ whenever $\beta_U \leq \beta$

$\Leftrightarrow x_\beta \in \prod_{\alpha \in I} U^{(\alpha)}$

$\Leftrightarrow x_\beta^{(\alpha)} \in U^{(\alpha)}$

$\Leftrightarrow x_\beta^{(\alpha)} \to x^{(\alpha)}$ for each $\alpha \in I$

Theorem :

A topological space X is a Housdorff space if and only if each convergent net in X has only one limit.

Proof :

Let (X, T) be a Hausdorff space. We have to show that every convergent net has a unique limit.

If possible suppose a net (x_α) converges to two different limits x and y.

Now, (x_α) converges to x $\Rightarrow$ For each nbhd U of x, there is an α_u in I such that

$x_\alpha \in U$ whenever $\alpha_u \leq \alpha$.

(x_α) converges to y $\Rightarrow$ For each nbhd V of y, there is an α_V in I such that

$x_\alpha \in V$ whenever $\alpha_v \leq \alpha$.

Then $\exists$ an $\alpha_{U\cap V}$ in I s.t $\alpha_{U\cap V} \geq \alpha_U$ and α_V.

Then $x_\alpha \in U$ and $x_\alpha \in V$ if $\alpha = \alpha_{U\cap V} \geq \alpha_U$ and α_V.

$\therefore U\cap V \neq \phi$, which is acontradiction, as (X, T) is a Hausdorff space.

So limit of a convergent net in a Hausdorff space is unique.

Conversely, suppose that every convergent net in (X, T) has a unique limit. We have to show that (X, T) is Hausdorff.

If possible suppose (X, T) is not Hausdorff. Let x and y be distinct elements of X that cannot be separated by open sets. If U_1 and U_2 are nbhds of x and V_1 and V_2 are nbhds of y such that

$U_1 \supseteq U_2$ and $V_1 \supseteq V_2$.

Define, $(U_1, V_1) \leq (U_2, V_2)$ if $U_1 \supseteq U_2$, $V_1 \supseteq V_2$.

For each nbhd U of x and each nbhd nbhd V of y let $x_{(U,V)}$ be an element of $U\cap V$. Then the net $(x_{(U,V)})$ converges to both x and y.

This is a contradiction to the fact that every convergent net in (X, T) has a unique limit.

Hence, (X, T) is a Hausdorff space.

Remember :

In a metric space (x, d), $x \in \bar{A} \Leftrightarrow \exists$ a sequence $(x_n) \subseteq A$ s.t $x_n \to x$. for $A \subseteq X$.

Propsition :

Let S be a subset of a topological space X and let x be an element of X. Then $x \in \bar{S} \Leftrightarrow$ some net in S converges to x.

Proof :

Let (x_α) be a net in S s.t $x_\alpha \to x \in S$.

To show that $x \in \bar{S}$.

For this we have to show that every nbhd of x intersects S.

Since (x_α) converges to x, so for a nbhd U_x of x.

$\exists\ \alpha_0 \in I$ s.t $x_\alpha \in U_x$ whenever $\alpha_0 \leq \alpha$.

In particular, $x_{\alpha_0} \in U_x$ and $x_{\alpha_0} \in S$.

$\Rightarrow S \cap U_x \neq \phi$.

$\therefore x \in \bar{S}$.

Conversely, let $x \in \bar{S}$.. Let I be the collection of all nbhds of x directed by declaring that

$U \leq V$ when $U \supseteq V$.

For each U in I, let x_U be a member of $U \cap S$.

Then (x_U) is a net in S converging to x.

1.4 Definition :

Let S be a subset of a topological space (X, T). An element x in X is called a limit point of S if and only if $x \in \overline{S - \{x\}}$.

Proposition :

Let S be a subset of a topological space X. Then an element x of X is a limit point of S if and only if there is a net in $S - \{x\}$ converging to x.

Proof :

Let S be a subset of a topological space X.

A net (x_α) in $S - \{x\}$ converges to x.

$\Leftrightarrow x \in S - \{x\}$.

$\Leftrightarrow$ Every nbhd U_x of x intersects $S \setminus \{x\}$

i.e. $U_x \cap (S - \{x\} \neq \phi$

$\Leftrightarrow$ x is a limit point of S.

Proposition :

A subset S of a topological space (X, T) is closed if and only if limit of a convergent net in S is in S.

Proof :

Let S be a subset of a topological space (X, T).

Now S is closed $\Leftrightarrow \overline{S} = S$.

Now, $x \in S = \overline{S} \Leftrightarrow \exists$ a net $(x_\alpha) \subseteq S$ s.t

$$x_\alpha \to x.$$

$\therefore$ S is closed $\Leftrightarrow$ limit of a convergent net in S is in S.

Proposition :

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if and only if $x_\alpha \to x_0 \Rightarrow f(X_\alpha) \to f(X_0)$.

Proof :

Let $f : X \to Y$ be continuous at $x_0 \in X$.

Let (x_α) be a net in X converging to x_0.

Let U be an open nbhd of $f(x_0)$. By the continuity of f, $f^{-1}(U)$ is a nbhd of x_0.

Since (x_α) converges to x_0, so $\exists\ \alpha_U \in I$ s.t

$x_\alpha \in f^{-1}(U)$ whenever $\alpha_U \leq \alpha$

$\Rightarrow f(x_\alpha) \in U$ whenever $\alpha_U \leq \alpha$

$\Rightarrow f(x_\alpha) \to f(x_0)$.

Conversely, suppose that $f(x_\alpha) \to f(x_0)$ if $x_\alpha \to x_0$.

To show that f is continuous at $x_0 \in x$.

If possible suppose that f is not continuous at x_0. Let V be a nbhd of $f(x_0)$ such that no nbhd U of x_0 has the property that $f(U) \subseteq V$.

Let I be the collection of all nbhds of x_0 directly by declaring that $U_1 \leq U_2$ when $U_1 \supseteq U_2$.

For each U in I, let x_U be an element of U such that $f(x_U) \notin V$.

Then the net (x_U) converges to x_0, but $(f(x_U))$ does not converges to $f(x_0)$, a contradiction.

Hence f must be continuous at $x_0 \in X$.

Corollary : Let X and Y be topological spaces. A function $f : X \to Y$ is continuous in X.if and only if $f(x_\alpha) \to f(x)$ whenever (x_α) is a net in X converging to x in X.

Proof : Let $f : X \to Y$ be continuous

$\Leftrightarrow$ f is continuous at each $x_0 \in X$.

$\Leftrightarrow f(x_\alpha) \to f(x_0)$, whenever $x_\alpha \to x_0$

since x_0 is an arbitrary point of X, it holds for all $x \in X$.

Subnet : Suppose that X is a set, that I is a directed set, and that $f : I \to X$ is a net. Suppose furthermore that J is a directed set and that $g : J \to I$ is a function such that.

1. $g(\beta_1) \leq g(\beta_2)$ in I whenever $\beta_1 \leq \beta_2$ in J.

2. g(J) is cofinal in I.

Then the net $f \circ g : J \to X$ is called a subnet of f.

Definition : A subset J of a directed set I is cofinal in I if for each α in I there is a β_α in J such that $\alpha \leq \beta_\alpha$.

Proposition : Let (x_α) be a net in a set X.

(a) The net (x_α) is a subnet of itself.

(b) Every subnet of (x_α) is a net in X.

(c) Every subnet of a subnet of (x_α) is a subnet of (x_α).

(d) If X is a topological space and (x_α) converges to an element x of X, then every subnet of (x_α) converges to x.

(e) If X is a topological space and there is an element x of X such that every subnet of (x_α) has a subnet converging to x, then $x_\alpha \to x$.

Proof : (a) Let $I = \{\alpha\}$ be an index set of the net (x_α). where $f(\alpha) = x_\alpha$

Let $J = I$ and $g : J \to I$, $g(\alpha) = \alpha$.

$$g(I) = I.$$

Since I is a directed set, for $\alpha \in I$, $\exists\, \beta \in I$, such that $\alpha \leq \beta$.

$\therefore$ g(I) is cofinal.

So, $(f \circ g)(\alpha) = f(g(\alpha)) = f(\alpha) = x_\alpha$.

$\therefore (x_\alpha)$ is a subnet of itself.

(b) Let $f : I \to X$ be a net and $g : J \to I$ be such that

(i) $g(\beta_1) \leq g(\beta_2)$ if $\beta_1 \leq \beta_2$.

(ii) g(J) is a cofinal in I.

and $f \circ g : J \to X$ is a subnet, J is a directed set.

Now, $\alpha \in J \subseteq I \Rightarrow \alpha \leq \alpha$

$\alpha, \beta, \gamma \in J \Rightarrow \alpha \leq \beta,\ \beta \leq \gamma \Rightarrow \alpha \leq \gamma$

For $\alpha, \beta \in J \Rightarrow \alpha, \beta \in I$

$\Rightarrow \exists\, \gamma \in I$ s.t $\alpha, \beta \leq \gamma$

$\Rightarrow \exists\, \beta_\gamma \in J$ s.t $\gamma \leq \beta_\gamma$

$\therefore \alpha, \beta \leq \beta_\gamma$

So, J is a directed set and hence $f \circ g$ is a net.

(c) Let (x_α) be a subnet $\Rightarrow (x_\alpha)$ is a net.

Now, every subnet of the net (x_α) is a net

But we know that every net is a subnet itself.

Hence, every subnet of a subnet is a subnet.

(d) Let (X, T) be a topological space and the net (x_α) converges to an element x X.

To show every subnet $(x_{g(\beta)})$, of (x_α) converges to x.

Let U be a nbhd of x. Then $\exists\, \alpha_0 \in I$ s.t

$x_{g(\beta)} = x_\alpha \in U$ for all $\alpha \geq \alpha_0$

Now, $\alpha_0 \in I \Rightarrow \beta_{\alpha_0} \in J$ s.t $\alpha_0 \leq \beta_{\alpha_0}$

$\therefore x_{g(\beta)} \in U$ whenever $\beta_{\alpha_0} \leq \alpha \leq \beta_\alpha$, when $g(\beta_{\alpha_0}) \leq g(\beta_\alpha)$.

$\therefore x_{g(\beta)} \to x$.

(e) Suppose X is a topolgoical space and that x is an element of X that is not a limit of (x_α). Then there is a nbhd U of x with this property.

For every α in the index set I for (x_α), there is a β_α in I such that $\alpha \leq \beta_\alpha$ and $x_{\beta_\alpha} \notin U$.

Let $J = \{\beta : \beta \in I, x_\beta \notin U\}$, a cofinal subset of I, and let (x_β) be the restriction of (x_α) to J. Then (x) is a subnet of (x_α) that clearly has not subnet converging to x.

This proves (e).

1.5 Technique of construction of a subnet :

Suppose that (x_α) and (y_β) are nets with respective index set I and J. It is often useful to be able to find subnets (x_γ) and (y_γ) of (x_α) and (y_β) respectively that have the same index set K. To do this, let K = I × J, directed by declarig that (α_1, β_1) (α_2, β_2) when $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Let $g : K \to I$ and $h : K \to J$ be the projection mappings, that is the mapping defined by the formulas $g(\alpha, \beta) = \alpha$ and $h(\alpha, \beta) = \beta$. Then $(x_{g(\alpha,\beta)})$ and $(y_{h(\alpha,\beta)})$ are subnets of (x_α) and (y_β) respectively having same index set.

If (x_α) lies in a topological space, then $(x_{g(\alpha,\beta)})$ converges to some x if and only if (x_α) converges to x and similarly for $(y_{h(\alpha,\beta)})$ and (y_β).

Accumulation point :

Let $(x_\alpha)_{\alpha \in I}$ be a net in a topological space X and let x be an element of X. Then (x_α) accumulates at x, and x is called an accumulation point of (x_α), if for each nbhd U of x and each α in I, there is a $\beta_{\alpha,u}$ in I such that $\alpha \leq \beta_{\alpha,u}$ and $x_{\beta_{\alpha,u}} \in U$.

Proposition :

Suppose that (x_α) is a net in a topological space X and that $x \in X$.

(a) If (x_α) converges to x, then (x_α) accumulates x.

(b) If (x_α) has a subnet that accumulates at x, then (x_α) accumulates at x.

Proof : (a) $x_\alpha \to x$ means if U is a nbhd of x, $\exists\, \alpha_0 \in I$ such that

$x_\alpha \in U$ if $\alpha_0 \leq \alpha$.

If $\alpha \in I$, then $\exists\, \beta_{\alpha,u} \in I$ such that

$\alpha \leq \beta_{\alpha,u}$ and $x_{\beta_{\alpha,u}} \in U$.

$\therefore$ x is accumulation point of (x_α).

(b) Let, (x_β) be a subnet of (x_α) and $(x_{(\beta)})$ accumulates at x.

To show (x_α) accumulates at x.

If U be a nbhd of x and $\beta \in J$. Then $\exists\ \beta_0 \geq \beta$ such that $x_{\beta_0} \in U$.

Let $\alpha_0 \in I$, $\exists\ \beta_0 \in J$ s.t $\alpha_0 \leq \beta$ and hence $\alpha_0 \leq \beta_0$ and $x_{\beta_0} \in U$.

$\therefore$ (x_α) accumulates at x.

Proposition : A net in a topological space accumulates at a point $\Leftrightarrow$ the net has a subnet converging to that point.

Proof : Let $(x_\alpha)_{\alpha\in I}$ be a net in a topological space. If (x_α) has a subnet converging to a point x, then that subnet accumulates at x, so (x_α) accumulates at x.

Conversely suppose that (x_α) accumulates at x. Let J be the collection of all ordered pairs (α, U) such that $\alpha \in I$ and U is a nbhd of x containing x_α.

Define a relation on J by declaring that

$(\alpha_1, U_1) \leq (\alpha_2, U_2)$ when $\alpha_1 \leq \alpha_2$ and $U_1 \supseteq U_2$

If $(\alpha_1, U_1), (\alpha_2, U_2) \in J$, then the fact that (x_α) accumulates at x assuries that there is an α_3 such that $\alpha_1 \leq \alpha_3$; $\alpha_2 \leq \alpha_3$ and

$x_{\alpha_3} \in U_1 \cap U_2$, which implies that

$(\alpha_1, U_1) \leq (\alpha_3, U_1 \cap U_2)$ and

$(\alpha_2, U_2) \leq (\alpha_3, U_1 \cap U_2)$.

It follows that this relation defined on J makes J into a directed set.

Let $g(\alpha, U) = \alpha$, whenever $(\alpha, U) \in J$.

Then $(x_{g(\alpha,u)})$ is a subnet of (x_α) converging to x.

Corollary : A subset S of a topological space is closed.

$\Leftrightarrow$ S contains every accumulation point of every net whose terms lie in S.

Proof : A subset S of a topological space is closed.

$\Leftrightarrow$ S contains every limit of every convergent subnet of every net whose terms lie in S.

$\Leftrightarrow$ S contains every accumulation point of every net whose terms lie in S.

Proposition : A subset S of a topological space is compact

$\Leftrightarrow$ each net in S has a subnet with a limit in S, that is, if and only if each net in S has an accumulation point in S.

Proof : Suppose that (x_α) is a net in S with no accumulation point in S.

For each x in S, let Ux be a nbhd of x that excludes the entire portion of the net from some term onward.

Let $\mathfrak{S} = \{U_x : x \in S\}$ be an open covering for S. Since every finite subcollection of $\mathfrak{S}$ excludes the entire net from some term onward, it follows that $\mathfrak{S}$ cannot be thinked to a finite subcovering for S, so, S is not compact.

Hence S is compact $\Rightarrow$ each net in S has an accumulation point in S.

Conversely, let S is not compact. Then S has an open covering $\mathfrak{S}$ that cannot be thinned to a finite subcovering for S. It can be assumed that $\mathfrak{S}$ is closed under the operation of taking finite unions of its elements. It follows that $\mathfrak{S}$ can be made into a directed set by declaring that $U \leq V$ when $U \subseteq V$.

For each U in $\mathfrak{S}$, let x_u be a member of X – U. Then (X_U) is a net in S with the properly that $X_{U_2} \notin U_1$, when $U_1 \leq U_2$.

It follows that (x_u) has no accumulation point in S.

Hence, each net in S has an accumulation point in S.

$\Rightarrow$ S is compact.

1.6 Topological Group :

Suppose that X is a set with a group (multiplication) operation, that is, an operation $(x, y) \rightarrow x.y$ from $X \times X$ into X such that

1. $(x.y).z = x. (y.z)$, whenever $x, y, z \in X$.

2. there is an identity element e in X such that $x.e = e.x = x$, whenever $x \in X$.

3. each element x of X has an inverse x^{-1} in X such that $x.x^{-1} = x^{-1}.x = e$

Then (x, .) is a group. Suppose furthermore that T is a topology for X such that the mappings $(x, y) \rightarrow x.y$ from $X \times X$ into X and $x \rightarrow x^{-1}$ from X into X are both continuous.

Then (X, T,.) is a topological group.

Remark : Let X be a group and that $x \in X$, and $A, B \subseteq X$.

Then 1. $x.X = \{x.g \mid g \in X\}$

2. $A.x = \{a.x \mid a \in A \subseteq X\}$

3. $x.A = \{x.a \mid a \in A \subseteq X\}$

4. $A.B = \{a.b \mid a \in A, b \in B\}$

5. $A^{-1} = \{a^{-1} \mid a \in A\}$

Proposition :

(a) Suppose that X is a topological group and $x_0 \in X$.

Then $x \to x_0.x$, $x \to x.x_0$ and $x \to x^{-1}$ are homeomorphisms from X onto itself.

Proof :

Let $f : x \to X$ be defined by $f(x) = x_0.x$; $x_0 \in X$

1. f is one-one : Let $p(x) = f(y)$

$\Rightarrow x_0.x = x_0.y$

$\Rightarrow x = y$ [By Left c.Law]

$\therefore$ f is one-one

2. f is onto : Let $y \in X$ (codomain). Then

$x_0^{-1}\, y \in X$.

$\therefore f(x_0^{-1}\, y) = x_0.(x_0^{-1}y) = (x_0.x_0^{-1})y = y$

$\therefore$ f is onto and $f^{-1}(y) = x_0^{-1}\, y$

3. f is continuous :

Let W be an open nbhd of x_0x. By the continuity of $(x_0, x) \to x_0.x$ at (x_0,x), for a nbhd W of $x_0.x$, $\exists$ a nbhd $U_{x_0} \times V_x$ of (x_0, x) s.t $U_{x_0}\, V_x \subseteq W$

$\Rightarrow x_0.v_x \subseteq U_{x_0}\, V_x \subseteq W$ [$\because\ x_0 \in U_{x_0}$)

$\Rightarrow f(v_x) \subseteq W$

$\therefore$ f is continuous.

Similarly $f^{-1}(x) = x_0^{-1}.x$ is also continuous.

Hence f is a homeomorphism.

Next let $f : X \to X$ be defined by $f(x) = x.x_0$; $x_0 \in X$.

1. f is one-one :

Let $f(x) = f(y)$

$\Rightarrow x.x_0 = y.x_0$

$\Rightarrow x = y$ [R.C. Law]

$\therefore$ f is one-one.

2. f is onto :

Let $y \in x$. Then $y_0 x_0^{-1} \in X$

$\therefore f(yx_0^{-1}) = (yx_0^{-1}).x_0$

$= y\,(xx_0^{-1}.x_0)$

$= y$

and $f^{-1}(y) = yx_0^{-1}$

$\therefore$ f is onto.

3. By let $x \in X$ be any point. Let W be an open nbhd of $x.x_0$.

By continuity of $(x, x_0) \to x.x_0$ at (x, x_0), for a bhhd W of $x.x_0$, $\exists$ a nbhd $U_x \times V_{x_0}$ of (x, x_0) such that $Ux.V_{x_0} \subseteq W$

$\Rightarrow Ux.x_0 \subseteq W$ [$\because x_0 \in V_{x_0}$]

$\Rightarrow f(U_x) \subseteq W$.

$\therefore$ f is continuous.

Similarly $f^{-1}(x) = x.x_0^{-1}$ is also continuous.

Hence f is homomorphism.

Next let $\phi : X \to X$ be defined by $\phi(x) = x^{-1}$

1. is one-one $\phi(x) = \phi(y)$

$\Rightarrow x^{-1} = y^{-1} \Rightarrow (x^{-1})^{-1} = (y^{-1})^{-1}$

$\Rightarrow x = y$

$\therefore \phi$ is one-one.

2. is onto : Let $\in$ Y codomain X. Then $y^{-1} \in X$.

$\therefore \phi(y^{-1}) = (y^{-1})^{-1} = y$

$\therefore \phi$ is onto.

3. By definition of topological group

$\phi(x) = x^{-1}$ iscontinuous.

Also $\phi^{-1}(x) = x^{-1}$ is also continuous.

Hence ϕ is a homoemorphism.

(b) If X is a topological group. $A \subseteq X$, $B \subseteq X$.

Then **(i)** A is open $\Rightarrow x_0$. A is open and A^{-1} is open.

(ii) A is closed $\Rightarrow x_0.A$ is closed and A^{-1} is closed.

(iii) A is compact $\Rightarrow x_0.A$ is compact and A^{-1} is compact.

(iv) A or B is open $\Rightarrow$ A.B is open and B.A is open.

Proof :

(i) We have $f : X \to X$ defined by $f(x) = x_0.x$ is a homeomorphism.

We know, that homomorphic image of an open set is open.

Now, A is open $\Rightarrow f(A) = x_0.A$ is open.

Again, we have $\phi : X \to X$ defined by $\phi(x) = x^{-1}$ is a homeomorphism.

$\therefore$ A is open $\Rightarrow \phi(A) = A^{-1}$ is open.

(ii) A is closed $\Rightarrow$ f(A) = x_0.A is closed

A is closed $\Rightarrow$ f(A) = ϕA^{-1} is closed, as f and ϕ are homeomrphisms.

(iii) A is compact $\Rightarrow$ f(A) = x_0. A is compact

A is compact $\Rightarrow$ ϕ(A) = A^{-1} is compact, as f and ϕ are homeomorphisms.

(iv) Let A and B be subsets of X and B is open.

Then A.B = U{a.B : a $\in$ A}

Now, B is open $\Rightarrow$ a.B is open, where a $\in$ A.

$\Rightarrow$ U{a.B : a $\in$ A} is open

$\Rightarrow$ A.B is open.

Again, B.A = U{B.A : a $\in$ A}

Now, B is open $\Rightarrow$ B.a is open, where a $\in$ A.

$\Rightarrow$ U{B.a : a $\in$ A} is open.

$\Rightarrow$ B.A is open.

(c) For each x_0 in X, the nbhds of x_0 are exactly the sets x_0.U such that U is a nbhd of e, which are in turn exactly the sets U.X_0 such that U is a nbhd of e.

Proof :

If U and U_{x_0} are nbhds of $\bar{e}$ and x0 respectively, then x_0.U and x_0^{-1}. U_{x_0} are nbhds of x_0 and e respectively, which together with the fact that $U_{x_0} = x_0.(x_0^{-1}.U_{x_0})$ easily yields that x_0.U is a nbhd of x_0.

Similarly, U.x_0 is a nbhd of x_0.

(d) For each nbhd U of e, there is a nbhd V of e such that V = V^{-1} and V.V $\subseteq$ U

Proof : Let U be a nbhd of e. By the continuity of (x, y) $\rightarrow$ x.y at (e, e), for nbhd U of e.e = e, $\exists$ nbhds V_1 of e and V_2 of e such that

V_1, $V_2 \subseteq$ U.

Let $V = V_1 \cap V_2 \cap V_1^{-1} \cap V_2^{-1}$, another nbhd of e.

Then V^{-1} = V. Also V.V $\subseteq V_1.V_2 \subseteq$ U

$\Rightarrow$ V = V^{-1} and V.V $\subseteq$ U.

Proposition : A subset S of a topological group X is relatively compact (ie S is compact) if and only if each net in S has a subnet with a limit in X (not assumed to be in S); that is, if and only if each net in S has an accumulation point in X.

[Remark : S is relatively compact $\Rightarrow$ $\bar{S}$ is compact.]

Proof : If S is relatively compact, then $\bar{S}$ is compact, so every net in S has a subnet with a limit in $\bar{S}$ and therefore in X.

For the converse, suppose that every net in S has a subnet with a limit in X. Let $(x_\alpha)_{\alpha\in I}$ be a net in S. It is enough to show that (x_α) has a convergent subnet.

For each $\alpha \in I$ and each nbhd U of the identity e of X, let $y_{(\alpha, U)}$ be an element of $(x_{\alpha,U}) \cap S$. Then $(Y_{(\alpha,U)})$ is a net if its index set J is preordered by declaring that $(\alpha_1, U_1) \leq (\alpha_2, U_2)$ when $\alpha_1 \leq \alpha_2$ and $U_1 \supseteq U_2$. [J = {(α, U) : $\alpha \in I$, U is a nbhd of e]

Furthermore, a corresponding subnet $(x_{(\alpha,\cup)})$ of (x_α) is obtained by lettig $x_{(\alpha,\cup)} = x_\alpha$, for each (α, U) in J.

It is enough to show that $(x_{(\alpha,\cup)})$ has a convergent subnet. Since the net $(y_{(\alpha,\cup)})$ has a subnet (y_β) with a limit x, it is enough to show that the corresponding subnet (x_β) of $(x_{(\alpha,\cup)})$ also converges to x.

Suppose that U_0 is a nbhd of e. Let α_0 be any element of I.

If $(\alpha_0, U_0) \leq (\alpha, U)$, then

$Y_{(\alpha,u)} \in x_{(\alpha,u)} . (\alpha ,U_0$, so

$x^{-1}_{(\alpha,u)}.y_{(\alpha,u)} \in U_0$.

It follows that $x^{-1}_{(\alpha,\cup)}.y_{(\alpha,\cup)} \to e$ and

therefore that $x^{-1}_\beta . y_\beta \to e$. The continuity of the group operations then assures that

$y^{-1}_\beta . x_\beta = (x^{-1}_\beta . y_\beta)^{-1} \to e^{-1} = e$

and that $x_\beta = y_\beta . (y^{-1}_\beta . x_\beta) \to x.e = x$ as required [$\because$ $Y_\beta \to x$ and $Y_\beta^{-1} . x_\beta \to e$].

Cauchy Net : A net $(x_\alpha)_{\alpha\in I}$ in an abelian topological group 'X' is (topologically) cauchy if for every nbhd U of the identity element 0 of X, there is an α_u in I such that

$x_\beta \to x_\gamma \in U$ whenever $\alpha_\cup \leq \beta$ and $\alpha_\cup \leq \gamma$.

Invariant metric : A metric d on a group X is left-invariant (right invariant) if $d(z.x, z.y) = d(x, y)$ $[d(x.z, y.z) = d(x, y)]$ whenever $x, y, z \in X$.

If d is both left-invariant and right-invariant, then d is invariant.

A metric d on an abelian group X is invariant if and only if $d(x + z, y + z) = d(x, y)$ $\forall$ $x, y, z \in X$.

Proposition : If a topology for a group is induced by an invariant metric, then the group is a topological group when given this topology.

Proof : Suppose, that a group X is given a topology that is induced by an invariant, metric d. Let e be the identity of X. If sequence (x_n) and (y_n) converges to x and y respectively in X, then

$d(x_n.y_n, x.y) = d\ (x^{-1}x_n .y_n . y^{1}_n, x^{-1}x.y\ y_n^{-1})$

$= d\ (x^{-1}.x_n, y.y_n^{-1})$

$\leq d\ (x^{-1}.x_n, e) + d(e, y.y_n^{-1})$

$= d(x_n, x) + d(y_n, y)$

$\to 0$ as $n \to \propto$ ·

Thus $d(x_n.y_n, x.y) \to 0$ as $n \to \infty$

Again, $d(x_n^{-1}, x^{-1}) = d(x_n.x_n^{-1}.x, x_n.x^{-1}.x)$

$= d(x, x_n) \to 0$ as $n \to \infty$

Hence, $x_n.y_n \to x.y$ and $x_n^{-1} \to x^{-1}$.

If follows that group multiplication and inversion are both continuous. So, X is a topological group.

Definition : An abelian topological group is complete if each Cauchy net in the group converges.

1.7 Vector topology or Topological vector space :

Suppose that X is a vector space with a topology T such that addition of vectors is a continuous operation from $X \times X$ into X and multiplication of vectors by scalars is a continuous operation from $F \times X$ into X. Then T is a vector or linear topology for X and the ordered pair (X, T) is a topological vector space (TVS) or a linear topological space (LTS).

If T has a basis consisting of convex sets, then T is a locally convex topology and the TVS (X, T) is a locally convex space (LCS).

Theorem : (X, T) is a TVS $\Leftrightarrow$ (X, T) is a topological group w.r.t addition and multiplication of vectors by scalars is a continuous operation.

Proof : Let (X, T) be a T.V.S. Let $x_n \to x$ and $y_n \to y$ clearly (X, +) is a group.

Now, $x_n \to x$ and $y_n \to y$

$\Rightarrow x_n + y_n \to x + y$

$\therefore$ Addition is continuous.

Again, since (X, T) is a TVS, so

$x \to \alpha x$ is continuous for $\alpha \in K$

$\Rightarrow x \to -x$ is continuous.

$\therefore$ Multiplication of vectors by a scalar is continous. Converse part is similar.

Proposition : Every norm topology is a locally convex topology.

Proof : Let $(X, T_{\|\cdot\|})$ be a norm topology.

Let $x_n \to x$ and $y_n \to y$ in X. Then

$\| (x_n + y_n) - (x + y) \| = \| x_n - x + y_n - y \|$

$\leq \| x_n - x \| + \| y_n - y \|$

$\to 0$ as $n \to \infty$

$\Rightarrow x_n + y_n \to x + y$

$\Rightarrow$ Addition is continuous.

Again, let $\alpha_n \Rightarrow \alpha$ in K and $x_n \to x$ in X.

Then $\| \alpha_n x_n - \alpha x \| = \| \alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x \|$

$\leq \| \alpha_n x_n - \alpha_n x \| + \| \alpha_n x - \alpha x \|$

$= | \alpha_n | \| x_n - x \| + | \alpha_n - x | \| x \|$

$\to 0$ as $n \to \infty$

$\Rightarrow \alpha_n x_n \to \alpha x$

$\Rightarrow$ Scalar multiplication is continuous.

So, (X, T) is a TVS.

Let, $B_n = \{x : \| x \| < \frac{1}{n}\}$. If G is an open set containing 0, then $\exists\ \varepsilon > 0$ such that

$B(0, \varepsilon) \subseteq G$. Then we can choose $n \in N$ s.t

$\frac{1}{n} < \varepsilon$. So, $B(0, \frac{1}{n}) \subseteq B(0, \varepsilon) \subseteq G$

Hence $\{B_n\}$ is a local base at the origin.

If $x, y \in B_n$, so $\| x \| < \frac{1}{n}$, $\| y \| < \frac{1}{n}$

Then $\| tx + (1 - t)y \| \leq t \| x \| + (1 - t) \| y \|$

$< t\frac{1}{n} + (1 - t)\frac{1}{n} = \frac{1}{n}$, for $0 < t < 1$

$\Rightarrow tx + (1 - t)y \in B_n$.

$\therefore B_n$ is convex.

So, every norm topology is a locally convex topology.

Definition : A subset A of a TVS is bounded if for each nbhd U of 0, there is a positive S_u such that $A \subseteq tU$ whenever $t > S_u$.

Note : A locally convex space is not a NLS.

If $X \neq \phi$ is a vector space, then $\{X, \phi\}$ is a LCS, but $\{X, \phi\}$ is not a NLS.

1. Addition is continuous :

X is the only nbhd of x + y

X is the only nbhd of x

X is the only nbhdof y.

and X + X = X.

So, addition $(x, y) \to x + y$ is continuous.

2. Scalar multiplication $(\alpha, x) \to \alpha x$ is continuous.

X is the only nbhd of $\alpha.x$

X is the only nbhd of x.

Let $t \in B(\alpha, \varepsilon) = \{x . |x - \alpha| < \varepsilon\}$

Then tX = X.

So scalar multiplication is continuous.

Also {X} is the unique local base of $x \in X$ and X is convex.

Hence $\{X, \phi\}$ is a LCS.

But it is not normable. In fact X is not Hausdorff. So an LCS is not NLS.

Example : If a Cauchy net in an abelian topological group has a convergent subnet, then the entire net converges to the limit of the subnet.

Proof : Suppose $(x_\alpha)_{\alpha\in I}$ is a Cauchy net in an abelian topological group X and that (x_α) has a subnet with a limit x_0. Let U be a nbhd of 0 in X, let V be a nbhd of 0,

such that $V + V \subseteq U$ and let α_0 be a member of I

such that $x_\gamma - x_\delta \in V$ whenever $\alpha_0 \leq \gamma$ and $\alpha_0 \leq \gamma$.

Since (x_α) accumulates at x_0, there is an $\alpha_\cup$ in I

Such that $\alpha_0 \leq \alpha_\cup$ and $x_{\alpha\cup} \in x_0 + V$. If $\alpha_\cup \leq \alpha$,

then $x_\alpha = (x_\alpha - x_{\alpha\cup}) + x_{\alpha\cup} \in V + (x_0 + V)$

$$= x_0 + (V + V)$$

$$\subseteq x_0 + U$$

$\Rightarrow x_\alpha \to x_0$.

Example : Every convergent net in an abelian topological group is Cauchy.

Proof : Let X be an abelian topological group and let (x_α) be a net in X converging to some element x_0 of X. Let U be a nbhd of 0 in X and let V be a nbhd of 0 such that $V = -V$ and $V + V \subseteq U$. Let $\alpha_\cup$ be a member of I such that $x_\alpha \in x_0 + V$ whenever U . If $\alpha_\cup \leq \alpha$ and $\alpha_\cup \leq \gamma$, then

$x_\beta - x_\gamma \in (x_0 + V) - (x_0 + V) = V - V = V + V \subseteq U$

$\therefore (x_\alpha)$ is Cauchy net.

Theorem : Suppose that X is a topological vector space.

(a) Let (β_α) be a net in IF and let (x_α) and (y_α) be nets in X such that all three nets have the same index set and $\beta_\alpha \to \beta$, $x_\alpha \to x$, $y_\alpha \to y$. Let γ and Z be elements of F and X respectively. Then

(i) $x_\alpha + y_\alpha \to x + y$

(ii) $\beta_\alpha x_\alpha \to \beta x$

(iii) $x_\alpha + z \to x + z$

(iv) $\gamma x_\alpha \to \gamma x$

(v) $\beta_\alpha z \to \beta z$.

Proof : We know that $f : X \to Y$ is continuous at x

$\Leftrightarrow x_\alpha \to x \Rightarrow f(x_\alpha) \to f(x)$.

(i) In a T.V.S, $+ : X \times X \to X$, $(x, y) \to x + y$ is continuous

$\therefore (x_\alpha, y_\alpha) \to (x, y)$ in $X \times X$

$\Rightarrow x_\alpha + y_\alpha \to x + y$.

Thus $x_\alpha \to x$, $y_\alpha \to y \Rightarrow x_\alpha + y_\alpha \to x + y$

(ii) In a T.V.S, $\because$ FXX $\to$ X, $(\alpha, x) \to \alpha.x$ is continuous.

$\therefore (\beta_\alpha, x_\alpha) \to (\beta, x)$ in FXX

$\Rightarrow \beta_\alpha x_\alpha \to \beta x$

Thus, $\beta_\alpha \to \beta$, $x_\alpha \to x \Rightarrow \beta_\alpha x_\alpha \to \beta x$.

(iii) For a nbhd W_{x+z}, $\exists$ a nbhd, Ux of x and v_z of y_z such that

$U_x + v_z \subseteq W_{x+z}$ [Definition of continuity of addition]

$\Rightarrow U_{x+z} \subseteq W_{x+z}$

$\Rightarrow T_z (Ux) \subseteq W_{x+z}$

$\therefore T_z$ is continuous.

$\therefore x_\alpha \to x \Rightarrow T_z (x_\alpha) \to T_z (x)$

$\Rightarrow x_{\alpha+z} \to x + z$

(iv) For a nbhd $W_{\gamma x}$, $\exists$ a nbhd U_x of x and anbhd (,) of , such that

$\gamma U_x \subseteq W_{\gamma x}$ [Definition of continuity of scalar multiplication]

$\Rightarrow M_y (U_x) \subseteq W_{yx}$

$\therefore M_y$ is continuous.

$\therefore x_\alpha \to x \Rightarrow M_y (x_\alpha) \to M_y (x)$

$\Rightarrow yx_\alpha \to yx.$

(v) For a nbhd $W_{\beta z}$, $\exists$ a nbhd U_z of z and anbhd $B(\beta, \varepsilon)$ of β such that $B(\beta, \varepsilon)\, U_z \subseteq W_{\beta z}$

$\Rightarrow B(\beta, \varepsilon)z \subseteq W_{\beta z}$

$\Rightarrow M_z (B(\beta, \varepsilon))\, W_{\beta z}$

$\therefore M_z$ is continuous.

$\therefore \beta_\alpha \to \beta \Rightarrow M_z (\beta_\alpha) \to M_z (\beta)$

$\Rightarrow \beta_\alpha z \to \beta z.$

(b) If f and g are continous functions from a topological space into X and α is a scalar, then f + g and αf are continuous.

Proof : Let (x_α) be a net in a topological space z,

such that $x_\alpha \to x$.

Then $f(x_\alpha) \to f(x)$

$g(x_\alpha) \to g(x)$

$\therefore f(x_\alpha) + g(x_\alpha) \to f(x) + g(x)$

$\Rightarrow (f + g)(x_\alpha) \to (f + g)(x)$

$\therefore$ f + g is continuous.

Let (x_β) be a net in z (topo.space) such that

$x_\beta \to x$. and $\alpha \in F$.

$\therefore f(x_\beta) \to f(x)$

$\Rightarrow \alpha f(x_\beta) \to \alpha f(x)$

$\Rightarrow (\alpha f) \to (\alpha f)(x)$

$\therefore \alpha f$ is continuous.

(c) Let x_0 be an element of X and let α_0 be a non zero scalar. Then the maps $x \to x + x_0$ and $x \to \alpha_0 x$ are homeomovphisms from X onto itself. Conseqenty, if A is a subset of X that is open / closed / compact, then $x_0 + A$ and $\alpha_0 A$ also have that property. If A and U are subsets of x and u is open, then A + U is open.

Proof : 1. Let $T_{x_0} : X \to X$ be defined by

$T_{x_0}(x) = x + x_0$.

(i) Let $T_{x_0}(x) = Tx_0(y)$

$\Rightarrow x + x_0 = y + x_0$

$\Rightarrow x = y$

$\Rightarrow Tx_0$ is one-one.

(ii) Let $x \in$ codomain, X.

Then $Tx_0\ (x - x_0) = x - x_0 + x_0 = x$.

Thus for $x \in X$, ∃ $x - x_0 \in X$ s.t $Tx.(x - x_0) = x$.

$\therefore Tx_0$ is onto and $Tx_{0-1}(x) = x - x_0$.

(iii) Let W_{x+x_0} be an open nbhd of $x + x_0$. Then ∃ U_x and V_{x_0}, open nbhds of x and x_0 respectively such that $U_x + V_{x_0} \subseteq W_{x+x_0}$

$\Rightarrow U_{x+x_0} \subseteq W_{x+x_0}$

$\Rightarrow T_{x_0}(U_x) \subseteq W_{x+x_0}$

$\therefore T_{x_0}$ is continuous at $x \in X$

$\Rightarrow T_{x_0}$ is continuous on X.

(iv) $T_{x_0}^{-1}(x) = x - x_0 = T_{-x_0}(x)$

$\Rightarrow T_{x_0}$ is continuous for any $x \in X$.

$\therefore T_{-x_0}$ is a homeomorphism.

2. Let $M_{\alpha_0} : X \rightarrow X$ be defined by $M_{\alpha_0}(x)\ \alpha_0 X$.

(i) Let $M_{\alpha_0}(x) = M_{\alpha_0}(y)$

$\Rightarrow \alpha_0 x = \alpha_0 y$

$\Rightarrow x = y\ [\because\ \alpha_0 \neq 0]$

$\therefore M_{\alpha_0}$ is one-one.

(ii) Let $y \in X$. Then $\alpha_0^{-1} Y \in X$ and

$M_{\alpha_0}(\alpha_0^{-1} y) = \alpha_0\ \alpha_0^{-1} y = y$

$\therefore M_{\alpha_0}$, is onto and $M_{\alpha}^{-1}(y) = \alpha_0^{-1} y = M_{\alpha_0}^{-1}(y)$

(iii) Let $W_{\alpha_0 x}$ be a nbhd of $\alpha_0 x$.

$\Rightarrow \exists\ \varepsilon > 0$ and some nbd V_x of x such that t $V_x \subseteq W_{\alpha_0} x$ a nbhd V_x of x such that $tv_x \subseteq W_{\alpha_0} x$ whenver i.e. $|\ t - \alpha_0\ | < \varepsilon$.

$\Rightarrow \alpha_0\ V_x \subseteq W_{\alpha_0} x$

$\Rightarrow M_{\alpha_0}(V_x) \subseteq W_{\alpha_0}x.$

$\therefore M_{\alpha_0}$ is continuous.

Again, $M_{\alpha_0}^{-1}(x) = M_{\alpha_0}^{-1}(x)$ is continuous. Since M_{α}^{0} is continuus for each $\alpha \in F$.

$\therefore M_{\alpha_0}$ is a homeomorphism.

3. We knwo that, under homeomorphism open / closed / compact set goes to open / closed / compact.

We know that, $Tx_0(x) = x_0 + x$ is a homeomorphism and A is open / closed / compact.

$\therefore T_{x_0}(A) = x_0 + A$ is open / closed / compact.

Again, $M_{\alpha_0}(x) = \alpha_0 x$ is a homeomorphism and A is open / closed / compact

$\therefore M_{\alpha_0}(A) = \alpha_0 A$ is open/closed/compact.

4. Here $A, U \subseteq X$ such that U is open

Now, $A + U = \bigcup_{a \in A} \{a + u\}$

= Union of open sets

= an open set.

$\Rightarrow A + U$ is open set.

(d) Suppose that A and B are subsets of X, that $x_0 \in X$ and let α_0 is a nonzero scalar. Then

1. $\bar{A} + \bar{B} \subseteq \overline{A + B}$, $x_0 + \bar{A} = \overline{x_0 + A}$, $\alpha_0\bar{A} = \overline{\alpha_0 A}$

2. $A^0 + B^0 \subseteq (A + B)^0$, $x_0 + A^0 = (x_0 + A)^0$, $\alpha_0 A^0 = (\alpha_0 A)^0$

Proof : 1. Let $x \in \bar{A} + \bar{B}$

$\Rightarrow x = y_0 + z_0$, where $y_0 \in \bar{A}$, $z_0 \in \bar{B}$

Let (y_α) and (z_β) be nets in A and B respectively such that $y_\alpha \to y_0$, $z_\beta \to z_0$.

Then, there are subnets (y_γ) and (z_γ) of (y_α) and (z_β) respectively having the same index set.

Then $y_\gamma + z_\gamma \to y_0 + z_0$

$\Rightarrow y_0 + z_0 \in \overline{A + B}$

$\Rightarrow x \in \overline{A + B}$

$\therefore \bar{A} + \bar{B} \subseteq \overline{A + B}$

Lemma : If $f : X \to Y$ is a homeomorphism, then for $A \subseteq X$, $f(\bar{A}) = f(\bar{A})$ and $f(A^0) = [f(A)]^0$

Proof : Here, $\bar{A}$ is a closed set and f is a closed map

$\Rightarrow f(\bar{A})$ is a closed set.

Also, $f(A) \subseteq f(\bar{A})$.

$\Rightarrow f(\bar{A})$ is a closed set containing f(A).

But $\overline{f(A)}$ is the smallest closed set containing f(A).

$\therefore \overline{f(A)} \subseteq f(\bar{A})$ .....(1)

Again, $\overline{f(A)}$ is a closed set and f is continuous

$\Rightarrow f^{-1}(\overline{f(A)})$ is a closed set

Also, $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$

$\therefore f^{-1}(\overline{f(A)})$ is closed set containing A and $\bar{A}$ is the smallest closed set containing A.

So $\bar{A} \subseteq f^{-1}(\overline{f(A)})$

$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$ ..... (2)

From (1) and (2), $f(\bar{A}) = \overline{f(A)}$

We know, $T_{x_0}(x) = x_0 + x$ is a homeomorphism.

$\Rightarrow T_{x_0}(A) = x_0 + A$ is a homeomorphism.

So, $T_{x_0}(\bar{A}) = \overline{T_{x_0}(A)}$

$\Rightarrow x_0 + \bar{A} = \overline{x_0 + A}$

Lemma : If $f : X \to Y$ is a homeomorphism, then for $A \subseteq X$, $f(A) = [f(A)]^0$.

Again, A^0 is open and f is open open

$\Rightarrow f(A^0)$ is an open set.

Also, $f(A^0) \subseteq f(A)$.

$\therefore f(A^0)$ is an open set contained in f(A)

But, $[f(A)]^0$ is the largest open set contained in f(A)

$\therefore f(A^0) \subseteq [f(A)]^0$ .... (1)

Again $[f(A)]^0$ is an open set and f is

continuous.

$\Rightarrow f^{-1}[f(A)]^0$ is an open set.

Also, $[f(A)]^0 \subseteq f(A)$

$\Rightarrow f^{-1}[f(A)]^0 \subseteq A$.

$\therefore f^{-1}[f(A)]^0$ is open set containg A and A^0 is the largest open set containing A.

$\therefore f^{-1}[f(A)]^0 \subseteq A^0$

$\Rightarrow [f(A)]^0 \subseteq f(A^0)$ ....(2)

From (1) and (2), $f(A^0) = [f(A)]^0$

We know that

$M_{\alpha_0}(x) = \alpha_0 x$ is a homeomerphism

$\Rightarrow M_{\alpha_0}(A) = \alpha_0 A$ is a homeomorphism

$\therefore M_{\alpha_0}(A^0) = [M_{\alpha_0}(A)]^0$

$\Rightarrow \alpha_0 A^0 = [\alpha_0 A]^0$

Again, $T_{x_0}(x) = x_0 + x$ is a homeomorphism

$\therefore T_{x_0}(A) = x_0 + A$ is a homeomorphism.

$T_{x_0}(A^0) = [T_{x_0}(A)]^0$

$\Rightarrow x_0 + A^0 = [x_0 + A]^0$

Also, $M_{\alpha_0}(x) = \alpha_0 x$ is a homeomorphism

$\Rightarrow M_{\alpha_0}(A) = \alpha_0 A$ is a homeomorphism

$\therefore M_{\alpha_0}(\bar{A}) = \overline{M\alpha_0(A)}$

$\Rightarrow \alpha_0 \bar{A} = \overline{\alpha_0 A}$

Next, A^0 is open $\Rightarrow A^0 + B^0$ is open.

Now, $A^0 \subseteq A$, $B^0 \subseteq B$ $A^0 + B^0 \subseteq A + B$

$\therefore A^0 + B^0$ is an open set contain in $A + B$.

But $(A + B)^0$ is the largest open set contain in $A + B$.

$\therefore A^0 + B^0 \subseteq (A + B)^0$

(e) For each x_0 in x, the nbhds of x_0 are exactly the sets $x_0 + U$ such that U is a nbhd of 0.

Proof : We know, $x \to x_0 + x$ is a homeomorphism

So, U is open containing 0

$\Rightarrow x_0 \in x_0 + U$ and $x_0 + U$ is open

$\Rightarrow x_0 + U$ is a nbhd of x_0.

Conversely, V is a nbhd of x_0

$\Rightarrow 0 \in -x_0 + V$ and $-x_0 + V$ is open.

Then $U = -x_0 + V$ is a nbhd of 0

$\Rightarrow V = x_0 + u$, where u is a nbhd of 0.

(f) Each nbhd of 0 in X is absorbing

[In a T.V.S 'X', $A \subseteq X$ is said to be absorbing if $x \in X$, there is a positive number to such that $x \in tA$, $\forall\ t > t_0$.]

Proof : Let U be a nbhd of 0 in X. Let $x \in X$

Then $(t, x) \to t \circ x$ is continuous at $(0, x)$

If U is a nbhd of 0, $\exists$ a nbhd V of x such that $t_V \subseteq U$ for $|t| < \varepsilon$, for $\varepsilon > 0$

$\Rightarrow tx \in U$ if $-\varepsilon < t < \varepsilon$. $(0 < t < \varepsilon)$

$\Rightarrow x \in t^{-1}U$ if $t^{-1} > \frac{1}{\varepsilon}$

$\Rightarrow x \in s\,U$ if $s > s_0$, where $s = t^{-1}$, so $= \frac{1}{\varepsilon}$

$\therefore$ U is an absorbing set.

(g) For each nbhd U of 0 in x, there is a balanced nbhd V of 0 in X such that $V \subseteq \bar{V} \subseteq \overline{V + V} \subseteq U$.

Proof : Suppose that U is a nbhd of 0 in X. The continuity of vector addition yields nbhds u1 and u2 of 0 such that $U_1 + U_2 \subseteq U$.

Let $U_3 = U_1 \cap U_2 \cap (-U_1) \cap (-U_2)$. Then U_3 is a nbhds of 0 such that $U_3 = -U_3$ and $U_3 + U_3 \subseteq U$.

The same procedure applied to U_3 instead of U yields a nbhd U_4 of 0 such that $U_4 = -U_4$ and $U_4 + U_4 + U_4 + U_4 \subseteq U$.

It follows that $U_4 + U_4 + U_4$ does not intersect $X - U$, so the fact that $U_4 = -U_4$ implies that $U_4 + U_4$ does not intersect

$(X - U) + U_4$.

Since $(X - U) + U_4$ is open, it follows that $\overline{U_4 + U_4}$ does not intersect

$(X - U) + U_4$, so $\overline{U_4 + U_4} \subseteq U$.

The continuity of multiplication of vectors by scalars produces a $s > 0$ and a nbhd U_5 of 0 in X such that $\alpha\, U_5 \subseteq U_4$ whenever $|\alpha| < 8$.

Let $V = U\ \{\alpha U_5 : |\alpha| < \delta\}$. Then V is a balanced nbhd of 0 lying in U_4.

[V is a balanced $\Rightarrow tv \subseteq v$ if $|t| \leq 1$.]

Let $|t| \leq 1$, $y \in tV$ $\Rightarrow y = tx, x \in V$

$\Rightarrow y = t\,\alpha\,u,\ x = \alpha u,\ u \in U_5$

$|\alpha| < \delta$

$\Rightarrow y = (t\alpha),\ u\ ,\ |t\alpha| = |t||\ | < \delta$

$\therefore\ y = (t\alpha).u,\ |t\,\alpha| < \delta$

$\Rightarrow y \in V$

$\therefore\ tV \subseteq V$ for $|t| \leq 1$.

Thus V is a balanced nbhd of 0 lying in U_4 and

$V \subseteq V = \overline{V} + \{0\} \subseteq \overline{V + V} \subseteq \overline{U_4 + U_4} \subseteq U.$

(h) Suppose that A is a bounded subset of X, that x_0 is an element of X, and $\alpha_0 \in F$. Then x_0 + A and α_0 + A are bounded.

Proof : Let A be a bounded subset of X. To show that x_0 + A is bounded.

Let U be a nbhd of 0 in X and let V be a balanced nbhd of 0 in X such that $V + V \subseteq U$.

Let s_v be a positive number such that $A \subseteq t_v$ and $x_0 \in t_v$ whenever $t > s_v$.

If $t > s_v$, then $x_0 + A \subseteq t_v + t_v \subseteq t_u$

$\therefore\ x_0 + A$ is bounded.

Next, to show that $\alpha_0 A$ is bounded.

Let $t > (|\alpha_0| + 1)$ sv.

Then $\dfrac{t}{1+|\alpha_0|} > s_v$

$\Rightarrow A \subseteq \dfrac{t}{1+|\alpha_0|} V$ [$\because$ A is bounded]

$\Rightarrow \alpha_0 A \subseteq t\,(|\alpha_0| + 1)^{-1} \alpha_0 V \subseteq tV \subseteq t_U$

Thus, $\alpha_0 A \subseteq t_U$ whenever $t > (|\alpha_0| + 1)$ sv.

$\therefore\ \alpha_0$ A is a bounded.

(i) Let A be a subset of X. Then

1. $[A] = \langle\overline{A}\rangle$

2. $\overline{CO}(A) = \overline{CO(A)}$

3. A is a subspace of X $\Rightarrow \overline{A}$ is a subspace of X.

4. A is balanced $\Rightarrow \overline{A}$ is balanced.

5. A is balanced $\Rightarrow A^0$ is balanced provided that $0 \in A^0$.

6. A is bounded $\Rightarrow \bar{A}$ is bounded.

7. A is convex $\Rightarrow \bar{A}$ is convex.

8. A is bounded $\Rightarrow A^0$ is bounded.

9. A is convex $\Rightarrow A^0$ is convex.

Proof : Let A be a subspace of X. To show that $\bar{A}$ is a subspace of X.

Let $x, y \in \bar{A}$ and $\alpha, \beta \in F$. To show $\alpha x + \beta y \in \bar{A}$

Now $x \in \bar{A} \Rightarrow \exists$ a net (x_α) in A such that $x_\alpha \to x$.

$y \in \bar{A} \Rightarrow \exists$ a net (y_β) in A such that $y_\beta \to y$.

Then there exists subnets (x_γ) and (y_γ) with same index set such that $x_\gamma \to x$, $y_\gamma \to y$.

$\Rightarrow \alpha x_\gamma + \beta y_\gamma \to \alpha x + \beta y$.

$\therefore$ A is a subspace, so $\alpha x_\gamma + \beta y_\gamma$ A

$\therefore \alpha x + \beta y \in \bar{A}$

$\Rightarrow \bar{A}$ is a subspace of X.

4. Let A be a balanced subset of X. To show that $\bar{A}$ is balanced.

A is balanced $\Rightarrow tA \subseteq A$, whenever $|t| \leq 1$.

$\Rightarrow tA \subseteq \bar{A}$, whenever $|t| \leq 1$

$\Rightarrow t\bar{A} \subseteq \bar{A}$, whenever $|t| \leq 1$

$\therefore \bar{A}$ is balanced.

5. Let A be a balanced subset of X. To show that A^0 is also balanced provided $0 \in A^0$

A is balanced $\Rightarrow tA \subseteq A$ for $0 < |t| \leq 1$.

Now $tA^0 \subseteq tA \subseteq A$ for $0 < |t| \leq 1$.

Thus tA^0 is an open set contained in A. But A^0 is the largest open set contained in A.

$\therefore tA^0 \subseteq A^0$ for $0 < |t| \leq 1$.

If $t = 0$, then $tA^0 = \{0\} \subseteq A^0$ ($\because 0 \in A^0$).

$\therefore A^0$ is balanced.

6. Let A be bounded subset of X. To show that $\bar{A}$ is a bounded subset of X.

Let U be a nbhd of 0 in X and let V be a nbhd of 0 in X such that $\bar{V} \subseteq U$.

Let s_v be a positive number such that $A \subseteq tV$, when $t > s_v$.

Then $\bar{A} \subseteq \overline{t_v}$

$\Rightarrow \bar{A} \subseteq t_{\bar{v}} \subseteq tU$, when $t > s_v$.

$\Rightarrow \bar{A} \subseteq tU$, when $t > s_v$.

$\therefore$ $\bar{A}$ is a bounded set.

7. Let A be a convex subset of X. To show that $\bar{A}$ is a convex subset of X.

For $0 < t < 1$, we have

A is convex $\Leftrightarrow tA + (1 - t)A \subseteq A$

$\Leftrightarrow tx + (1 - t)y \in A$, where

$x, y \in A$.

Let $x, y \in \bar{A} \Rightarrow \exists$ nets $(x_\alpha), (y_\beta) \subseteq A$ such that,

$x_\alpha \to x, y_\beta \to y$

$\Rightarrow \exists$ subnets (x_γ) and (y_γ) in A

such that $x_\gamma \to x, y_\gamma \to y$

Then for $0 < t < 1$,

$t x_\gamma + (1 - t)y_\gamma \to tx + (1 - t)y$

But as A is convex, so $tx_\gamma + (1 - t)y_\gamma \in A$

$\therefore tx + (1 - t)y \in \bar{A}$

$\Rightarrow \bar{A}$ is a convex set.

8. Let A be a bounded subset of X. To show that A^0 is bounded. Let U be a nbhd of 0.

A is bounded $\Rightarrow \exists\, t_U > 0$ such that $A \subseteq t_U, \forall\ t > t_U$

Then $A^0 \subseteq (t_U)^0 = t_U^{\ 0} \subseteq t_U, \forall\ t > t_U$

$\Rightarrow A^0 \subseteq t_U$ for all $t < t_U$.

$\therefore A^0$ is bounded.

9. Let A be a convex subset of X. To show that A^0 is convex set.

Let $0 < t < 1$. Since A is convex, so

$tA^0 + (1 - t)A \subseteq A$.

Now, $tA^0 + (1 - t)A^0 \subseteq tA + (1 - t)A \subseteq A$.

$\therefore$ $tA^0 + (1 - t)A^0$ is an open set contained in A.

But A^0 is the largest open set contained in A.

$\therefore$ $tA^0 + (1 - t)A^0 \subseteq A^0$.

$\Rightarrow$ A^0 is a convex set.

(j) Let Y be a subspace of X. Then the relative topology that Y inherits from X is a vector topology. If the topology of X is locally convex, then so is the relative topology of Y.

Proof : Let (X, T) be a topological vector space and we know that

$T_y = \{Y \cap U : U \in T\}$ is a topology on Y.

We have to show **(i)** $+ : Y \times Y \to Y$ is continuous.

Lt $Y \cap U$ be an open subset of Y.

But U is open in (X, T) and $+ : X \times X \to X$ is continuous.

So, $\exists$ $V \in T$ such that $V + V \subset U$

$\Rightarrow Y \cap V + Y \cap V \subset Y \cap U$.

$\therefore$ $+ : Y \times Y \to Y$ is continuous.

(ii) To show : $K \times Y \to Y$, (α, y) $\alpha.y$ is continuous.

Let $Y \cap U$ be an open nbhd of (Y, T_y)

$\Rightarrow$ U is open in (X, T) and $(\alpha, x) \to \alpha x$ is continuous.

So, $\exists$ a nbhd $B = \{t : \alpha\ |t - \alpha| < S\}$ and an open subset V of x such that

$tV \subset U$ for $|t - \alpha| < s$.

$\Rightarrow$ $t(Y \cap V) \subset y \cap U$ for $|t - \alpha| < s$.

$\Rightarrow$ Scalar multiplication is continuous in (Y, T_y)

$\therefore$ (Y, T_y) is a T.V.S.

If (X, T) is locally convex TVS, then $\exists$ a basis

$\beta_x = \{U : U$ is open and $x \in U\}$ such that U is convex.

We can show that $\beta_x Y = \{Y \cap U : U \in \beta_x\}$ is a local base at x w.r.t (Y, T_y).

Let $Y \cap U \in T_y$ and $x \in Y \cap U$ and $x \in G$ be open in (X, T)

$\Rightarrow \exists\ U \in \beta_x$ such that $U \subset G$

$\Rightarrow Y \cap U \subset Y \cap G$.

$\therefore$ $\beta_x Y$ is a local base at x.

Again, u is convex $\Rightarrow Y \cap U$ is convex.

$\therefore$ (Y, T_y) is locally convex TVS.

Theorem : If U is a convex nbhd of 0 in a TVS 'X', then there is a balanced convex nbhd v of 0 such that $V \subset V \subset \overline{V + V} \subset U$

Proof : Let U be a convex nbhd of 0 in X. We have to show that a balanced convex nbhd V of 0 such that

$V \subset \bar{V} \subset \overline{V + V} \subset U$

WLOG, we can assume that $- u = u$.

Since U is convex, so

$3^{-1}U + 3^{-1}U - 3^{-1}U = U.$

$\therefore$ $3^{-1}U + 3^{-1}U$ does not intersect the open set $(X - U) + 3^{-1}u$ that includes $X \setminus U$. So,

$\overline{3^{-1}U + 3^{-1}U} \subseteq u.$

It is enough to find a convex balanced nbhd V of 0 such that $V \subseteq 3^{-1} U$.

Let $W = \cap\{3^{-1} \alpha U : \alpha \in F, |\alpha| = 1\}$

Then W is a subset of 3^{-1} U and is convex as the intersection of convex sets.

Step 1 : To show $0 \in W^0$

Let B be a balanced nbhd of 0 included in $3^{-1}U$. If is a scalar such that $|\alpha| = 1$, then

$B = \alpha B \subseteq 3^{-1} \alpha U$

and so $B \subseteq W$

$\therefore$ It follows that $0 \in W^0$.

Step 2 : To show that W^0 is balanced.

Let β be a scalar such that $|\beta| \leq 1$ and let t and be scalars such that $0 \leq t \leq 1$, $|\gamma| = 1$ and $\beta = t\gamma$.

$\beta w = t(\cap\{3^{-1} \alpha \gamma U : \alpha \in F : |\alpha| = 1\})$

$tw = tw + (1 - t) \{0\} \subseteq w$

$\Rightarrow$ W is balanced and $0 \in W^0$.

$\Rightarrow$ W^0 is a balanced nbhd of 0.

$\therefore$ W^0 is a balanced convex nbhd of 0.

Let $V = W^0 \subseteq 3^{-1}$ U.

Then V is a balanced, convex nbhd of 0 such, that

$V \subseteq \overline{V} \subseteq \overline{V + V} \subseteq U.$

Definition : Let A be a subset of TVS (X, T).

Then **(i) convex hull co(A)** = the smallest convex set containing A.

= the intersection of all convex sets containing A.

(ii) closed convex hull

co(A) = the smallest closed convex set containing A.

= the intersection of all closed set containing A.

(iii) Linear hull $\langle A \rangle$ = The smallest subspace of X contuing A.

= intersection all subspaces containing A.

(iv) Closed linear span of A,

[A] = the smallest closed, subspace of X containing A.

Theorem : (i) $[A] = \overline{\langle A \rangle}$

(ii) $\overline{CO}(A) = \overline{CO(A)}$

Proof : (i) $\langle A \rangle$ is the smallest subspace of X containing A.

$\overline{\langle A \rangle}$ is a closed subspace containing A.

$\therefore [A] \subseteq \overline{\langle A \rangle}$, as [A] is the smallest closed subspace containing A.

Again,

$\langle A \rangle \subseteq [A]$

$\Rightarrow \overline{\langle A \rangle} \subseteq \overline{[A]} = [A]$, as [A] is a closed subspace of X.

$\Rightarrow \overline{\langle A \rangle} \subseteq [A]$

$\therefore [A] = \overline{\langle A \rangle}$

(ii) We know, CO(A) = the smallest convex set containing A.

$\overline{CO(A)}$ = the closed convex set containing A.

But, $\overline{CO}(A)$ = The smallest closed convex set containing A.

$\therefore \overline{CO}(A) \subseteq \overline{CO(A)}$ ....(1)

Again,

$CO(A) \subseteq \overline{CO}(A)$

$\Rightarrow \overline{CO(A)} \subseteq \overline{CO}(A)$, as $\overline{CO}(A)$ is closed.

$\Rightarrow \overline{CO(A)} \subseteq \overline{CO}(A)$ ....(2)

From (1) and (2), we get

$\overline{CO}(A) = \overline{CO(A)}$

Remark : 1. A TVS has a local base at 0 such that each member of the local base is balanced.

2. An locally compact TVS has a local base at 0 such that each member of the local base is balanced and convex.

Proof : 1. Let $\beta_0 = \{U_n\}$ be a local base of 0 of a TVS 'X'.

Let $\beta_1 = \{V_n : V_n$ is balanced and $V_n \subset U_n\}$

Then $\{V_n\}$ is a local base at 0 such that each V_n is balanced.

Proposition : A subset A of a TVS X is bounded if and only if it has this property. For each balanced nbhd U of 0 in X, there is a positive 's_U' such that $A \subseteq s_u U$.

Proof : Let A be bounded $\Rightarrow$ A is absorbed by every nbhd U of 0.

$\Rightarrow$ A is absorbed by every balanced nbhd U of 0.

$\Rightarrow \exists\ s_u > 0$ such that $A \subseteq tU \ \forall\ t \geq s_U$

$\Rightarrow A \subseteq s_U U$.

Conversely suppose that, for each balanced nbhd U of 0 in X. $\exists\ s_u > 0$ such that $A \subseteq s_u U$.

Let V be a balanced nbhd of 0 such that $V \subset U$.

Let s_v be a positive number such that

$A \subseteq s_v V$

If $t > s_v$. Then $\frac{s_v}{t} \leq 1$

$$\Rightarrow \frac{s_v}{t} V \subseteq V$$

$$\Rightarrow s_v V \subseteq tV$$

$\therefore A \subseteq s_v V \subseteq tV \subseteq tU, \ \forall\ t > s_u$

$\therefore$ A is bounded.

Proposition : Every compact subset of a TVS is bounded. Thus every convergent sequence in a TVS is bounded.

Proof : Let K be a compact subset of A TVS 'X' and let U be a balanced nbhd of 0 in X.

Since U is absorbing, the collection

$\{tU : t > 0\}$ is an open covering four K.

So, there are positive numbers $t_1, t_2, ..., t_n$

Such that $t_1 < t_2 < .... < t_n$ and

$K \subseteq \bigcup_{j=1}^{n} t_j U$

Since $t_j U = t_n(t_n^{-1} t_j U) \subseteq t_n U$, for each j.

$\therefore K \subseteq t_n U$ for $t_n > 0$

$\Rightarrow$ K is bounded.

2nd part :

Next, supose (x_n) is a sequence in TVS X

Such that $x_n \to x$

Put $K = \{x_n\} \cup (x)$

Then we can show K is compact.

Let $\{U_j\}$ be an open cover of K. Let U_{n_0} be an open set containing x.

Then $x_i \in U_{n_0}$, $\forall$ $i \geq n_1$

Let $x_1 \in u_1, x_2 \in u_2, ..., x_{n-1} \in U_{n-1}$

Then $K \subseteq U_{n_0} \cup U_1 \cup. ... \cup U_{n-1}$

$\Rightarrow$ K is compact and hence bounded.

Theorem : Every To vector topology is completely regular.

Proof : Suppose, that x is a TVS whose topology is To.

Let x, and y be two distinct elements of X.

Then there is a nbhd. U of 0 such that either

$x \notin y + U$ or $y \notin x + U$

Suppose, $x \notin y + U$. Then $y \notin x - U$

It not $y = x - u$, where $u \in U$ and

$x = y + u \in y + U$, a contradiction.

$\therefore$ It follows that the topology of X is T_1.

Now, let x_0 be an element of X and let F be a closed subset of X such that $x_0 \in X - F$.

Then $0 \notin -x_0 + F$ (otherwise $x_0 \in F$)

$\Rightarrow 0 \notin \overline{-x_0 + F}$

$\therefore$ There is a batanced nbhd B of 0 such that

$B \cap (-x_0 + F) = \phi$

$\therefore$ There is continous function $f : X \to [0, +\infty)$

Such that $f(0) = 0$ and $f(x) \geq 1$ whenever $x \in X - B$.

$\Rightarrow f(x) \geq 1$ whenever $x \in -x_0 + F$.

Let $g(x) = \min \{1, f(x - x_0)\}$ whenever $x \in X$.

Then g is a continous function from X into [0, 1] such that $g(x_0) = 0$ and $g(x) = 1$, whenever $x \in F$.

$\therefore$ The topology of X is completely regular.

Remark : $g(x_0) = \min \{1, f(x_0 - x_0)\} = 0 \Rightarrow g(x_0) = 0 \ \forall \ x \in F$

If $x \in F$, then $x - x_0 \in - x_0 + F$

$\Rightarrow x - x_0 \in X - B$

$\Rightarrow f(x - x_0) \geq 1$

$\therefore g(x) = \min \{1, f(x - x_0)\} = 1$

$\therefore g(x) = 1 \ \forall \ x \in F$

Theorem : Suppose that x* is a linear functional on a TVS X. Then the following are equivalent.

(a) The functional x* is continuous.

(b) There is a nbhd U of 0 in X such that x*(U) is bounded subset of K.

(c) The kernel of x* is a closed subset of X.

(d) The kernel of x* is not a proper dense subset of X.

Proof : We shall show that (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a).

If $x^* = 0$, the theorem is true trivially. So we assume $\exists \ x_0 \in X$ such that $x^* (x_0) \neq 0$.

(a) $\Rightarrow$ (c) : Suppose that x* is continuous.

$\Rightarrow (x^*)^{-1} (\{0\})$ is a closed subset of X.

[$\because$ x* is continuous and $\{0\}$ is closed set]

$\Rightarrow$ Ker x* is a closed subset of X.

(c) $\Rightarrow$ (d) : Suppose ker x* is a closed subset of X.

So, $\overline{\ker x^*} = \ker x^* \subsetneq X$.

So, ker x* is not a proper dense subset of X.

(d) $\Rightarrow$ (b) : Suppose, ker x* is not a proper dense subset of X.

$\therefore X \setminus \overline{\ker x^*} \neq \phi$

Fix, $x_0 \in X \setminus \overline{\ker x^*}$. Then there is a balanced nbhd U of 0 such that $x_0 + U \subseteq \overline{X \setminus \ker (x^*)}$

Then $u \in U \Rightarrow x^*u \neq -x^*(x_0)$

If $x^*(u) = -x^*(x_0)$, then $x^*(u + x_0) = 0$

$\Rightarrow u + x_0 \in \ker x^* \subseteq \overline{\ker(x^*)}$

$\Rightarrow u + x_0 \notin X - \overline{\ker(x^*)}$,

a contradiction.

$\therefore x^*(u) \neq x^*(x_0)$ for $u \in U$

Now, U is balanced $\Rightarrow x^*(U)$ is balanced

$\Rightarrow t\, x^*(U) \subseteq x^*(U)$ for $|t| \leq 1$.

If $x^*(U)$ contains an element, then it contains all elements, with absolute value smaller than the absolute value of element.

So, $-x^*(u) \in x^*(U) \Rightarrow x^*(U)$ does not contains elements with absolute value bigger than $|-x^*(x_0)|$

Thus $|x^*(u)| \leq |-x^*(x_0)| \;\forall\; u \in U$

$\Rightarrow x^*(U)$ is bounded.

(b) ⇒ (a) : Suppose, there is a nbhd U of 0 in X such that $x^*(U)$ is bounded subset of K.

W.L.O.G, we assume, $|x^*(x)| < 1$ whenever $x \in U$

Then $|x^*x| < \varepsilon$ whenever $\varepsilon > 0$ and $x \in \varepsilon U$

If $x \in X$ and $\varepsilon > 0$, then

$|x^*y - x^*x| = |x^*(y-x)|$

$< \varepsilon$ whenever $y \in x + \varepsilon U$

$\Rightarrow |x^*y - x^*x| < \varepsilon$ whenever $y \in x \varepsilon U$

$\Rightarrow x^*$ is continuous at $x \in X$.

$\Rightarrow x^*$ is continuous on X.

Lemma : Suppose that C is a convex subset of a TVS X. If $x \in C$, $y \in C^0$ and $0 < t < 1$, then $tx + (1-t)y \in C^0$.

Proof : C is convex $\Rightarrow tC + (1-t)C \subseteq C$ for $0 < t < 1$

$\Rightarrow tC + (1-t)C^0 \subseteq C$

Now, $tC + (1-t)C^0$ is an open set and C^0 is the largest open subset of C.

Hence, $tC + (1-t)C^0 \subseteq C^0$

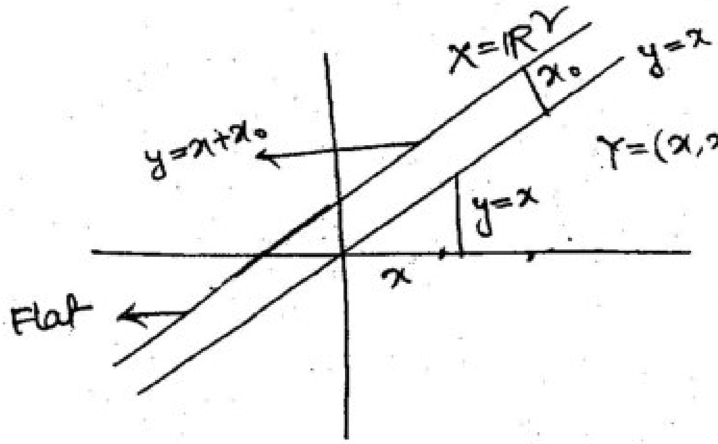
i.e. $tx + (1-t)y \in C^0$ for $x, \in C$

$y \in C^0$.

Flat or Affine Subset of a v.s :

Let X be a v.s. A flat or affine subset of X is a translate of a subspace of X.

That is, a set of the form x + Y, where x X and Y is a subspace of X.



Mazur's Seperation Theorem :

Let X be a TVS and let F and C be subsets of X such that F is tlat and C is conves with non-empty interior (i.e. $C^0 \neq \phi$). If $F \cap C^0 = \phi$, then there is an x* in X* and a real number 's' such that

1. Re $x^*x = s$ for each $x \in F$

2. Re $x^*x \leq s$ for each $x \in C$.

3. Re $x^*x < s$ for each x in C^0.

Proof : Case 1: Let X be a real vector space and $0 \in C^0$.

Let us prove that $C^0 = \{x : x \in X, p(x) < 1\}$,

where $p = U_C$ = minkowski functial

$= \inf \{t > 0 : x \in t_C\}$

Then C is a convex and absorbing subsets of X, so the Minkowski's functional p of C is a sublinear and

$\{x : x \in X, p(x) < 1\} \subseteq C \subseteq \{x : x \in X, p(x) \leq 1\}$

The continuity of multiplication of vectors by scalars implies that for each $x \in C^0$, there is an $s_x > 1$ and $s_x x \in C^0 \subseteq C$.

$\therefore s_x p(x) = p(s_x x) \leq 1$

$\Rightarrow p(x) \leq \frac{1}{s_x} < 1 \Rightarrow p(x) < 1.$

$\therefore\ x \in C^0 \Rightarrow p(x) < 1$

$\therefore\ C^0 \subseteq \{x \in X : p(x) < 1\}$ .....(1)

Conversely $p(x) < 1\ \forall\ x \in X$

$\Rightarrow \inf \{t_x > 0 : x \in t_x C\} < 1$

$\Rightarrow \exists\ t_x > 0$ such that $x \in t_x C$ and $t_x > 1$

Then $p(t_x x) = t_x\, p(x) < 1$

$\Rightarrow t_x x \in C$ and $0 \in C^0$

Since $x = f_x^{-1} (t_x x) + (1 - t_x^{-1})\, 0$, so

$x \in C^0$

$\therefore\ \{x \in X : p(x) < 1\} \subseteq C^0$ .....(2)

From (1) and (2), we get

$C^0 = \{x \in X : p(x) < 1\}$

Let Y be a subspace of X and $x_0 \in X$ such that $F = x_0 + Y$.

By given condition, $F \cap C^0 = \phi$ and $0 \in C^0$

$\Rightarrow 0 \notin F$

Then the subspace Y contuns neither '$-x_0$' nor its negative x_0.

$\therefore$ Each element of the subspace $Y + <\{x_0\}>$ of X has a unique representation of the form $y + \alpha x_0$ where $y \in Y$ and $\alpha \in R$.

Define, $x_0^* : Y + <\{x_0\}> \rightarrow R$ by $x_0^* (y + \alpha x_0) =$, whenever $y \in Y$ and $\alpha \in R$.

Then x_0^* is a linear functional on $Y + <\{x_0\}$.

If $\alpha \geq 0$ and $y \in Y$, then $\alpha^{-1}y + x_0$ is in F and so it is not in C^0.

$\therefore\ x_0^* (y + \alpha x_0) = \alpha \leq \alpha\ p(\alpha^{-1}y + x_0) = p(y + \alpha x_0)$

$\Rightarrow x_0^* (y + \alpha x_0) \leq p(y + \alpha x_0)$ [$\because\ \alpha^{-1}y + x_0 \notin c^0$

$\Rightarrow p(\alpha^{-1}y + x_0) \geq 1$]

Since $p(x) \geq 0$ for each $x \in X$, it is also true that $x_0^* (y + \alpha x_0) \leq p(y + \alpha x_0)$ whenever $y \in Y$ and $\alpha \leq 0$.

So, x_0^* is dominated by p on $Y + <\{x_0\}>$.

By the vector space versionof the Hahn-Branch extension theorem, the functional x_0^* can be extended to a linear functional x^* on X such that

$x^*x \le p(x) \ \forall \ x \in X.$

Now, C^0 is an open nbhd of 0, and so it contains a balanced nbhd U of 0.

Now, $|x^*(u)| = \max \{x^*(-u), x^*(u)\}$

$\le \max \{p(-u), p(u)\} < 1$

whenever $u \in U$

$\therefore$ x^* is bounded in a balanced nbhd U of 0 [$x^*(U)$ is bounded subset of R]

$\Rightarrow x^* \in X^*.$

Finally, $x^*(x) \le p(x) \le 1 \ \forall \ x \in C.$

$x^*(x) \le p(x) < 1 \ \forall \ x \in C^0$

Since $F = x_0 + Y$, it follows that

$x^*x = x_0^*x = 1$ when $x \in C.$

$\therefore$ x^* satisfies the conclusion of the theorem when $s = 1$.

Case 2 : Assume $0 \notin C^0$. Given, $C^0 \neq \phi \ \exists \ x_1 \in C^0$

Then the interior $-x_1 + C^0$ of the convex set $-x_1 + C$ contains 0 and does not intersect the flat subset $-x_1 + F$ of X.

i.e. $(-x_1 + C^0) \cap (-x_1 + F) = \phi$

So, there is an x^* in X^* such that

$x^*(x) = 1 \ \forall \ x \in -x_1 + F$

$x^*(x) \le 1 \ \forall \ x \in -x_1 + C$

$x^*(x) < 1 \ \forall \ x \in -x_1 + C^0$

$\therefore$ It follows that

$x^*(-x_1 + y) = 1 \ \forall \ y \in F$

$x^*(-x_1 + y) \le 1 \ \forall \ y \in C$

$x^*(-x_1 + y) < 1 \ \forall \ y \in C^0$

$\Rightarrow x^*(y) = 1 + x^*(x_1) \ \forall \ y \in F$

$x^*(y) \le 1 + x^*(x_1) \ \forall \ y \in C$

$x^* (y) < 1 + x^* (x_1) \ \forall \ y \in C^0$

$\therefore$ x^* satisfies the conclusion of the theorem when $s = 1 + x^* (x_1)$

Case 3 : Let X be a complex vector space. Let X_r be the real TVS obtained by restricting multiplication of vectors by scalars to R × X. Since every subspace of X is also a subspace of X_r, the set F is flat in X_r.

It follows that there is a continuous real linear functional z^* on X and a real number s such that

$z^*(x) = \forall \ x \in F$

$z^*(x) \leq s \ \forall \ x \in C$

$z^*(x) < \forall \ x \in C^0$

Define, $x^* : X(C) \to C$ by

$x^*(x) = z^*(x) - iz^* (ix) \ \forall \ x \in X.$

Then x^* is a complex-linear functional on X with $Rex^* = z^*$

The continuity of z^* and the vector space operations of X and C gives that $x^* \in X^*$

Hence, $Re \ x^* (x) = s \ \forall \ x \in F$

$re \ x^*(x) \leq s \ \forall \ x \in C$

$Re \ x^*(x) < s \ \forall \ x \in C^0$.

Hence proved.

Corollarly : Let Y be a closed subspace of a locally convex space X. Suppose that $x \in X \setminus Y$. Then there is an x^* in X^* such that $x^*x = 1$ and $Y \subseteq \ker (x^*)$.

Proof : Given Y is closed $\Rightarrow X - Y$ is open

$\Rightarrow X - Y$ is an open nbhd of x.

Since X is locally convex, $\exists$ a convex nbhd c of x such that $C \subseteq X \setminus Y$.

$\therefore C \cap Y = \phi$

$\Rightarrow C^0 \cap Y = \phi$

Also Y is flat in X.

Then by Mazur's theorem, $\exists \ x_0^* \in X^*$ and a real number 's' such that $Re \ x_0^*(x) < s$ when $x \in C$ $(= C^0)$

$Re \ x_0^*(x) = s$ when $x \in Y$.

Since $0 \in Y$, it follows that $s = \text{Re } x^*(0)$

$$\Rightarrow s = 0$$

$\therefore x_0^*(x) = \text{Re } x_0^*(x) - i \text{ Re } x_0^*(ix) = 0 \ \forall \ x \in Y.$

$\therefore Y \subseteq \ker(x_0^*)$

Let $x^* = (x_0^* x)^{-1} x_0^*$. Then $x^*(x) = (x_0^*x)^{-1}(x_0^*x)$

$$\Rightarrow x^*(x) = 1.$$

Again, $x^*(y) = (x^*x)^{-1} x_0^*(y) \ \forall \ y \in Y$

$= (x^*x)^{-1}.0, \ \forall \ y \in Y$

$= 0$

$\therefore \ker(x^*) \supseteq Y$. [$\because \ker(x^*) = \ker(x_0^*) \supseteq Y$]

Thus, $x^*x = 1$ and $Y \subseteq \ker(x^*)$

Corollarly : Suppose that Y is a subspace of an LCS 'X' and that $y \in {}^*Y^*$. Then there is a x^* in X^* whose restriction to Y is y^*.

Proof : If $y^* = 0$, then the zero element of x^* extends the zero element of y^* to X.

So, let us assume that, $y^* \neq 0$. Then there is a $y_0 \in Y$ such that $y^*(y_0) = 1$

Let $Z = \overline{\ker(y^*)}$, where the closure is taken in X. The continuity of y^* and the fact that the topology of Y is inherited from X together imply that $y_0 \notin z$.

So, there is an x^* in X^* such that

$$x^*(y_0) = 1 \text{ and } z \subseteq \ker(x^*)$$

We have to show $x^*|y = y^*$ (i.e. $x^*(y) = y^*(y), \ \forall \ y \in Y$).

If $y \in Y$, then

$y^*[(y^*y)\, y_0 - y] = (y^*y)(y^*y_0) - y^*y$

$= y^*y - y^*y$ [as $y^*y_0 = 1$]

$= 0$

$\Rightarrow (y^*y)y_0 - y \in \ker(y^*) \subseteq \overline{\ker(y^*)} = z$

$\Rightarrow (y^*y)y_0 - y \in \ker(x^*)$

$\therefore x^*(y) = x^*(y) + x^*[(y^*y)\, y_0 - y]$

$= x^*(y) + x^*(y_0)yx(y) - x^*(y)$

$= x^*(y_0)y^*(y)$

$= y^*(y) \ \forall \ y \in Y \quad$ as $x^*(y_0) = 1$

$\therefore x^*/_Y = Y^*$.

Corollarly : If x and y are different elements of a Hausdorff LCS X, then there is an xx in Xx such that $x^*(x) \neq x^*(y)$.

Proof : Here $x \neq y \Rightarrow x - y \neq 0$

$\therefore$ x – y is not in the closed subspace {0} of X.

Thus, $\exists \ x^* \in X^*$ such that $x^* (x - y) = 1$

$$\Rightarrow x^*(x) - xx(y) = 1$$

$$\Rightarrow x^*(x) \neq x^*(y).$$

1.8 Metrizable Vector Topology :

Definition 1 : A topological space (X, T) is said to be **topologically complet** if the topology T is induced by complete metricd.

$$\text{i.e. } T_d = T.$$

Definition 2 : A TVS (X, T) is called F-space if T is topologically complete.

Definition 3 : A locally convex F-space is called a Frechet space.

Metrization Theorem :

Suppose that X is a Hausdorff TVS whose topology has a countable local basis. Then the topology of Xis induced by an invariant metric such that the open balls centred at the origin are balanced.

If X is locally convex TVS, then its topology is induced by metric which is an invariant metric such that the open balls centred at the origin are convex and balanced.

Proof : Let T be the given vector topology for X.

Then the topology T has a countable local basis $\{B(2^{-n}) : n = 0, 1, 2, ...\}$ such that, for each n, the set $B(2^{-n})$ is balanced (and, if X is an LCS, then convex) and $B(2^{-n-1}) + B(2^{-n-1}) \subseteq B(2^{-n})$.

Let $B_0 = \{B(2^{-n}) : n = 0, 1, 2, ...\}$

Let B = B(1), then let B and $\{B(2^{-n}) : n \in N\}$ be such that $B_0 = B \cup \{B(2^{-n}) : n \in N\}$

Then the map $t \rightarrow B(t)$ form $\{2^{-n} : n = 0, 1, 2, ...\}$ into T is extended to $(0, \infty)$ in such a way that each B(t) is a T-nbhd of 0 that is balanced (and convex if each member of B_0 is convex, as X is LCS) and $B(s) \subseteq B(t)$ whenever $0 < s < t$.

Then the formula $f(x) = \inf \{t : t > 0, x \in B(t)\}$ defines a T-continuous non-negative-real-valued function on X such that $f(0) = 0$ and such that $f(x) = f(-x)$ and

$f(x + y) \leq f(x) + f(y)$ whenever $x, y \in X$.

If x is a nonzero member of X, then the fact that B_0 is a local basis for the Hausdorff topology T implies that there is a non-negative integer n such that $x \notin B(2^{-n})$.

Hence, $f(x) \geq 2^{-n} > 0$

Define, $d(x, y) = f(x - y)$. Then to show that of is an invariant metric on X.

M_1) $d(x, y) = f(x - y) \geq 0$

M_2) $d(x, y) = f(x - y) = f(y - x) = d(y, x)$

M_3) $x = y \Leftrightarrow x - y = 0 \Leftrightarrow f(x - y) = 0 \Leftrightarrow d(x, y) = 0$

or [and if possible let $x \neq y$

Then $x - y$ 0

$\therefore$ $f(x - y) \geq 2^{-n} > 0$, a contradiction

$\therefore$ $d(x, y) = 0 \Rightarrow x = y$.]

M4) $d(x, y) + d(y, z) = f(x - y) + f(y - z)$

$$\geq f(x - y + y - z)$$

$$= f(x - z)$$

$$= d(x, z)$$

$\Rightarrow d(x, y) + d(y, z) \geq d(x, z)$

$\therefore$ d is a metric on X. Also,

$d(x + z, y + z) = f(x + z - y - z)$

$$= f(x - y) = d(x, y)$$

$\therefore$ d is an invariant metric on X.

Let T_d be the topology induced by metric d. We have to show that $T = T_d$.

Fo this we shall show that T and T_d have the same local base at origin.

For each positive t, let $V(t) = U \{B(s) : 0 < s < t\}$. Then each V(t) is a T-nbhd of 0 that is balanced (and convex if each member of B_0 is convex) and $\{V(t) : t > 0\}$ is a local basis for T.

Now if $t > 0$, $V(t) = \{x : x \in X, f(x) < t\}$

$$= \{x \in X : d(x, 0) < t\}$$

$$= B(0, t)$$

$\therefore$ Each v(t) is the d-open ball of radius t centred at 0, so the d-open balls centred at 0 are all balanced (and convex if each member of B_0 is convex).

$$\begin{aligned} \text{Let } |s| \le 1 \quad \therefore sV(t) &= \{s\, x \in X : f(x) < t\} \\ &= y \in x : f\left(\frac{y}{s}\right) < t\} \\ &= \{y \in x : f(y) < s\, t \le t\} \\ &= V(t) \end{aligned}$$

$\Rightarrow sV(t) \subseteq V(t) \therefore V(t)$ balanced.

Let U (s, r) be the d-open ball of radius r centred at x. Then

$$\begin{aligned} U(x, r) &= \{y \in x : f(y-x) < r\} \\ &= x + \{y : y \in X\ f(y) < r\} \\ &= x + V(r) \text{ for each +ve 'r'} \end{aligned}$$

Now, $\{U(x, r) : x \in X, r > 0\}$ is a basis for T_d

and $\{x + V(r) : x \in X, r > 0\}$ is a basis for T.

$\therefore T = T_d$.

Finally, if T is a locally convex topology, then the selection of each member of B_0 to be convex assures that each d-open ball x + V(r) is convex.

Hence the theorem.

Eidenheit's Theorem :

Let X be a TVS and C_1 and C_2 be two convex subsets of X such that C_2 has non-empty interior. If $C_1 \cap C_2^0 = \phi$, then tthere is a x* X* and a real s such that

1) $\operatorname{Re} x^*(x) \ge s \ \forall\ x \in C_1$.

2) $\operatorname{Re} x^*(x) \le s \ \forall\ x \in C_2$

3) $\operatorname{Re} x^*(x) < s \ \forall\ x \in C_2^0$.

Proof : Since the flat subset {0} of X does not intersect the non-empty convex open subset $C_2^0 - C_1$ of X, then Mazurs seperation theorem, $\exists$ an $x^* \in X^*$ such that for each $x_2 \in C_2^0$ and $x_1 \in C_1$,

$$\begin{aligned} \operatorname{Re} x^*(x_2) - \operatorname{Re} x^*(x_1) &= \operatorname{Re} x^*(x_2 - x_1) \\ &< \operatorname{Re} x^* 0 = 0 \end{aligned}$$

$\therefore$ There is a real number s such that

$\sup \{Re\ x^*(x) : x \in C_2^0\} \leq s \leq \inf \{Re\ x^*x : x \in C_1\}$

$\therefore$ $Re\ x^*(x) \geq s\ \forall\ x \in C_1$.

Now fix an $x_2 \in C_2^0$ and $x_1 \in C_1$. The continuity of vector space operations of X implies that there is a to such that $0 < t_0 < 1$ and

$t_0x_1 + (1 - t_0)\ x_2 \in C_2^0$.

$\therefore$ $s \geq Re\ x^* [t_0x_1 + (1 - t_0)\ x_2$

$= t_0\ Re\ x^*\ (x_1) + (1 - t_0)\ Re\ x^*\ (x_2)$

$> t_0\ Re\ x^*\ (x_2) + (1 - t_0)\ Re\ x^*\ (x_2)$

$= Re\ x^*\ (x_2)$

$\therefore$ $Re\ x^*\ (x) < s\ \forall\ x \in C_2^0$.

Finally, let $x \in C_2$ and $x_2 \in C_2^0$ that previously fixed. If $0 < t < 1$, then

$t_x + (1 - t)x_2 \in C_2^0$ and so

$tRex^*(x) + (1 - t)\ Re\ x^*(x_2)$

$= Re\ x^*(f_x + (1 - t)x_2) \leq s$

Lettiy $t \rightarrow 1$, $Re\ x^*\ (x) < s\ \forall\ x \in C_2$.

Unit 2
Bounded linear Operator

2.1 Definition :

Let X and Y be TVS. A liear operator T from X into Y is bounded if T(B) is a bounded subset of Y whenever B is a bounded subset of X.

Theorem :

Let $T : X \to Y$ from a TVS X into a TVS Y is a linear operator.

Then the following are equivalent

(a) The operator T is continuous.

(b) The operator T is continuous at 0.

Further each of (a) and (b) implies

(c) The operator T is bounded.

If X is metrizable, then (a), (b), (c) are equivalent.

Proof : (a) ⇒ (b) : T is continuous on X.

$\Rightarrow$ T is continuous at every $x \in X$

$\Rightarrow$ T is continuous at $x = 0$

(b) ⇒ (a) : Let $T : X \to Y$ be continuous at 0

Let $x \in X$. Let (x_α) be a net in X such that

$\Rightarrow x_\alpha \to x$

$\Rightarrow x_\alpha - x \to 0$

$\Rightarrow T(x_\alpha - x) \to T(0) = 0$

$\Rightarrow T_{x_\alpha} - T_x \to 0$

$\Rightarrow T_{x_\alpha} \to T_x$

So, T is continuous at x. Since x is orbitrary, T is continuous on X.

(a) ⇒ (c) Let $T : X \to Y$ be continuous. Let B be a bounded subset of X. Let V be a nbhd of 0 in Y.

Then continuity of T implies that thre is a nbhd U of 0 in X such that $T(U) \subseteq V$.

Again, B is bounded $\Rightarrow \exists\ s > 0$ such that $B \subseteq tU,\ \forall\ t > s$

$\Rightarrow T(B) \subseteq T(tU)\ \forall\ t > s$

$\Rightarrow T(B) \subseteq tT(U) \subseteq tV\ \forall\ t > s$

$\Rightarrow T(B) \subseteq tV,\ \forall\ t > s.$

$\therefore$ T(B) is a bounded subset of Y.

$\Rightarrow$ T is a bounded linear operator.

$\therefore$ (a) $\Rightarrow$ (c) $\therefore$ (b) $\Rightarrow$ (a) $\Rightarrow$ (c)

$\therefore$ (a) or (b) $\Rightarrow$ (c)

(c) $\Rightarrow$ (b)

Finally, suppose that the topology of X is induced by a metric (invariant)

Suppose T is bounded linear operator. Let (x_n) be a sequence in X converging to 0.

For each positive integer k, there is a positive integer n_k such that kx_n lies in the open ball of radius k^{-1} centred at 0 whenever $n \geq n_k$.

Therefore, there is a nondecreasing sequence (k_n) of positive integers such that $k_n \to \infty$ and $k_n x_n \to 0$. Since the set $\{k_n x_n : n \in N\}$ and the operator T are bounded, so is the set $\{k_n T_{x_n} : n \in N\}$. Let W be a nbhd of 0 in Y and let 's' be a positive number such that $\{k_n T_{x_n} : n \in N\} \subseteq tW\ \forall\ t > s$.

Hence there is a positive integer n_w such that

$k_n T_{x_n} \in K_n W$ whenever $n \geq n_w$

$\Rightarrow T_{x_n} \in W$ for all sufficiently large n.

$\Rightarrow$ The sequence (T_{x_n}) converges to 0.

$\Rightarrow$ The operator T is continuous at 0.

$\therefore$ (c) $\Rightarrow$ (b) but (a) $\Leftrightarrow$ (b).

$\therefore$ (c) (a) also.

2.2 Definition :

A vector topology T on X is said to be locally bounded on X if some nbhd of the origin in X is bounded.

Theorem (Metrizability) :

Every locally bounded Hausdorff vector topology is induced by a metric.

Proof : Suppose that a Hausdorff TVS 'X' has a bounded nbhd V of 0.

It follows that if U is a nbhd of 0, then there is a positive integer n_U such that

$V \subseteq n_U\, U\, n_U^{-1}\, V \subseteq U$.

$\therefore$ $\{n_u^{-1} V : n \in N\}$ is a countable local basis at 0 for the topology of X. Hence the topology of X is metrizable.

Theorem (Normability) : A topology for a vector space is induced by a norm if and only if it is a Hausdorff vector topology that is locally bounded and locally convex.

Proof : Suppose TVS (X, T) is normable. Then $\exists$ a norm $\|.\|$ on X such that $T = T_{\|.\|}$.

Norm topology is Hausdorff, as $x \neq y$

$$\Rightarrow B(x, r) \cap B(y, r) = \phi$$

$$\text{where } r < \frac{\|x - y\|}{2},$$

It is locally bounded, since B(x, r) is a nbhd of x and B(x, r) is bounded, as it is absorbed by

$B(x, t) = x + B(0, t), \ \forall\ t > r$.

Also, $\{B(0, r) : r > 0\}$ is convex local base at 0,

since $B(0, r) = \{x : \|x\| < r\}$ is convex.

Hence a normable TVS is Hausdorff, locally bounded and locally convex.

Conversely, suppose that the topology T of a Hausdorff TVS 'X' is locally bounded and locally convex.

$\Rightarrow$ There is a nbhd V of 0 that is bounded, balanced, convex and absoring.

We have, the Minkowski functional

$p(x) = \inf \{t > 0 : x \in tv\}$ is a seminorm on X.

We have to show that (i) p is a norm on X

(ii) $T_p = T$.

(i) Let $x(\neq 0) \in X \Rightarrow X - \{x\}$ is a nbhd of 0, as T is Hausdorff.

Again, V is bounded $\Rightarrow V \subseteq s^{-1} (X - \{x\})\ \forall\ s^{-1} > t^{-1} > 0$

$\Rightarrow sV \subseteq X - \{x\}$ whenever, $0 < s < t$

$\Rightarrow x \notin s\,V$, whenever, $0 < s < t$.

$\therefore$ $p(x) = \inf \{s > 0 : x \in sV\}$ and $x \notin sV$

$\Rightarrow p(x) > s > 0$.

Thus $x \neq 0 \Rightarrow p(x) \neq 0$

$\Rightarrow p(x) = 0$ gives $x = 0$

$\therefore$ p is a norm on X.

(ii) We have,

$\{x \in X : p(x) < 1\} \subseteq V \subseteq \{x \in X : p(x) \leq 1\}$.

Let $x \in V$, then it follows from the T-continuty of multiplication of vectors by scalars and the fact that V is a T-open, that there is some $r > 1$ such that $rx \in V$

$\Rightarrow rx \in V \subseteq \{x \in X : p(x) \leq 1\}$

$\Rightarrow p(rx) \leq 1$

$\Rightarrow p(x) \leq \frac{1}{r} < 1$

$\Rightarrow p(x) < 1$

$\therefore V = \{x \in X : p(x) < 1\} = B(0, 1)$ is an p-open unit ball.

$\therefore \{n^{-1} V : n \in N\}$ is a local base at 0 for p-to pology of X.

Also, V is bounded $\Rightarrow$ For a nbhd U of 0, $\exists\, n > 0$

such that

$n^{-1} v \subseteq U$

$\therefore \{n^{-1}V : n \in N\}$ is a local base at 0 for T.

$\therefore$ T and T_p are the same topology i.e. $T = T_p$.

2.3 Definition : A family of linear operators from a TVS X into a TVS Y is uniformly bounded if $\cup\{T(B) : T \in \mathfrak{F}\}$ is a bounded subset of Y, whenever B is a bounded subset of X.

Theorem (The Uniform Boundness Principle for F-space)

(Banach Steinhaus Theorem)

Let $\mathfrak{F}$ be a family of bounded linear operators from an F-space X into a TVS Y. Suppose that $\{T_x : T \in \mathfrak{F}\}$ is bounded for each x in X. Then $\mathfrak{F}$ is uniformly bounded. In short, the pointwise boundness of implies its uniform boundness.

Proof : Assume that $\mathfrak{F} \neq \phi$. Let B be a bounded subset of F-space X and U a nbhd of the origin 0_y of Y.

The theorem will be proved once a positive 's' is found such that $\cup\{T(B) : T \in \mathfrak{F}\} \subseteq t\,U$ whenever $t > s$.

Let V be a balanced nbhd of 0_Y such that $\overline{V + V} \subseteq U$.

Let $S = \cap\{T^{-1}(\overline{V}) : T \in \mathfrak{F}\}$. Then S is closed subset of X, because of the continuity of each T in $\mathfrak{F}$

If $x \in X$, then the boundness of $\{Tx : T \in \mathfrak{F}\}$ gives that

$\exists\, n_x > 0$ such that $\{Tx : T \in \mathfrak{F}\} \subseteq n_x V$

$$\Rightarrow T_x \in n_x V \subseteq n_x \overline{V}$$

$$\Rightarrow x \in n_x T^{-1}(\overline{V}),\ \forall\ T \in \mathfrak{F}$$

$$\Rightarrow x \in n_x S$$

$\therefore$ It follows that $X = \cup\{nS : n \in N\}$.

By the Baire category theorem, one of the closed sets nS, and hence S itself, must have nonempty interior. i.e. $S^0 \neq \phi$.

Let x_0 be any point in S^0 and let $W = x_0 - S^0$ be a nbhd of the origin of X.

For each T in $\mathfrak{F}$,

$$\begin{aligned} T(W) &\subseteq T(x_0) - T(S^0) \\ &\subseteq T(x_0) - T(S)\ [\because\ S^0 \subseteq S] \\ &\subseteq \overline{V} - \overline{V} \\ &\subseteq \overline{V + V} \subseteq U \\ &\Rightarrow T(W) \subseteq U. \end{aligned}$$

The boundness of B yields a positive s such that

$B \subseteq tW$ whenever $t > s$

$\Rightarrow T(B) \subseteq tT(W) \subseteq tU$ whenever $t > s$. $\forall\ T \in \mathfrak{F}$

$\Rightarrow T(B) \subseteq tU$ whenever $t > s\ \forall\ T \in \mathfrak{F}$

$\Rightarrow \cup\{T(B) : T \in \mathfrak{F}\} \subseteq t\,U$ whenever $t > s$.

$\therefore \cup\{T(B) : T \in \mathfrak{F}\}$ is a bounded subset of Y, whenver B is a bounded subset of X.

Corollary : Let (T_n) be a sequence of bounded linear operators from an T-space X into a Hausdorff TVS Y such that $\lim_n T_n x$ exists for each x in X. Define $T : X \to Y$ by $T_x = \lim_n T_n x$. Then T is a bounded linear operator from X into Y.

Proof :

Here $T_x = \lim_n T_n(x)$

Let $x, y \in X$ and $\alpha, \beta \in F$.

$$\therefore T(\alpha x + \beta y) = \lim_n T_n(\alpha x + \beta y)$$
$$= \alpha \lim_n T_n(x)\, \beta \lim_n T_n(y)$$
$$= \alpha T(x) + \beta T(y).$$

$\therefore$ T is linear.

Let B be a bounded subset of X. To show T(B) is a bounded subset of Y.

Now, $\lim_n T_n x$ exists $\Rightarrow (T_n x)$ is a convergent sequence.

$\Rightarrow (T_n x)$ is a cauchy sequence.

$\Rightarrow (T_n x)$ is a bounded sequence.

$\Rightarrow (T_n)$ is pointwise bounded.

$\therefore$ By the uniform boundness theorem, (T_n) is uniformly bounded.

$\therefore \bigcup_n \{T_n(B) : n \in N\}$ is a bounded subset of Y.

$\Rightarrow \overline{\bigcup_n \{T_n(B) : n \in N\}}$ is a bounded subset of Y.

Clearly, $T(B) \subseteq \overline{\bigcup_n \{T_n(B) : n \in N\}}$, which is a bounded set

$\Rightarrow$ T(B) is bounded subset of Y.

$\therefore$ T is a bounded linear operator.

Remark

$y \in T(B) \Rightarrow y = T_x,\ x \in B.$

$\Rightarrow y = \lim_n T_n x \Rightarrow y \in \overline{T_n(x)} \subseteq \overline{\bigcup_n \{T_n(B) : n \in N\}} \Rightarrow y \in \overline{\bigcup_n \{\text{In}(B)\}}$

The Open Mapping Theorem for F-Spaces :

Every bounded linear operator from an F-space onto an F-space is an open mapping.

Proof : We use the fact that if d is an invariant metric on (X, T) and $x_1, x_2, \ldots x_n \in X$,

then $d\left(\sum_{j=1}^{n} x_j, 0\right) \le \sum_{j=1}^{n} d(x_j, 0)$

For n = 2, $d(x_1 + x_2, 0) \leq d(x_1 + x_2, x_2) + d(x_2, 0)$

$\leq d(x_1, 0) + d(x_2, 0)$

$$\left[\begin{array}{l} d(x_1 + x_2, 0) = d(x_1 - x_2) \leq d(x_1, 0) + d(0, -x_2) \\ \qquad = d(x1, 0) + d(x2, 0) \end{array}\right]$$

Let $d(x_1 + x_2 + ... + x_{n-1}, 0) \leq \sum_{j=1}^{n-1} d(x_j, 0)$

$$\therefore d(x_1 + x_2 + ... + x_n, 0) \leq d(x_1 + x_2 + ... + x_{n-1}, 0) + d(x_n, 0)$$

$$\leq \sum_{j=1}^{n-1} d(x_j, 0) + d(x_n, 0)$$

$$= \sum_{j=1}^{n} d(x_j, 0)$$

$$\therefore d\left(\sum_{j=1}^{n} x_j, 0\right) \leq \sum_{j=1}^{n} d(x_j, 0)$$

Let T be a bounded linear operator from an F-space X onto an F-space Y., and let N be a nbhd of the origin O_x in X. Suppose that it were shown that T(N) must include a neighbourhood of the origin O_y of Y.

It would follow that if G is an open subset of X and $x \in G$, then

$T(G) = T_x + T(-x + G) \supseteq T_x + [T(-x + G)]^0$, and so T(G) is an open subset of Y, since it would include a nbhd of each of its points.

$\therefore$ It is enough to prove that $O_y \in [T(N)]^0$

We first show that $O_y \in [T(N)]^0$

Let V be a balanced nbhd of O_x such that $V + V \subseteq N$.

If $(\overline{T(V)})^0 \neq \phi$, then $O_y \in (\overline{T(V)})^0 - (\overline{T(V)})^0$

$\subseteq \overline{T(V)} - \overline{T(V)}$

$= \overline{T(V)} + \overline{T(V)}$

$\subseteq \overline{T(N)}$

$\therefore \overline{T(N)}$ includes that nbhd $(\overline{T(V)})^0 - (\overline{T(V)})^0$ of O_Y.

$\therefore O_Y \in (T(N))^0$, if $(T(V))^0 \neq \phi$

Now, we shall show that $[T(V)]^0 \neq \phi$

Since T is onto, so T(X) = Y and since V is absorbing

We get $x \in X \Rightarrow x \in nV$ whenever $n > n_1 > 0$

$$\Rightarrow x \in \bigcup_n nV$$

$\therefore X = \bigcup_n (nV)$.

$$\therefore Y = T(X) = T(\bigcup_n (nV))$$

$$= \bigcup_n T(nV)$$

So, by Baire Calegory Theorem, there must exists a positive integer n_0 such that

$T(n_0V)$ is not nowwhere dense in Y.

$\therefore [\overline{T(n_0V)}]^0 \neq \phi$

$\Rightarrow n_0 [\overline{T(v)}]^0 \neq \phi$

$\Rightarrow [\overline{T(v)}]^0 \neq \phi$

Hence, $O_y \in [\overline{T(N)}]^0$

Let d_X, d_Y be the complete invariant metrices inducing the topologies of X and Y respectively. Let $U_x(r)$ and $\cup_Y(r)$ denote the open balls of radius r centred at O_x and O_Y respectively when $r > 0$, and let ε be a positive number such that $U_x(\varepsilon) \subseteq N$.

Then there is a sequence (δ_n) of positive reals converging to 0 such that $\cup_Y(\delta_n) \subseteq \overline{T(U_x(2^{-n}\varepsilon))}$ whenever $n \in N$.

Let Y_0 be an arbitrary element of $\cup_Y(\delta_1)$. The tTheorem will be proved once it is shown that there is an x_0 in $U_x(\varepsilon)$ such that $Tx_0 = y_0$.

Since $y_0 \in \cup_y(\delta_1) \subseteq T(U_x(2^{-1}\varepsilon))$, there is an x_1 in $U_x(2^{-1}\varepsilon)$ such that $d_Y(y_0, Tx_1) < \delta_2$.

Since $y_0 - Tx_1 \in \cup_Y(\delta_2) \subseteq T(U_x(2^{-2}\varepsilon))$, there is an x_2 in $U_x(2^{-2}\varepsilon)$ such that $d_Y(y_0, Tx_1 + Tx_2)$

$= d_Y(y_0 - Tx_1, Tx_2) < \delta_2$.

Continuing in this way, we get a sequence (x_n) in X such that $x_n \in U_x(2^{-n}\varepsilon)$ and $d_Y\left(y_0, \sum_{j=1}^{n} Tx_j\right)$

$< \delta_{n+1} \ \forall \ n \in N$.

If $m_1, m_2 \in N$ and $m_1 < m_2$, then

$$d_X\left(\sum_{j=1}^{m_2} x_j, \sum_{j=1}^{m_1} x_j\right) = d_X\left(\sum_{j=m_1+1}^{m_2} x_j, O_x\right)$$

$$\leq \sum_{m_1+1}^{m_2} d_x(x_j, O_x)$$

$$< \sum_{m_1+1}^{\infty} \frac{\varepsilon}{2^j}$$

$$= 2^{-m_1}\varepsilon \to 0 \text{ as } m_1 \to \infty$$

$\therefore$ The partial sums of the formal series $\sum_n x_n$ form a cauchy sequence and so that $\sum_n x_n$ converges.

Let $x_0 = \sum_n x_n$

Since, $\lim_n d_Y\left(y_0, T\left(\sum_{j=1}^{n} x_j\right)\right) = 0$, it follows

that

$$Tx_0 = T\left(\lim_n \sum_{j=1}^{n} x_j\right)$$

$$= \lim_n T\left(\sum_{j=1}^{n} x_j\right) = y_0$$

$$\Rightarrow Tx_0 = y_0$$

Finally, $d_x(x_0, O_x) = \lim_n d_x\left(\sum_{j=1}^{n} x_j, O_x\right)$

$$\leq \sum_{j=1}^{\infty} d_x(x_j, O_x)$$

$$< \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon$$

$$\Rightarrow x_0 \in U_x(\varepsilon) \subseteq N$$

$\therefore y_0 = Tx_0 \in T(U_x(\varepsilon)) \subseteq T(N)$

$\Rightarrow y_0 \in [T(N)]^0$ Hence proved.

The Closed Graph Theorem for F-spaces :

Let T be a linear operator from an F-space X into an F-space Y. Suppose that whenever a sequence (x_n) in X converges to some x in X and (Tx_n) converges to some y in Y, it follows that $y = T_x$. Then T is bounded.

Proof : Let d_X and d_Y be complete invariant metrics inducing the topologies of X and Y respectively. For each pair of elements (x_1, y_1) and (x_2, y_2) of $X \times Y$, let

$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = [(d_x(x_1, x_2))^2 + (d_Y(y_1, y_2))^2]^{1/2}$

Then $d_{X \times Y}$ is a complete invariant metric that induces the product topology of $X \times Y$. Hence $X \times Y$ is an F-space when given its product topology and the usual vector space operations for a vector space sum.

Let $G = \{(x, T_x) : x \in X\}$ i.e. let G be the graph of T in $X \times Y$.

Let $((x_n, Tx_n))$ be a sequence in G such that

$(x_n, Tx_n) \to (x, y)$ in $X \times Y$

We have to show $(x, y) \in G$.

Now,

$(x_n, Tx_n) \to (x, y)$ in $X \times Y$

$\Rightarrow d_{X \times Y}((x_n, Tx_n), (x, y)) \to 0$

$\Rightarrow [(d_x(x_n, x))^2 + (d_Y(Tx_n, y))^2]^{1/2} \to 0$

$\Rightarrow d_x(x_n, x) \to 0$ and $d_Y(Tx_n, y) \to 0$

$\Rightarrow x_n \to x$ and $Tx_n \to y$

$\Rightarrow y = Tx$ (by Given condition)

$\therefore (x, y) = (x, Tx) \in G$.

$\therefore$ G is a closed subspace of F-space $X \times Y$

$\Rightarrow$ G is also an F-space.

Since the map $(x, Tx) \mapsto x$ from G onto X is one-to-one bounded linear operator, its inverse is bounded. So the map $x \mapsto (x, Tx) \mapsto Tx$ is itself a bounded linear operator.

Proposition : Let X be a set let $\mathfrak{F}$ be a family of functions and $\{(Y_f, \mathfrak{F}_f) : f \in \mathfrak{F}\}$ a family of topological spaces such that each f in $\mathfrak{F}$ maps X into the corresponding Y_f. Then there is a smallest, topology for X w.r.t which each member of $\mathfrak{F}$ continuous.

That is, there is a unique topology $T_{\mathfrak{F}}$ for X such that–

1. Each f in $T_{\mathfrak{F}}$ continuous.

2. if T is any topology for X such that each f in $\mathfrak{F}$ is Y-continuous, then $T_{\mathfrak{F}} \subseteq T$

The topology $T_{\mathfrak{F}}$ has $\{f^{-1}(U) : f \in \mathfrak{F}, U \in T_{\mathfrak{F}}\}$ as a subbasis.

Proof : Let $\mathfrak{S} = \{f^{-1}(U) : f \in \mathfrak{F}, U \in T_{\mathfrak{F}}\}$.

Let $T_{\mathfrak{F}}$ be the topology generated by the subbasis .

Since $\mathfrak{S} \subseteq T_{\mathfrak{F}}$, every member of $\mathfrak{F}$ is $T_{\mathfrak{F}}$ continuous.

Now, suppose that T is a topology for X such that every member of $\mathfrak{F}$ is T-continuous.

Then $\mathfrak{S} \subseteq T$, so $T_{\mathfrak{F}} \subseteq T$

Let T_1 be another topology

Then $T_1 \subseteq T_{\mathfrak{F}}$ but $T_{\mathfrak{F}} \subseteq T_1$

$\Rightarrow T_1 = T_{\mathfrak{F}}$

$\therefore$ $T_{\mathfrak{F}}$ is the unique topology for X.

2.4 Definition : Let all notion be as in the proceding proposition. Then the set $\mathfrak{F}$ is a topologizing family of functions for X, and the topology $T_{\mathfrak{F}}$ is the $\mathfrak{F}$ topology of X or the topology σ (X, $\mathfrak{F}$) or the **weak topology** of X induced by $\mathfrak{F}$.

The collection $\{f^{-1}(U) : f \in \mathfrak{F}, U \in Y_f\}$ is the standard subbasis for this topology.

Theorem : Let X be a set and $\mathfrak{F}$ be a topologizing family of functions for X. Suppose that (x_α) is a net in X and x is a member of X. Then $x_\alpha \to x$ with respect to the $\mathfrak{F}$-topology if and only if $f(x_\alpha) \to f(x)$ $\forall$ $f \in \mathfrak{F}$.

Proof : Let $x_\alpha \to x$ w.r.t $\mathfrak{F}$-topology for X.

Now, $f \in \mathfrak{F} \Rightarrow$ f is continuous w.r.t $\mathfrak{F}$-topology on X

$\therefore$ $f(x_\alpha) \to f(x)$ $\forall$ $f \in \mathfrak{F}$

Conversely suppose that $f(x_\alpha) \to f(x)$ for each f in $\mathfrak{F}$

If $f \in \mathfrak{F}$ and U is a nbhd of f(x), then $f^{-1}(U)$ is a nbhd of x.

So, there exists $\alpha_{f,U}$ such that

$f(x_\alpha) \in U$ if $\alpha_{f,U} \leq \alpha$

$\Rightarrow x_\alpha \in f^{-1} U$ if $\alpha_{f,u} \leq \alpha$

$\Rightarrow x_\alpha \to x$ w.r.t the $\mathfrak{F}$-topology.

2.5 Weak Topology : Let X be a normed space. Then the topology for X induced by the topologizing family X* is the weak topology of X or the X* topology of X or the topology σ (X, X*).

Theorem : A subset of a normed space is bounded if and only if it is weakly bounded.

Proof : Let B be a norm bounded subset of $(X, \|.\|)$

$\Rightarrow$ B is absorbed by every $\|.\|$ nbhd of 0.

$\Rightarrow$ B is absorbed by every weak nbhd of 0.

$\Rightarrow$ Be is weakly bounded.

Conversely, suppose that A is a weakly bounded subset of a normed space X.

It may be assumed that A is nonempty.

Let Q be the natural map from X into X^{**}.

Then Q(A) is a non-empty collection of bounded linear functionals on the Banach space X^*.

For each x^* in X^*,

$$\sup \{|(Qx)(x^*)| : x \in A\} = \sup \{|x^*(x)| : x \in A\} < \infty$$

It follows from the uniform boundness principle that

$$\sup \{\|x\| : x \in A\} = \sup \{\|Q x\| : x \in A\} < \infty$$

$\therefore$ A is norm bounded.

Weak* topology

Let X be a normed space and let Q be the natural map from X into X^{**}. Then the topology for X^* induced by the topologizing family Q(X) is the weak* topology of X^* or the X topology of X^* or the topology $\sigma(X^*, X)$.

The Weak* topology of the dual space of a normed space X is the smallest topology for X^* such that for each x in X, the linear functional $x^* \to x^*x$ on $X^*$is continuous w.r.t that topology.

The Banach-Alaoglu Theorem

If V is a nbhd of 0 in a TVS X and if $K = \{\wedge \in X^* : |\wedge x| \le 1$, for every $x \in V\}$, then K is weak*-compact.

Proof : Since every nbhd of 0 is absorbing, so V is absorbing nbhd of 0.

Then for each $x \in X$, $\exists\ \gamma(x) < \infty$ such that $x \in \gamma(x)V$.

Hence, $|\wedge x| \le \gamma(x)\ (x \in X, \wedge \in K)$ .....(1)

Let $D_x = \{\alpha \in F : |\alpha| \le \gamma(x)\}$. Let T be the product topology on P, the cartesian product of all D_x, one for each $x \in X$.

Since each D_x is compact, so P is also compact. The elements of P are the functions f on X (linear or not) that satisfy $|f(x)| \le \gamma(x)\ (x \in X)$ .....(2)

Thus, $K \subset X^* \cap P$. If follows that K inherits two topologies : one from X^* and the other, T from P.

We will see that

(a) These two topologies coincide on K and

(b) K is a closed subset of P.

Since P is compact, (b) implies that K is T-compact and then (a) implies that K is weak*-compact.

Fix some $\wedge_0 \in K$. Choose $x_i \in X$, for $1 \leq i \leq n$, choose $\delta > 0$.

Put $W_1 = \{\wedge \in X^* : |\wedge x_i - \wedge_0 x_i| < \delta \text{ for } 1 \leq i \leq n\}$

and $W_2 = \{f \in P : |f(x_i) - \wedge_0 x_i| < \delta \text{ for } 1 \leq i \leq n\}$

Let n, x_i, and δ range over all admissible values. The resulting sets W_1, then form a local base for the weak* topology of X^* at $\wedge_0$ and the sets W_2 form a local base for the product topology Y of P at $\wedge_0$.

Since, $K \subset P \cap X^*$, we get

$W_1 \cap K = W_2 \cap K$

$\Rightarrow$ (a) is proved.

Next, suppose f_0 is in the T-closure of K. Choose $x \in X$, $y \in X$, scalars α and β and $\varepsilon > 0$. The set of all $f \in P$ such that $|f - f_0| < \varepsilon$ at x, at y and at $\alpha x + \beta y$ is a T-nbhd of f_0. Therefore K contains such an f.

Since this f is linear, we have,

$$f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)$$
$$= (f_0 - f)(\alpha x + \beta y) + \alpha(f - f_0)(x) + \beta(f - f_0)(y)$$

$\Rightarrow |f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\,\varepsilon$

Since $\varepsilon > 0$ was arbitrary, we get

$f_0(\alpha x + \beta y) = \alpha f_0(x) - \beta f_0(y)$

$\Rightarrow f_0$ is linear.

Finally, if $x \in V$ and $\varepsilon > 0$, the same argument shows that there is an $f \in K$ such that

$|f(x) - f_0(x)| < \varepsilon$

Since $|f(x)| \leq 1$, by definition of K, if follows that $|f_0(x)| \leq 1$.

$\therefore f_0 \in K$

$\Rightarrow$ K is a closed subset of P, i.e. (b) is proved.

Hence the theorem.

(B) Theorem : Suppose that X is a vector space and X' is a subspace of the vector space $X^{\#}$ of all linear functionals on X. Then to topology of X is a locally convex topology and the dual space of X w.r.t this topology is X'.

Proof : For the prove of this all allusions to a topology for X refer to the X' topology.

Suppose (x_β), (y_β) are nets in X and (α_β) is a net in F such that three nets have the same index set. Let $x_\beta \to x$, $y_\beta \to y$, $\alpha_\beta \to \alpha$.

The continuity of addition and multiplication of F, assures that for each fin X',

$$f(\alpha_\beta x_\beta + y_\beta) = \alpha_\beta f(x_\beta) + (y_\beta)$$

$$\to \alpha f(x) + f(y) = f(\alpha x + y)$$

$\Rightarrow \alpha_\beta x_\beta + y_\beta \to \alpha x + y.$

$\Rightarrow$ The vector space operations of X are continuous.

It is easy to see that

$\{f^{-1}(U) : f \in X', U$ is an open ball in F$\}$ is a subbasis for the topology of X that generates a basis for that topology consisting of convex sets, so X is an LCS.

Let f_0 be a continuous linear functional on X.

$\Rightarrow$ Then there is a nbhd of 0 in X that is mapped by f_0 into the open unit ball of F.

$\Rightarrow$ There is a nonempty finite collection $f_1, ...f_n \in X'$ and corresponding collection $U_1, ...U_n$ of nbhds of 0 in F, such that

$f_1^{-1}(U_1) \cap .... \cap (f_n^{-1}(U_n)$ is mapped by f_0 into the open unit ball of F.

Let $x \in \ker(f_1) \cap ... \cap \ker(f_n)$.

Then $mx \in f^{-1}(U_1) \cap .... \cap f^{-1}(U_n) \ \forall \ m \in N$.

$\therefore m|f_0(x)| = |f_0(mx)| < 1 \ \forall \ m \in N$.

$\Rightarrow x \in \ker f_0$.

$\therefore f_0$ is a linear combination of $f_1.....f_n$, so

$f_0 \in X'$

$\Rightarrow$ The dual space of X included in X'

The reverse inclusion follows from the definition of the X' topology of X.

Hence proved.

Proposition : Suppose that X is a V.S and X' is a supspace of $X^{\#}$. Then a subset A of X is bounded w.r.t the X' topology $\Leftrightarrow$ f(A) is bounded in F for each $f \in X'$.

Proof : Throughout the proof, the topology of X is the X' topology.

Let A be a subset of X.

Suppose that A is bounded.

Let f be any member of X', and let U be an open unit ball in F.

$\Rightarrow$ There exists $t > 0$ such that

$A \subseteq t\, f^{-1}(U)$

$\Rightarrow f(A) \subseteq tU$

$\Rightarrow f(A)$ is bounded.

Conversely, let f(A) is bounded, whenever $f \in X'$

Let U_0 be a nbhd of 0 in X and let $f_1, f_2, \ldots f_n \in X'$ and $V_1, \ldots V_n$ nbhds of 0 in F such that

$f_1^{-1}(V_1) \cap \ldots \cap f_n^{-1}(V_n) \subseteq U_0$

The boundness of each $f_j(A)$ yields a $s > 0$ such that $f_j(A) \subseteq t\, V_j \ \forall\ j$ when $t > s$.

$\Rightarrow A \subseteq t[f_1^{-1}(V_1) \cap \ldots \cap f_n^{-1}(V_n)] \subseteq t\, U_0,\ t > s.$

$\Rightarrow A \subseteq t\, U_0,\ t > s$

$\Rightarrow$ A is bounded.

Proposition : Let X be a normed space and let A and B be subsets of X and X* respectively.

(a) The set $A^{\perp}$ is a weakly* closed subspace of X*.

(b) $({}^{\perp}B)^{\perp} = [B]^{w^*}$

(c) If B is a subspace of X*, then $({}^{\perp}B)^{\perp} = \overline{B}^{w^*}$

Proof : Let Q be the natural map from X into X**.

(a) Then $A^{\perp} = \{x^* : x^* \in X^*, x^*x = 0\ \forall\ x \in A\}$

$= \cap\ \{\ker(Qx) : x \in A\}$

For each $x \in A$, the linear functional Qx is weakly* continuous on X*.

$\Rightarrow \cap\{\ker(Qx) : x \in A\}$ is a weakly* closed subspace of X*.

$\Rightarrow A^{\perp}$ is a weakly* closed subspace of X*.

(b) We have $({}^{\perp}B)^{\perp}$ is a weakly* closed subspace of X* that includes B.

So, $[B]^{w^*} \subseteq ({}^{\perp}B)^{\perp}$

Now, suppose that $x_0^* \in X^* - [B]^{w^*}$

$\Rightarrow \exists\ x_0 \in X$ such that $x_0^*(x_0) = 1$

and $[B]^{w^*} \subseteq \ker(Qx_0)$

Since $x_0 \in {}^{\perp}B$, so $x_0^* \notin ({}^{\perp}B)^{\perp}$

$\therefore ({}^{\perp}B)^{\perp} \subseteq [B]^{w^*}$

$\therefore ({}^{\perp}B)^{\perp} = [B]^{w^*}$

(c) We have, $< \overline{B} >^{w^*} = [B]^{w^*}$, then

$({}^{\perp}B)^{\perp} = \overline{B}^{w^*}$.

Chapter - 3
Locally Convex spaces

The Hahn Banach Theorms

3.1.1. Definition :

The dual space X^* of a topological vector space X is the vector space of all continuous linear functionals on X.

The addition and scalar multiplication are defined in X^* as follows

$(\wedge_1 + \wedge_2)x = \wedge_1 x + \wedge_2 x$, $(\alpha\wedge_1)x = \alpha\wedge_1 x$

($\wedge_1, \wedge_2 \in X^*$ and α is a scalar)

3.1.2. Proposition :

If u is the real part of a complex linear functional f on X, then u is real linear and

$f(x) = u(x) - i\,u(ix)$ ......(i)

Conversely, if $u : X \to \mathbb{R}$ is real-linear an a complex vector space and if f is defined (i), then f is complex linear on X.

Proof :

Let $f : X \to K$ be a complex linear functional and let

$f(x) = u(x) + iv(x)$,

where u(x) and v(x) are real and imaginary parts of f(x).

Now, for any real α,

$f(\alpha x) = u(\alpha x) + iv(\alpha x)$.

But every real number is also a complex number.

and so $f(\alpha x) = \alpha f(x)$

$\Rightarrow u(\alpha x) + iv(\alpha x) = \alpha u(x) + i\alpha v(x)$

Equating real parts we get

$u(\alpha x) = \alpha u(x)$.

This shows that u is real linear.

Further,

$f(ix) = u(ix) + iv(ix)$

or $iu(x) - v(x) = u(ix) + iv(ix)$

Equating real parts, we get

$v(x) = -u(ix)$.

$\therefore \quad f(x) = u(x) + i(-u(ix))$

$= u(x) - iu(ix).$

Conversesly, suppose u(x) is a real linear functional on a complex vector space X. Define a map $f : X \to K$ by

$$f(x) = u(x) - iu(ix).$$

Then, for any complex scalar α and $x, y \in X$, we have

$$f(x + y) = u(x + y) - iu(ix + iy)$$
$$= u(x) + u(y) - i\{u(ix) + u(iy)\} \quad [\because \quad u \text{ is additive}]$$
$$= [u(x) - iu(ix)] + [u(y) - iu(iy)]$$
$$= f(x) + f(y).$$

$\therefore \quad$ f is additive.

Now, if $\alpha = a + ib$, then

$$f(\alpha x) = u(\alpha x) - iu(i\alpha x)$$
$$= u((a + ib)x) - iu(iax + i^2bx)$$
$$= u(ax + ibx) - iu(iax + i^2bx)$$
$$= u(ax) + u(ibx) - iu(iax) + iu(bx) \qquad [\because \quad u \text{ is additive}]$$
$$= au(x) + bu(ix) - iau(ix) + biu(x)$$
$$[\because \quad u \text{ is real linear}]$$
$$= (a + ib)\, u(x) - i(a + ib)\, u(ix)$$
$$= (a + ib) \quad [u(x) - iu(ix)]$$
$$= \alpha f(x)$$

$\therefore \quad$ f is complex linear.

Corollary :

Let X be a TVS. Then

(i) every complex linear function on X is in X^* if and only if its real part is continuous

(ii) every continuous real linear $u : X \to R$ is the real part of a **unique** $f \in X^*$.

3.2. Theorem :

Suppose,

(a) M is a **subspace** of a **real vector space** X,

(b) $p : X \to R$ satisfies

$$p(x + y) \leq p(x) + p(y) \text{ and } p(tx) = tp(x)$$

if $x \in X$, $y \in Y$, $t \geq 0$.

(c) $f : M \to \mathbb{R}$ is **linear** and $\mathbf{f(x) \leq p(x)}$ on M.

Then there exists a **linear** $\wedge : X \to R$ such that

$$\wedge x = f(x), \ \forall x \in M$$

and $-p(-x) \leq \wedge x \leq p(x), (x \in X)$.

Proof :

If $M = X$, then f itself is the required extension and hence the result is obvious.

If $M \neq X$, then consider $x_0 \in X - M$.

and define

$$M_0 = \{x + tx_0 : t \in \mathbb{R}\}$$

Then M_0 is a vector subspace of X, such that $M \subseteq M_0$.

For $x, y \in M$

$$f(x) + f(y) = f(x + y) \leq p(x + y) \qquad \text{[by (c)]}$$

$$\Rightarrow f(x) + f(y) \leq p(x - x_0 + x_0 + y)$$

$$\Rightarrow f(x) + f(y) \leq p(x - x_0) + p(x_0 + y)$$

$$\Rightarrow f(x) - p(x - x_0) \leq p(x_0 + y) - f(y), \ \forall \ x, y \in M$$

$$\Rightarrow f(x) - p(x - x_0) \leq \sup_{y \in M} \{p(y + x_0) - f(y)\}, \ \forall x \in M$$

$$\Rightarrow \inf_{x \in M} \{f(x) - p(x - x_0)\} \leq \sup_{y \in M} \{ p(y + x_0) - f(y) \}$$

Let α be a real number, such that

(1) $\inf_{x \in M} \{f(x) - p(x - x_0)\} \leq \alpha \leq \sup_{y \in M} \{ p(x + x_0) - f(y) \}$

Define $\quad f_0 : M_0 \to R$ by

(2) $f_0(x + tx_0) = f(x) + t\alpha, \qquad (x \in M, t \in \mathbb{R})$

Then, f_0 is well defined and linear on M_0.

Also, for any $x \in M$

$$f_0(x) = f_0(x + 0.x_0) = f(x) + 0.\alpha = f(x)$$

$\therefore \quad f_0 = f$ on M.

We are left to show that

$$f_0 \leq p \text{ on } m_0.$$

Consider, $t > 0$ and $x, y \in M$. Since m is a subspace, $t^{-1}x, t^{-1}y \in M$.

Now, from (1)

$$f(t^{-1}x) - p(t^{-1}x - x_0) \leq \alpha \leq p(t^{-1}y + x_0) - f(t^{-1}y)$$

$$\Rightarrow t^{-1}f(x) - t^{-1}p(x - tx_0) \leq \alpha \leq t^{-1}p(y + tx_0) - t^{-1}f(y)$$

$[\because \quad f \text{ is linear and } f(tx) = tf(x) \ \ \forall \ t > 0]$

$$\Rightarrow f(x) - p(x - tx_0) \leq t\alpha$$

and $p(y + tx_0) - f(y) \geq t\alpha$

$$\Rightarrow f(x) - t\alpha \leq p(x - tx_0)$$

and $f(y) + t\alpha \leq p(y + tx_0)$

$$\Rightarrow f_0(x - tx_0) \leq p(x - tx_0) \qquad ......(3)$$

and $f_0(y + tx_0) \leq p(y + ty_0) \qquad \forall\, t > 0$

Again if $t = 0$, then

$$f_0(x + tx_0) = f_0(x) = f(x) \leq p(x) = p(x + tx_0) \qquad ......(4)$$

From (3) and (4), it follows that

$$f_0(x + tx_0) \leq p(x + tx_0), \ \forall\ t \in \mathbb{R}.$$

i.e. $f_0(x) \leq p(x), \ \forall\ x \in M_0$.

If $M_0 = X$, then we are done,

If $M_0 \neq X$, The continue the process.

We complete the proof using Haudorff maximality theorem :

"Every non-empty partially ordered set P contains a totally ordered subset Ω which is maximal w.r.t. the property of being totally ordered"

Let P be the collection of all ordered pairs(M', f'), where M' is a subspace of X that contains M and f' is a linear functional on M' that extends f and satisfies $f' \leq p$ on M.

We define a partial ordering "≤" on P by the rule,

$$(M', f') \leq (M'', f'')$$

iff $M' \subseteq M''$ and $f' = f''$ on M'.

Then (P, ≤) is **partially ordered** set.

By Hausdorff maximality theorem ther exists a maximal totally ordered subset Ω of P.

Let Φ be the collection of all M', such that $(M', f') \in \Omega$. Then Φ is totally ordered by set conclusion and the union $\widetilde{M}$ of all members of Φ i.e. $\widetilde{M} = \bigcup_{M' \in \Phi} M'$ is therefore a subspace of X.

If $x \in \widetilde{M}$, then $x \in M'$ for same $M' \in \overline{\Phi}$.

Define a map $\wedge : \widetilde{M} \to \mathbb{R}$by

$$\wedge(x) = f'(x)$$

where f' is the functional which occurs in the pair $(M', f') \in \Omega$.

Clearly $\wedge$ is well defined, as Φ is totally ordered.

$\wedge$ is Linera :

Let x, y be any two elements of $\widetilde{M}$. Then $x \in M'$, $y \in M''$ for some $M', M'' \in \Phi$. Since Φ is totally ordered, one of M', M'' must contain the other. Let $M' \subseteq M''$.

Then $x, y \in M''$

$\therefore \quad \wedge(x+y) = f'(x+y)$

$= f'(x) + f'(y)$

$= f(x) + f'(y) \qquad [\because \quad f = f' \text{ on } M']$

$= \wedge(x) + \wedge(y)$

and $\quad \wedge(\alpha x) = f'(\alpha x) = \alpha f(x) = \alpha \wedge(x)$

$\therefore \quad \wedge$ is a linear functional on $\tilde{M}$.

Also $x \in \tilde{M}. \Rightarrow x \in M'$ for some $M' \in \Phi$

$\Rightarrow f'(x) \leq p(x)$

$\Rightarrow \wedge(x) \leq p(x)$

Hence $\wedge \leq p, \ \forall \ x \in \tilde{M}$.

If $\tilde{M}$ were a proper subspace of X, the first part of the proof would give a further extension of $\wedge$, and this would contradict the maximality of Ω.

Thus $\tilde{M} = X$ and $\wedge$ is the required extension of f.

Finally, $\qquad \wedge \leq p$, on $\tilde{M} = X$.

i.e. $\quad \wedge(x) \leq p(x), \ \forall x \in X$.

Also $\ \wedge(-x) \leq p(-x), \ \forall x \in X$.

$\Rightarrow -\wedge(x) \leq (-x), \ \forall x \in X$.

$\Rightarrow -p(-x) \leq \wedge(x), \ \forall x \in X$..

Hence $\qquad -p(-x) \leq \wedge(x) \leq p(x) \ \forall x \in X$.

This completes the proof.

3.3. Theorem :

Suppose M is a subspace of a vector space X, p is a seminorm on X, and f is a linear functional on M such that

$$|f(x)| \leq p(x), \ (x \in M)$$

Then f extends to a linear functional $\wedge$ on X that satisfies

$$|\wedge x| \leq p(x), \ (x \in X)$$

Proof :

Given p is a seminorm, on X

i.e. $\quad p(x+y) \leq p(x) + p(y)$

and $p(tx) = |t|\, p(x), \ \forall$ scalar t.

and for $x \in X, y \in X$

Case I :

When X is a real vector space we observed that

(i) M is a subspace of the real vector space X.

(ii) $p : X \rightarrow \mathbb{R}$ satisfies

$$p(x + y) \leq p(x) + p(y)$$

and $p(tx) = tp(x), \ \forall \ t > o$ and $x, y \in X$.

(iii) $f : M \rightarrow \mathbb{R}$ is linear and $f \leq p$ on M.

Thus all the conditions of the theorem 3.2. are satisfied and hence we get an extension $\wedge$ of f on X, that satisfies

$$\wedge = f \text{ on } M$$

and $\quad -p(-x) \leq \wedge(x) \leq p(x)$

Since p is a seminorm

$\therefore \quad p(-x) = p(x), \ \forall \ x \in X$

and hence

$$|\wedge(x)| \leq p(x), \ \forall \ x \in X.$$

This proves the result when X is a real vector space.

Case II :

When X is a comple vector space.

In this case f is a complex linear functional.

Suppose, Ref = u.

Then $f(x) = u(x) - iu(ix), \ \forall \ x \in M$.

Also, $u : M \rightarrow \mathbb{R}$ is a real linear functional and

$$u(x) \leq p(x), \ \forall \ x \in M.$$

Therefore, by therom, 3.2., there is a real linear U on X such that

$$U = u \text{ on } m \text{ and}$$

$$|U(x)| \leq p(x), \ \forall \ x \in X.$$

Let us define, $\wedge : X \rightarrow \mathbb{C}$ by

$$\wedge(x) = U(x) - iU(ix), \ \forall \ x \in X.$$

Then, we can show that

i. $\wedge$ is complex linear

ii. $|\wedge(x)| \leq p(x)$.

iii. $\wedge(x) = f(x), \ \forall \ x \in M$

i. $\wedge$ is complex linear as seen in 3.1.2.

ii. $|\wedge(x)| \leq p(x)$ on X.

To every $x \in X$, there corresponds $\alpha \in ¢$, such that $|\alpha| = 1$ and

$$\alpha\wedge(x) = |\wedge x|.$$

Hence $\quad |\wedge x| = \wedge(\alpha x) = \text{Re}\wedge(\alpha x) = U(\alpha x) \leq p(\alpha x)$

$$\Rightarrow |\wedge(x)| \leq |\alpha|\, p(x) = p(x), \quad \because \quad |\alpha| = 1.$$

Hence $|\wedge(x)| \leq p(x), \ \forall\ x \in X.$

Finally, for $x \in M$

$$\begin{aligned} \wedge x &= U(x) - iU(ix) \\ &= u(x) - iu(ix) \\ &= f(x) \end{aligned}$$

$\therefore \quad \wedge = f$ on M.

This complete the proof.

Corollory :

If X is a normed linear space and $x_0 \in X$, then there exists $\wedge \in X^*$ such that

$$\wedge x_0 = \|x\| \text{ and } |\wedge x| \leq \|x\| \ \forall x \in X \qquad (1997, 1999)$$

Proof :

If $x_0 = 0$, then taking $\wedge = 0$ we can conclude the proof.

If $x_0 \neq 0$, then consider the subspace generated by x_0 as M.

i.e. $\quad M = \{\alpha x_0 : \alpha \in ¢\}$.

Consider $p : X \to \mathbb{R}$ as

$$p(x) = \|x\|, \ \forall\ x \in X.$$

Define, $f : M \to K$ by

$$f(\alpha x_0) = \alpha\|x_0\|, \ \forall\ \alpha x_0 \in M.$$

f is linear :

$$\begin{aligned} f(\alpha x_0 + \beta x_0) &= f((\alpha + \beta)x_0) \\ &= (\alpha + \beta)\|x_0\| \\ &= \alpha\|x_0\| + \beta(x_0) \\ &= f(\alpha x_0) + f(\beta x_0) \qquad \forall\, \alpha x_0, \beta x_0 \in M \end{aligned}$$

$\therefore \quad$ f is additive. Also, for any scalar a,

$$f(a(\alpha x_0)) = f((\alpha x)x_0) = a\alpha\|x_0\| = af(\alpha x_0)$$

$\therefore \quad$ f is linear.

f is bounded :

For any $x = \alpha x_0 \in M$

$$f(x) = \alpha \| x_0 \|$$

$$\Rightarrow | f(x) | = | \alpha \| x_0 \| = | \alpha | \, \| x_0 \| = \| \alpha x_0 \| = \| x \|$$

This shows that f is bounded.

Therefore by Hahn Banach Extension, $\exists$ a linear function $\wedge$ on X, such that

$$\wedge = f \text{ on } M$$

and $| \wedge(x) | \leq p(x), \ \forall \ x \in X$

i.e. $| \wedge x | \leq \| x \|, \ \forall \ x \in X.$

$\therefore$ $\wedge$ is bounded and hence a continuous linear functional on X.

$\therefore$ $\wedge \in X^*$.

Finally,

$$\wedge(x_0) = \wedge(1.x_0) = 1.\| x_0 \| = \| x_0 \|$$

i.e. $\wedge x_0 = \| x_0 \|$.

Further $\| \wedge \| = \sup\limits_{\|x\| \leq 1} \{ | \wedge x | : x \in X \}$

$$\leq \sup_{\|x\| \leq 1} \{ \|x\| : x \in X \}$$

$$\leq 1$$

$\therefore$ $\| \wedge \| \leq 1$ ......(1)

Again choosing, $\alpha = \dfrac{1}{\|x_o\|}$, we find that $x = \alpha x_0 \in M$.

Also

$$\| x \| = \| \alpha x_0 \| = | \alpha | \, \| x_0 \| = \frac{1}{\|x_o\|} \| x_0 \| = 1$$

$\therefore$ $| \wedge x | = | \wedge(\alpha x_0) | = | \alpha \| x_0 \| | = | \alpha | \, \| x_0 \| = \| \alpha x_0 \| = 1$

$\therefore$ $\operatorname{Sup} \{ | \wedge x | : \| x \| \leq 1 \} \geq 1$ and $\| \wedge \| \geq 1$ ......(2)

From (1) and (2), we get $\| \wedge \| = 1$.

3.4. Theorem(Separation theorem) :

Suppose A and B are disjoint, non-empty convex sets in a topological vector space X

(a) If A is open there exists $\wedge \in X^*$ and $\gamma \in R$ such that

$$\operatorname{Re} \wedge x < \gamma \leq \operatorname{Re} \wedge y$$

for every $x \in A$ and for every $y \in B$.

(b) If A is compact, B is closed and X is locally convex, then there exists $\wedge \in X^*$, $\gamma_1 \in R$, $\gamma_2 \in R$ such that

$$\text{Re} \wedge x < \gamma_1 < \gamma_2 \text{ Re} \wedge y$$

for every $x \in A$ and for every $y \in B$.

Proof :

If is enough to prove this for real scalars. For if the scalar field is complex and the real case has been proved then there is a real linear u on X, that gives the required separation; if $\wedge$ is the unique complex linear functional defined by

$$\wedge x = u(x) - iu(ix)$$

then $\wedge \in X^*$ and $\text{Re} \wedge x = u(x)$.

Assume that, all scalars are real.

(a) Fix $a_0 \in A$, $b_0 \in B$ and put

$$x_0 = b_0 - a_0 \text{ and } C = A - B + x_0.$$

Then C is a convex neighbourhood of o in X. Because,

$$0 = a_0 - b_0 + x_0 \in C$$

and C is open(as A is open).

Since every neighbourhood of 0 in X, is absorbing it follows that C is convex absorbing subset of X.

Let p be the Minkowski's functional defined on X. Then by Theorem 1.3.5., p satisfies,

$$p(x + y) \leq p(x) + p(y)$$

and $p(tx) = tp(x), \ \forall t \geq 0$

and for every $x \in X$, $y \in X$.

Since $A \cap B = \phi$, $x_0 \notin C$, and so $p(x_0) \geq 1$.

For if $x_0 \in C$, then

$$x_0 = a_1 - b_1 + x_0 \text{ for some } a_1 \in A, b_1 \in B..$$

$\Rightarrow a_1 = b_1$ which is not true.

Also, $0 \in C$, $x_0 \notin C \Rightarrow x_0 \neq 0$.

Define, $f(tx_0) = t$ on the subspace M of X generated by x_0.

Then $f : M \to \mathbb{R}$ is real linear. Further $f(x) \leq p(x), \ \forall \ x \in M$. Because, if $t \geq o$

$$f(tx_0) = t \leq tp(x_0) = p(tx_0)$$

and if $t < 0$, then

$$f(tx_0) = t < 0 \leq p(tx_0) \qquad [\because \ p(x) \geq 0]$$

$\therefore \quad f(x) \leq p(x) \ \forall \ x \in M.$

Thus all the conditions of 3.2. are satisfied and hence f extends to a linear functional $\wedge$ on X, satisfying

$\wedge \leq p$ on X.

Now, $x \in C \Rightarrow p(x) \leq 1 \Rightarrow \wedge(x) \leq 1$

Again $x \in (-C) \Rightarrow -x \in C \Rightarrow \wedge(-x) \leq 1 \Rightarrow -\wedge(x) \leq 1$

Thus $|\wedge x| \leq 1, \forall x \in C \cap (-C)$

This shows that $\wedge$ is bounded in the neighbourhood $C \cap (-C)$ of 0 and hence $\wedge$ is continuous and $\wedge \in X^*$.

If now $a \in A$ and $b \in B$, we have

$$\wedge(a) - \wedge(b) + 1 = \wedge(a - b + x_0) \leq p(a - b + x_0) < 1$$

Since $\wedge x_0 = 1$, $a - b + x_0 \in C$ and C is open.

Thus $\wedge a \leq \wedge b$.

It follows that $\wedge(A)$ and $\wedge(B)$ are disjoint convex subsets of $\mathbb{R}$, with $\wedge(A)$to the left $\wedge(B)$. Also $\wedge(A)$ is open set since A is an open set and every non-constant linear functional on X is an open mapping.

Let γ be the right end and point of $\wedge(A)$, then

$$\wedge x < \gamma \leq \wedge y, \ \forall \ x \in A \text{ and } \forall \ y \in B.$$

(b) There exists, a neighbourhood V of o on X such that

$$(A + V) \cap B = \phi.$$

By part (a), with A + V in place of A, thre exists $\wedge \in X^*$ such that $\wedge(A + V)$ and $\wedge(B)$ are disjoint convex subsets of $\mathbb{R}$, with $\wedge(A + V)$ open and to the left $\wedge(B)$.

Since $\wedge(A)$ is a compact subset of $\wedge(A + V)$; $\exists \gamma_1, \gamma_2 \in R$ such that

$$\wedge x < \gamma_1 < \gamma_2 < \wedge y$$

for every $x \in A$ and for every $y \in B$

Corollary :

If X is a **locally convex space** then X^* **separates** points on X.

Proof :

Let $x_1 \, x_2 \in X$ and $x_1 \neq x_2$.

Let us consider $A = \{x_1\}$, $B = \{x_2\}$.

Then A and B are disjoint subsets of X, with A is compact and B is closed. It follows that, (by part (b) of the theorent 3.4.), that $\exists \wedge \in X^*$ and $\gamma_1, \gamma_2 \in R$, such that

$$\wedge x_1 < \gamma_1 < \gamma_2 < \wedge x_2.$$

This implies that $\wedge x_1 \neq \wedge x_2$. Hence X^* separates points of X.

3.5. Theorem :

Suppose M is a subspace of a locally convex X, and $x_0 \in X$. If x_0 is not in the closure of M, then there exists $\wedge \in X^*$ such that $\wedge x_0 = 1$, but $\wedge x = 0$, for all $x \in M$.

Proof :

Consider $A = \{x_0\}$ and $B = \overline{M}$. Then A is compact and B is closed, with

$A \cap B = \phi$.

Therefore, there exists $\wedge_1 \in X^*$ such that $\wedge_1 A$ and $\wedge_1 B$ are disjoint and hence $\wedge_1 x_0$ and $\wedge_1 M$ are disjoint. thus $\wedge_1(M)$ is a proper subspace of the scalar field. This forces $\wedge_1(M) = \{0\}$ and hence $\wedge_1 x_0 \neq 0$.

Define

$\wedge : x \to K$ by

$$\wedge(x) = \frac{\wedge_1 x}{d}, \; d = \wedge_1 x_0$$

Then, $\wedge$ is a continuous linear functional defined on X and hence $\wedge \in X^*$.

Also $\wedge x_0 = \frac{\wedge_1 x_0}{d} = \frac{d}{d} = 1$.

and for $x \in M$, $\wedge x \in \wedge(M) \Rightarrow \wedge x = 0$

i.e. $\wedge x = 0, \; \forall \; x \in M$. This completes the proof.

3.6. Theorem(1999) :

(Hahn Banach Extension theorm on Locally Convex Space)

If f is a con tinuous linear functional on a subspace M of a locally convex space X, M then there exists $\wedge \in X^*$, such that $\wedge = f$ on M.

Proof :

If $f = 0$, there $\wedge = 0$, will meet our requirements.

So, without loss of generality, let us assume that f is not identically zero.

Define,

$$M_0 = \{x \in M : f(x) = 0\}$$

Thus, there exists $x_0 \in M - M_0$ such that $f(x_0) = 1$. Since f is linear.

By the continuity of f, M_0 is a closed linear subspace of M, w.r.t. the relative topology on M enherited from X.

Since $x_0 \notin M_0$ implies x_0 is not in the M-closure of M_0

and x_0 is not in the X-closure of M_0.

Therefore by theorem 3.5., there exists $\wedge \in X^*$ such that

$\wedge x_0 = 1$ and $\wedge x = 0 \; \forall \; x \in M_0$.

If $x \in M$, then $x - f(x)x_0 \in M_0$ because

$$f(x - f(x)x_0) = f(x) - f(x)f(x_0) = f(x) - f(x) = 0.$$

$\therefore \quad \wedge(x - f(x)x_0) = 0$

$\Rightarrow \wedge x - f(x) \wedge(x_0) = 0$

$\Rightarrow \wedge x = f(x)$

$\therefore \quad \wedge x_0 = 1$. Hence $\wedge = f$ on M.

3.7. Theorem :

Suppose B is a **convex, balanced, closed** set is a **locally convex space** X, $x_0 \in X$ but $x_0 \notin B$. then, there exists $\wedge \in X^*$ such that $| \wedge \mathbf{x} | \leq \mathbf{1}$ for all $x \in B$, but $\wedge \mathbf{x}_0 > \mathbf{1}$.

Proof :

Consider $A = \{x_0\}$. Then A and B are disjoint, non-empty convex subsets of X.

Also A is compact and B is closed. Therefore by theorem 3.4(6), $\exists \wedge_1 \in X^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, such that

$$\text{Re} \wedge_1 x_0 < \gamma_1 < \gamma_2 < \text{Re} \wedge_1 x \quad \forall \ x \in B.$$

This shows that $\wedge_1 x_0$ is not in the closure K of $\wedge_1(B)$ and K is a proper subset of the scalar field ¢.

Let $\quad \wedge_1 x_0 = re^{iv}, r > 0.$

Now, Since B is balanced, so is K. Hence K is a disc, because non-trivial balanced subsets of ¢ are that discs only.

Hence, there exists $0 < s < r$, so that $| z | \leq s$, for all $z \in K$.

Now, define

$$\wedge = s^{-1}e^{-i\theta} \wedge_1.$$

Then, $\wedge$ is also a continuous linear functional on X and so $\wedge \in X^*$.

Now

$$\wedge x_0 = s^{-1}e^{-i\theta} \wedge_1 x_0 = s^{-1}e^{-i\theta}.re^{-i\theta} = \frac{r}{s}$$

$\Rightarrow \wedge x_0 > 1$

and for $x \in B$

$$| \wedge x | = | s^{-1}e^{i\theta} \wedge_1 x | = \frac{1}{s} \ | \wedge_1 x | \leq \frac{1}{s}.s = 1$$

Thus $\wedge x_0 > 1$ and $| \wedge x | \leq 1$.

Weak Topologies

Let τ_1 and τ_2 be two topologies on a set X and assume $\tau_1 \subset \tau_2$; that is every τ_1 open set is also τ_2-open. Then we say that τ_1 is weaker than τ_2 and that τ_2 is finer than τ_1.

Proposition :

If $\tau_1 \subset \tau_2$ then the identity from mapping on X is continuous from (X, τ_2) to (X, τ_1) and is an open mapping from (X, τ_1) and (X, τ_2).

Proof :

Let $i : (X, \tau_2) \to (X, \tau_1)$ be the identity mapping, then for any $G \in \tau_1$

$$i^{-1}(G) = G \in \tau_2. \quad (\because \quad \tau_1 \subset \tau_2)$$

This shows that the inverse image under i of every τ_1-open set τ_2-open, and hence i is continuous from (X, τ_2) to (X, τ_1).

Again, let $i : (X, \tau_1) \to (X, \tau_2)$ be the identity mapping, then for any open set $G \in \tau_1$,

$$i(G) = G \in \tau_2, \qquad (\because \quad \tau_1 \subset \tau_2)$$

Thus, image under i of every τ_1-open set is τ_2-open and hence i is an open mapping from(X, τ_1) to (X, τ_2).

Proposition :

If $\tau_1 \subset \tau_2$ are topologies on a set X, if τ_1 is a Hausdorff topology, and if τ_2 is compact, then $\tau_1 = \tau_2$.

Proof :

It is enough to show that

$$\tau_2 \subset \tau_1.$$

To see this, let $F \subset X$, be τ_2-closed. Then F is τ_2-compact as X is τ_2-compact.

Since $\tau_1 \subset \tau_2$, it follows that every τ_1-open cover of F is also a τ_2-open cover of F and so F is τ_1-compact.

Since τ_1 is a Hausdorff space and compact subsets of a Hausdorff space are closed, F is τ_1-closed.

Hence $\tau_2 \subset \tau_1$ and consequently $\tau_1 = \tau_2$.

Quotient Topology

Consider the quotent topology τ_N of $\frac{X}{N}$, where $\tau_N = \{E \subset \frac{X}{N} : \pi^{-1}(E) \in \tau\}$

and $\pi : X \to \frac{X}{N}$ is the quotient map.

By its very definition, τ_N is the finest topology on $\frac{X}{N}$, that makes π-continuous and it is the weakest one that makes π an open mapping. Explicitly, if τ' and τ'' are topologies on $\frac{X}{N}$ and if π is continuous relative τ' and π is open relative to τ'', then

$$\tau' \subset \tau_N \subset \tau''.$$

$\mathcal{F}$ - topology

Suppose that X is a set and $\mathcal{F}$ is a non-empty family mappings

$f: X \rightarrow Y_f$, where Y_f is a topological space.

Let τ be the collection of all unions of intersections of sets $f^{-1}(V)$, with $f \in F$ and V is open in Y_f. Then τ is a topology on X, and it is in fact the weakest topology on X that makes every $f \in \mathcal{F}$ continuous. If τ' is another topology with that property, then $\tau \subset \tau'$. This topology τ is called **the weak-topology on X**, induced by $\mathcal{F}$ or **$\mathcal{F}$-topology of X**.

Proposition :

The $\mathcal{F}$-topology τ on X, is the weakest topology X that makes every $f \in \mathcal{F}$ continuous

Proof :

Let τ be the $\mathcal{F}$-topology on X, and τ' be any other topology of X w.r.t. which every $f \in \mathcal{F}$ is continuous. To show $\tau \subset \tau'$.

To see this let $G \subset X$ be τ-open. Then G is the union of the finite intersections of the sets $f^1(V)$ with $f \in \mathcal{F}$ and V is open on Y_f. By continuity of each $f \in \mathcal{F}$ w.r.t. τ', each $f^1(V)$ is τ'-open.

Since τ' is closed under finite insection and arbitrary union it follows that G is τ'-open.

So, $\tau \subset \tau'$. This completes the proof.

Proposition :

If $\mathcal{F}$ is a family of mapping $f: X \rightarrow Y_f$ where X is a set and each Y_f is a Hausdorff space anf if $\mathcal{F}$ separates points of X, then F-topology on X is a Hausdorff topology.

Proof :

Let $p, q \in X$ and $p \neq q$.

Since $\mathcal{F}$ separates points of X, $\exists f \in \mathcal{F}$ such that $f(p) \neq f(q)$.

Since Y_f is a Hausdorff space, $\exists$ Y_f-open sets V_1 and V_2 such that

$$p \in f^{-1}(V_1),\ q \in f^{-1}(V_2) \ \text{ and } V_1 \cap V_2 = \phi$$

and $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = f^{1}(\phi) = \phi$.

Hence X is a Hausdorff space w.r.t. the $\mathscr{F}$-topology.

Proposition :

If X is a compact topological space and if some sequence $\{f_n\}$ of continuous real valued functions separates points on X, then X is metrizable.

Solution :

Let τ be the given topology of X. We are to show that τ is compatible with some metric d on X.

Suppose, without loss generality, that $|f_n| \le 1$ for all n. Let us define

$d : X \times X \to \mathbb{R}$ by the rule

$$d(p,q) = \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)|$$

d is well defined since the series is convergent.

We show that, d is a metric on X.

M_1) $d(p, q) \ge 0$

M_2) $d(p, q) = 0$

$$\Leftrightarrow \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)| = 0$$

$$\Leftrightarrow f_n(p) = f_n(q), \quad \forall\ n \ge 1$$

$$\Leftrightarrow p = q \qquad [\because \quad \{f_n\} \text{ is a separating family}]$$

M_3) $d(p, q) = d(q, p)$

M_4) $$d(p,q) = \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)|$$

$$= \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(r) + f_n(r) - f_n(q)|$$

$$\le \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(r)| + \sum_{n=1}^{\infty} 2^{-n} |f_n(r) - f_n(q)|$$

$$= d(p, r) + d(r, q)$$

Hence d is a metric on X.

Let τ_d be the topology in duced on X by the metric d. We claim that $\tau_d = \tau$.

Now, for any $(p, q) \in X \times X$

$$|g_n(p, q)| = |2^{-n}| f_n(p) - f_n(q)|| \le 2^{-n} = M_n \text{ (say)} \quad [\because \quad |f_n| \le 1]$$

where $\sum_{n=1}^{\infty} g_n(p,q) = \sum_{n=1}^{\infty} 2^{-n} |f_n(p) - f_n(q)|$.

Since $\Sigma 2^{-n}$ is a convergent series of positive numbers, by Weirstress M-test, the series

$$\sum_{n=1}^{\infty} g_n(p,q) = \sum_{n=1}^{\infty} 2^{-n}|f_n(p) - f_n(q)| \text{ converges uniformly on } X \times X.$$

Also each f_n is τ-continuous. All these together imply that d is a τ-continuous function on $X \times X$. The balls

$B_r(p) = \{q : d(p, q) < r\}$ are therefore τ-open. Then $\tau_d \subset \tau$.

τ_d induced by a metric d is a Hausdorff topology and also τ is compact.

Therefore, $\tau_d = \tau$ follows from a preceding proposition. This completes the proof.

3.9. Lemma :

Suppose $\wedge_1, \wedge_2, ..., \wedge_n$ and $\wedge$ are **linear functionals** on a **vector space** X. Let

$N = \{x : \wedge_1 x = .. = \wedge_n x = 0\}$.

The following three properties **are equivalent.**

(a) There are scalars $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$\wedge = \alpha_1\wedge_1 + ... + \alpha_n\wedge_n$.

(b) There exists $\gamma < \infty$ such that

$|\wedge x| \le \gamma \max\limits_{1\le i\le n} |\wedge_i x|, (x \in X)$.

(c) $\wedge x = 0$ for every $x \in N$.

Proof :

(a) ⇒ (b)

Let $\wedge = \alpha_1\wedge_1 + ... + \alpha_n\wedge_n$.

For any $x \in X$,

$\wedge x = \alpha_1\wedge_1 x + ... + \alpha_n\wedge_n x$

$|\wedge x| \le (|\alpha_1| + |\alpha_2| + ... + |\alpha_n|) \max\limits_{1\le i\le n} |\wedge_i x|$

$= \gamma \max\limits_{1\le i\le n} |\wedge_i x|, \gamma = |\alpha_1| + |\alpha_2| + ... + |\alpha_n| < \alpha$

Thus, $\exists \gamma < \infty$ such that

$|\wedge x| \le \gamma \max\limits_{1\le i\le n} |\wedge_i x|, \quad (x \in X)$

(b) ⇒ (c)

Let $x \in N$ be any element.

Then $\wedge_i x = 0, \ \forall \ i = 1, 2, ..., n$.

$\Rightarrow \gamma \max\limits_{1\le i\le n} |\wedge_i x| = 0$

$\Rightarrow |\wedge x| = 0$

$\Rightarrow \wedge x = 0, \quad \forall\ x \in N.$

(c) ⇒ (a)

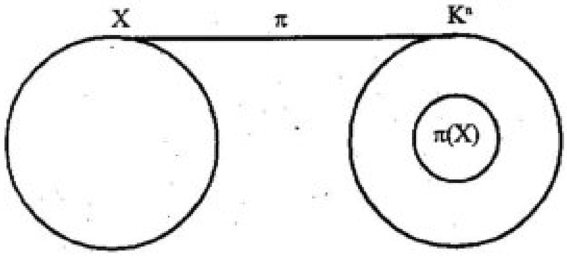
Let K be the scalar field. Define $\pi : X \to K^n$ by

$$\pi(x) = (\wedge_1 x, \wedge_2 x, \ldots, \wedge_n x).$$

So that

$$\pi(x) = \{(\wedge_1 x, \wedge_2 x, \ldots, \wedge_n x) \subset K^n : x \in X\}$$

Clearly $\pi(X)$ is a subspace of K^n.



Define $f : \pi(X) \to K$ by $f(\pi(x)) = \wedge x, \ \forall\ x \in X.$

f is well defined. For any $x, y \in X$,

$$\pi(x) = \pi(y)$$

$$\Rightarrow \pi(x - y) = 0$$

$$\Rightarrow \wedge_i(x - y) = 0, \quad \forall\ i = 1, 2, \ldots, n$$

$$\Rightarrow x - y \in N$$

$$\Rightarrow \wedge(x - y) = 0$$

$$\Rightarrow \wedge x = \wedge y$$

$$\Rightarrow f(\pi(x)) = f(\pi(y))$$

Thus f is well defined.

Also, f is linear.

Let us extend f to a linear functional F on K^n, so that

$$F = f \text{ on } \pi(X).$$

Then for any

$$(u_1, u_2, \ldots, u_n) \in K^n$$

$$F(u_1, u_2, \ldots, u_n) = F(u_1e_1 + u_2e_2 + \ldots + u_ne_n)$$

$$= u_1F(e_1) + \ldots + u_nF(e_n)$$

$$= \alpha_1u_1 + \ldots + \alpha_nu_n \qquad [\alpha_i = F(e_i)] \quad 1 \leq i \leq n$$

Thus for any $x \in X$.

$$\wedge x = f(\pi(x)) = F(\pi(x))$$
$$= F(\wedge_1 x, ..., \wedge_n x)$$
$$= \alpha_1 \wedge_1 x + ... + \alpha_n \wedge_n x$$
$$= (\alpha_1 \wedge_1 + ... + \alpha_n \wedge_n)x.$$

Hence $\wedge x = (\alpha_1 \wedge_1 + ... + \alpha_n \wedge_n)x, \quad \forall\ x \in X$

$\therefore \quad \wedge = \alpha_1 \wedge_1 + ... + \alpha_n \wedge_n$

This complete the Lemma.

3.10. Theorem :

Suppose X is a vector space and X' is a separating vector space of linear functionals on X. The the X'-topology τ' makes X into a locally convex space whose dual space is X'.

Proof :

Given X : Vector space

X' : Separating vector space linear functionals on X.

τ' : X'-topology on X.

To show, (X, τ') is a locally convex space.

Since R and ¢ are Hausdorff space, it follows the X'-topology τ' on X is Hausdorff.

The linearily of the members of X' shows that τ' is translation invariant.

If $\wedge_1, \wedge_2, ..., \wedge_n \in X'$, if $r_i > 0$, and if

$$V = \{ x : |\wedge_i x| < r_i, 1 \leq i \leq n\} = \bigcap_{i=1}^{n} \wedge_i^{-1}\left(D_{r_i} \right) \qquad ......(1)$$

then V is convex, balanced and $V \in \tau'$. In fact, the collection of all V of the form (1) is a local base for τ'.

Thus τ' is a locally convex topology on X. We are left to show that vector addition and scalar multiplication are τ'-continuous.

If $V = \{x : |\wedge_i x| < r_i, 1 \leq i \leq n\}$ then

$$\frac{1}{2}V + \frac{1}{2}V = V.$$

This proves that addition is continuous.

Now suppose $x \in X$ and α is a scalar. Then $x \in sV$, for some $s > 0$, since V is absorbing.

Now, if $r > 0$, $|\beta - \alpha| < r$ and $y - x \in rV$, then

$$\beta y - \alpha x = (\beta - \alpha)y + \alpha(y - x)$$
$$\subseteq |\alpha - \beta|(x + rV) + \alpha rV$$

$\subseteq |\alpha - \beta| (sV + rV) + \alpha rV$

$\subset (r(s + r) + |\alpha| r)V$

$\subset V$ provided

$r(s + r) + |\alpha| r < 1.$

Thus choosing r so small that $r(r + s) + |\alpha| r < 1$

we find that corresponding to every neighbourhood $\alpha x + V$ of αx, $\exists$ a neighbourhood $W = x + rV$ of x and $r > 0$ such that

$$\beta W \subseteq \alpha x + V, \text{ whenever } |\beta - \alpha| < r.$$

Hence scalar multiplication is continuous.

All these together imply that (X, τ') is a locally convex space.

Finally, we show that $(X, \tau')^* = X'$.

By definition, every $\wedge \in X'$ is τ'-continuous so that $X' \subset X^*$.

To prove the other inclusion we proceed as follows.

Suppose $\wedge$ is τ'-continuous linear functional on X. To show $\wedge \in X'$.

Then $|\wedge x| < 1$ for all x in some set V of the form (1).

Let $\wedge_1, \wedge_2, ..., \wedge_n \in X'$ and $\mu_i > 0$ such that

$$V = \{x : |\wedge_i x| < r_i, 1 \le i \le n\}$$

with $|\wedge x| < 1, \ \forall \ x \in V.$

Let $\alpha_x = \max_{1 \le i \le n} |\wedge_i x|$, for $x \in X$.

For any $\gamma > 0$, we have

$$\left|\wedge_i\left(\frac{x}{\gamma\alpha_x}\right)\right| = \frac{1}{\gamma\alpha_x}|\wedge_i (x)| < \frac{1}{\gamma} < r_i \quad 1 \le i \le n$$

provided, $\gamma > \dfrac{1}{r_i}, \ 1 \le i \le n$.

So, $\dfrac{x}{\gamma\alpha_x} \in V$ and $\left|\wedge\left(\dfrac{x}{\gamma\alpha_x}\right)\right| < 1 \Rightarrow \dfrac{1}{\gamma\alpha_x}|\wedge x| < 1$

$$\Rightarrow |\wedge x| < \gamma\alpha_x$$

$$\Rightarrow |\wedge x| < \gamma \max_{1 \le i \le n} |\wedge_i x|$$

Hence by Lemma 3.9, $\wedge = \alpha_i \wedge_i$ for some scalars α_i.

Since $\wedge_i \in X'$ and X' is a vector space $\alpha_i \wedge_i \in X'$ i.e., $\wedge \in X'$.

Consequently, $X^* \subset X'$. Hence $X^* = X'$ i.e., $(X, \tau')^* = X'$.

3.11. The Weak Topology of TVS:

Suppose X is a topological vector space with topology τ, whose dual X^* separates points on X. We know that this happens in every locally convex space.

The X^*-topologyof X is called the weak topology of X and is denoted by τ_w.

Then by Theorem 3.10. $X_w = (X, \tau_w)$ is locally convex whose dual space is X^*.

Since every $\wedge \in X^*$ is τ-continuous, and since τ_w is the weakest topology on X, with that property, we have $\tau_w \subset \tau$.

In this context the given topology τ will be often called the original topology of X.

Note :

(A) Subbasic members of τ_w are of the form

$$(x^*)^{-1}(D_r) = \{x : x^*(x) \in D_r\}$$
$$= \{x : |x^*(x)| < r\}$$

(B) τ_w is the coarsest topology on X w.r.t. which dual of X is X^*.

Suppose τ_1 is any other topology on X such that $(X, \tau_1)^* = X^*$.

To show $\tau_w \subset \tau_1$.

Let $G \in \tau_w$ **be arbitrary. Then G is the union of** finite intersection of the sets $\{x : |\wedge_i x| < r_i\} = \wedge_i^{-1}(D_{r_i})$

But $\quad \wedge_i \in X^*$

$\Rightarrow \wedge_i^{-1}(D_{r_i})$ is τ_1-open

$\Rightarrow$ G is τ_1-open $\qquad [\because \quad \tau_1$ is a topology].

$\therefore \quad G \in \tau_1$. Hence $\tau_w \in \tau_1$.

Convention :

Original(neighbourhood, open, closed, closure, compact etc) means the corresponding concept w.r.t. **original topology**.

Weak(neighbourhood, open, closed, closure concept etc.) means the corresponding concept w.r.t. **weak topology**.

Proposition :

A sequence $\{x_n\}$ **converges weakly** is a **tvs** (X, τ) to x iff $\wedge x_n \to \wedge x$ **for all** $\wedge \in X^*$.

Proof :

First suppose that

$x_n \to x$ weakly.

i.e. $y_n \to 0$ weakly where $y_n = x_n - x$.

Then for $\varepsilon > 0$, $\exists\ n_o \in N$ such that

$$y_n \in \wedge^{-1}(D_\varepsilon), \quad \forall n \geq n_o, \forall \wedge \in X^*$$

$$\Rightarrow \wedge y_n \in D_\varepsilon, \quad \forall n \geq n_o, \forall \wedge \in X^*$$

$$\Rightarrow |\wedge x_n - \wedge x| < \varepsilon, \forall n \geq n_o, \forall \wedge \in X^*$$

$$\Rightarrow \wedge x_n \to \wedge x, \quad \forall \wedge \in X^*$$

Conversely, let

$$\wedge x_n \to \wedge x, \quad \forall \wedge \in X^*.$$

To show $x_n \to x$ weakly.

i.e. to show $y_n \to 0$ weakly, $y_n = x_n - x$.

Let G be any weak neighbourhood of o then G contains a neighbourhood of the form

$$V = \{x : |\wedge_i x| < r_i \text{ for } 1 \leq i \leq k\}$$

$$= \bigcap_{i=1}^{K} \wedge_i^{-1}\left(D_{r_i}\right)$$

where $\wedge_i \in X^*$, and $r_i > 0$. $i = 1, 2, ..., k$.

Since $\wedge y_n \to 0 \quad \forall \wedge \in X^*$

$$\Rightarrow \wedge_i y_n \to 0 \quad \forall i = 1, 2, ..., k$$

$\Rightarrow$ for $r_i > 0$, $\exists m_i \in \mathbb{N}$ such that

$$|\wedge_i y_n| < r_i, \quad \forall n \geq m_i, 1 \leq i \leq K$$

$$\Rightarrow y_n \in V \quad \forall n \geq m_0 = \max_{1 \leq i \leq k}$$

$$\Rightarrow y_n \in \wedge_i^{-1}\left(D_{r_i}\right), \forall n \geq m_i, 1 \leq i \leq K$$

$$\Rightarrow y_n \in G \quad \forall n \geq n_0.$$

Thus for any weak neighbourhood of G of 0, $\exists n_0 \in \mathbb{N}$ such that

$$y_n \in G, \quad \forall n \geq n_0$$

So, $y_n \to 0$ weakly and $x_n \to x$ weakly.

Corollory :

Original convergent sequence, converges weakly.

Proof :

Let $x_n \to x$ originally

i.e. $y_n \to 0$ originally, $y_n = x_n - x$

To show $\wedge x_n \to \wedge x$, $\forall \wedge \in X^*$

Since any $\wedge \in X^*$ is originally continuous at 0, for $\varepsilon > 0$, $\exists$ a neighbourhood V of 0 such that

$$|\wedge y_n| < \varepsilon, \text{ whenever } y_n \in V \quad ......(1)$$

Again, $y_n \to 0$, originally. This implies, for the neighbourhood V of 0, $\exists\, n_0 \in \mathbb{N}$ such that

$$y_n \in V, \;\forall\; n \geq n_0 \qquad \text{......(2)}$$

Combining (1) and (2), we get, for $\varepsilon > 0$, $\exists\, n \geq n_0$, such that

$$|\wedge y_n| < \varepsilon, \qquad \forall\; n \geq n_0$$

$$\Rightarrow |\wedge x_n - \wedge x| < \epsilon, \qquad \forall\; n \geq n_0$$

$$\Rightarrow \wedge x_n \to \wedge x, \qquad \forall\; \wedge \in X^*$$

Hence $x_n \to x$ weakly.

Proposition :

A set $E \subset X$ in a tvs X is weakly bounded iff every $\wedge \in X^*$ is a bounded function on E i.e. iff for every $\wedge \in X^*$, $\wedge(E)$ is bounded.

Proof :

Suppose E is weakly bounded.

Then for every V of the form

$$V = \{x : |\wedge_i x| < r_i \text{ for } 1 \leq i \leq n\}$$

where $\wedge_i \in X^*$ and $r_i > 0$, there exists $t_0 = t_0(V) > 0$ such that

$$E \subset tV, \qquad \forall\; t \geq t_0.$$

Let $\wedge \in X^*$ be arbitrary and consider $V_0 = \{x \,|\, |\wedge x| < r\}$ for $r > 0$.

Then, $\exists\, n_0 = n_0(V) > 0$ such that

$$E \subset nV_0, \qquad \forall\; n \geq n_0$$

$$\Rightarrow n_0^{-1}x \in V_0, \qquad \forall\; x \in E$$

$$\Rightarrow |\wedge n_0^{-1}x| < r, \qquad \forall x \in E$$

$$\Rightarrow |\wedge x| < n_0 r, \qquad \forall x \in E$$

$\Rightarrow \wedge$ is bounded on E.

i.e., $\wedge(E)$ is a bounded subset.

Conversely, let $E \subset X$ and $\wedge(E)$ be bounded for all $\wedge \in X^*$. To show E is weakly bounded.

Suppose $V = \{x : |\wedge_i x| < r_i, \text{ for } 1 \leq i \leq k\}$,

be a weak neighbourhood of 0, where $r_i > o$ and $\wedge_i \in X^*$.

Then $\wedge_i(E)$ is bounded for $1 \leq i \leq k$

$$\Rightarrow |\wedge_i(x)| \leq M_i < \infty, \text{ for } 1 \leq i \leq k$$

$$\forall\; x \in E \text{ and for some } M_i > 0.$$

$$\Rightarrow \sup_{x \in E} |\wedge_i(x)| \leq M_i < \infty, \text{ for } 1 \leq i \leq k.$$

Let $M = \sup_{1 \leq i \leq k} M_i$. Then

$$\sup_{x \in E} |\wedge_i (x)| \leq M_i < \infty, \forall 1 \leq i \leq k$$

Now, for any $x \in E$

$$|\wedge_i x| \leq M, \ \forall 1 \leq i \leq k$$

$$\Rightarrow |\wedge_i x| \leq r_i \frac{M}{r_i}, \ \forall 1 \leq i \leq k$$

Let $n_0 > \max_{1 \leq i \leq k} \frac{M}{r_i}$, then

$$|\wedge_i x| < r_i n_0, \ \forall 1 \leq i \leq k$$

$$\Rightarrow \left| \wedge_i \frac{x}{n_0} \right| < r_i, \forall 1 \leq i \leq n$$

$$\Rightarrow \frac{x}{n_0} \in V, \forall x \in E$$

$$\Rightarrow x \in n_0 V, \ \forall \ x \in E$$

$$\Rightarrow E \subset n_0 V.$$

Now, if $n \geq n_0$, i.e., $\frac{n_0}{n} \leq 1$.

Then $\frac{n_0}{n} V \subset V$, [$\because$ V is balanced]

$$\Rightarrow n_0 V \subset nV, \ \forall \ n \geq n_0$$

$$\Rightarrow E \subset nV, \ \forall \ n \geq n_0$$

Thus, for any weak neighbourhood V of 0, $\exists \ n_0 \in \mathbb{N}$ such that

$$E \subset nV, \ \forall \ n \geq n_0.$$

Hence, E is weakly bounded.

3.12. Theorem :

Suppose E is a **convex subset** of a **locally convex space X**. Then the **weak closure** $\overline{E}_w$ of E is **equal** to its **original closure** $\overline{E}$.

Proof :

Let τ be the original topology and τ_w be the weak topology. Then

$$\tau_w \subset \tau.$$

$\overline{E}_w$ = intersection of all weak by closed sets containing E.

$\supseteq$ intersection of all original closed sets containing E. $[\because \tau_w \subset \tau]$

$= \overline{E}$

i.e. $\overline{E}_w \supseteq \overline{E}$ ......(1)

To obtain the opposite inclusion let us choose $x_0 \in X, x_0 \notin \overline{E}$. Then by separation theorem, $\exists \wedge \in X^*$ and $\gamma \in R$ such that $\text{Re} \wedge x_0 < \gamma < \text{Re} \wedge x, \ \forall \ x \in \overline{E}$.

Consider $V = \{x : \text{Re} \wedge x < \gamma\}$

Then V is a weak neighbourhood of x_0, which does not intersect E. Thus $x_0 \notin \overline{E}_w$.

Thus $x_0 \notin \overline{E} \Rightarrow x_0 \notin \overline{E}_w$.

$\Rightarrow \overline{E}_w \subseteq \overline{E}$ ......(2)

From (1) and (2), we get $\overline{E}_w = \overline{E}$.

Proposition :

For **convex subsets** of a **locally convex space**

(a) **Originally closed** equals **weakly closed**. and

(b) **Originally dense** equals **weakly dense**.(1995)

Proof :

Suppose, E is any convex subset of a locally convex space. Then

$\overline{E}_w = \overline{E}$.

(a) Now, if E is originally closed then

$\overline{E} = E$

$\Leftrightarrow \overline{E}_w = E$

$\Leftrightarrow$ E is weakly closed.

(b) If E is originally dense, then

$\overline{E} = X \Leftrightarrow \overline{E}_w = X$ $[\because \ \overline{E}_w = E]$

$\Leftrightarrow$ E is weakly dense.

3.13. Theorem :

Suppose X is a **metrizable locally convex space.** If $\{x_n\}$ is a sequence in X that **converges weakly** to some $x \in X$. Then there is a sequence $\{y_i\}$ in X such that

(a) each y_i is a **convex combination** of **finitely many x_n**. and

(b) $y_i \to x$ **originally**.

Proof :

Let $\{x_n\}$ be a sequence in X which converges weakly to $x \in X$.

i.e. $x_n \xrightarrow{w} x$.

Consider H, the convex hull of $\{x_n\}$.

Then each $y \in H$ is of the form

$$y = \sum_{i=1}^{\infty} \alpha_i x_i \text{ with } \Sigma\alpha_i = 1$$

and for each y, only finitely many α_n are $\neq 0$.

Also, H is a convex subset of the locally convex space X and hence

$$\overline{H}_w = \overline{H}.$$

Now $x_n \xrightarrow{W} x \Rightarrow x \in \overline{H}_w = \overline{H}$ $\quad [\because \{x_n\} \subset H]$

$\Rightarrow \exists$ a sequence $\{y_i\}$ in H such that $y_i \to x$ originally, where each y_i is a convex combination of finitely many x_n's.

3.14. The weak*-topology of a dual space :

Let X be TVS whose dual is X^* which may or may not separate points of X.

Further weak*-topology is defined on X^* where as weak-topology is defined on X.

The **important** observations to make is that every $x \in X$ **induces a linear functional** f_x on X^* defined by

$$f_x(\wedge) = \wedge x, \ \forall \ \wedge \in X^*$$

and that $\{f_x : x \in X\}$ **separates points** on X^*.

Let $\mathscr{F} = \{f_x : x \in X\}$.

Since $f_x : X^* \to K$ and K is Hausdorff space, so the $\mathscr{F}$-topology on X^* is a Hausdorff topology. Further w.r.t. this F-topology, X^* is a locally convex space.(since $\mathscr{F}$ is a separating family).

This, F-topology of X^* is called the **weak*-topology** of X^*.

Since there is an isometric isomorphism between X and F.

So the weak*-topology of X^* is also defined to be the X-topology of X^*.

Also every linear functional on X^* that is weak continuous has the form

$\wedge \to \wedge x$ for some $x \in X$.

For any $\wedge_0 \in X^*$, a weak*-neighbourhood of $\wedge_0$ is

$$W = \{\wedge \in X^* : |\wedge x_i - \wedge_0 x_i| < \delta_i, 1 \le i \le n\}$$

$$= \bigcap_{i=1}^{n} \{\wedge \in X^* : |\wedge x_i - \wedge_0 x_i| < \delta_i\}$$

where x_i's are in X and δ_i's > 0.

3.15. The Banach Alaoglu Theorem :

If **V is a neighbourhood** of 0 in a **TVS** and if

$$K = \{\wedge \in X^* : |\wedge x| \le 1 \ \forall \ x \in V\}$$

then K is **weak*-comapact.**

K is sometimes called the **polar** of V.

Proof :

Since a neighbourhood of 0 is absorbing there corresponds to each $x \in X$ a number $\gamma = \gamma(x) < \infty$ such that

$$x \in \gamma V.$$

Hence

$$\wedge \in K \Rightarrow \left|\wedge\left(\frac{x}{\gamma(x)}\right)\right| \le 1$$

$$\Rightarrow |\wedge x| \le \gamma(x)$$

Let $D_x = \{\alpha \in \mathbb{C} : |\alpha| \le \gamma(x)\}$.

For each $x \in X$, D_x being closed and bounded subset of $\mathbb{C}$, is compact.

Let $P = \prod_{x \in X} D_x$ and τ be the product topology on P.

Since each D_x is compact, so is P, by Tychonoff's theorem.

The elements of P are the functions f on X(linear or not) that satisfy

$$|f(x)| \le \gamma(x), x \in X.$$

Thus $K \subset X^* \cap P$. It follows that K inherits two topologies, one from X^*(its weak*-topology, to which the conclusion of the theorem refers) and the other from the product topology τ of P.

We will see that

(a) these two topologies coincide on K, and

(b) K is closed subset of P.

Since P is compact, (b) implies that K is τ-compact and then (a) implies that K is weak*-compact.

(a) Fix some $\wedge_0 \in K$, choose $x_i \in X$, for $1 \le i \le n$, choos $\delta > 0$. Put

$$W_1 = \{\wedge \in X^* : |\wedge x_i - \wedge_0 x_i| < \delta, \text{ for } 1 \le i \le n\}$$

$$W_2 = \{f \in P : |fx_i - \wedge_0 x_i| < \delta, \text{ for } 1 \le i \le n\}$$

Let n, x_i and δ range over all admissible values. The resulting sets W_1 then form a local base for the weak*-topology of X^* at $\wedge_0$ and the sets W_2 form a local base for the product topology τ of P at $\wedge_0$ since

$$K \subset P \cap X^*, \text{ we have}$$

$$W_1 \cap K = W_2 \cap K.$$

This proves that relative weak*- topology on K coincides with the relative produced topology on K. This completes the part (a).

(b) Suppose f_0 is in the τ-closure of K. To show $f_0 \in K$.

It is enough to show , f is linear and bounded.

We proceed by choosing $x \in X$, $y \in X$ scalars α and β and $\varepsilon > 0$.

Define

$$W_1 = \{f : | f(x) - f_0(x) | < \varepsilon\}$$

$$W_2 = \{f : | f(y) - f_0(y) | < \varepsilon\}$$

$$W_3 = \{f : | f(\alpha x + \beta y) - f_0(\alpha x + \beta y) | < \varepsilon\}$$

Put $W = W_1 \cap W_2 \cap W_3$

Then W is a τ-neighbourhood of f_0. Therefore

$$W \cap K \neq \phi \qquad [\because \; f_0 \in \tau\text{-closure of } K]$$

Let $f \in W \cap K$, then f is linear.

Now

$$\begin{aligned} f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y) &\leq | f_0(\alpha x + \beta y) - f(\alpha x + \beta y) | \\ &\quad + | \alpha | \, | f_0(x) - f(x) | + | \beta | \, | f_0(y) - f(y) | \\ &< \varepsilon + | \alpha | \varepsilon + | \beta | \varepsilon \\ &= (1 + | \alpha | + | \beta |)\varepsilon. \end{aligned}$$

Since ε was choosen arbifrarily small,

$$f_0(\alpha x + \beta y) = \alpha f_0(x) + \beta f_0(y) \text{ and } f_0 \text{ is linear.}$$

f_0 is bounded :

For $x \in V$, $\varepsilon > 0$, let

$$W_0 = \{f : | f(x) - f_0(x) | \varepsilon\}$$

W_0 is a neighbourhood of f_0, and

$$W_0 \cap K \neq \phi.$$

So, $\exists \, f \in W_0 \cap K$ such that

$$| f(x) - f_0(x) | < t.$$

But $f \in K \Rightarrow | f(x) | \leq 1$.

Now $| f_0(x) | - | f(x) | \leq | f_0(x) - f(x) | < \varepsilon$

$$\Rightarrow | f_0(x) | < \varepsilon + 1 \qquad [\because \; | f(x) | \leq 1]$$

$$\Rightarrow | f_0(x) | \leq 1 \qquad [\because \; t > 0 \text{ is arbitrary}]$$

Hence $f_0 \in K$. Consequently $\overline{K} = K$.

Then proves (b) and hence the theorem.

Theorem :

If X is a **separable topological vector space**, $K \subset X^*$ and K is **weak*-compact** then K is **metrizable**, in the **weak*-topology**.

Proof :

It is enough to show the existence of a countable family of continuous real valued linear functionals of K, which separates the points of K.

Since X is separable, it has a countable dense subset $\{x_n\}$ say.

Define, $f_n : X^* \to ¢$ by

$f_n(\wedge) = \wedge x_n$

Then each f_n is weak*-continuous by the definition of the weak*-topology.

Further, if $\wedge, \wedge' \in X^*$ such that

$\wedge \neq \wedge'$

$\Rightarrow \wedge x \neq \wedge' x$, for some $x \in X$.

$\Rightarrow \wedge^x n_0 \neq \wedge'^x n_0$, for some $^x n_0 \in \{x_n\}$ $[\because \quad \overline{\{x_n\}} = X]$

$\Rightarrow {}^f n_0(\wedge) \neq {}^f n_0(\wedge')$

$\therefore$ $\{f_n\}$ separates points of X^*.

Thus $\{f_n\}$ is a separating family of continuous linear functionals on X^*.

Now defining $g_n = \text{Re} f_n$ and noting that continuity of f_n implies the continuity of $\text{Re} f_n$, we find $\{g_n\}$ is a countable family of continuous real valued functionals on X^* which separates points of $K \subseteq X^*$. Hence K is metrizable.

Theorem :

If V is a neighbourhood of 0 in a **separable topological vector space** X, and if $\{\wedge_n\}$ is a **sequence in X^*** such that

$|\wedge_n x| \leq 1, x \in V, n = 1, 2, \ldots$

then there exists a **subsequence** $\{\wedge_{n_i}\}$ and **there is a** $\wedge \in X^*$ such that

$\wedge x = \lim\limits_{i \to \infty} \wedge n_i x \qquad (x \in X)$

Proof :

Let $K = \{\wedge_n\} = \{\wedge_n \in X^* : |\wedge_n| \leq 1, \ \forall \ x \in V\} \qquad n = 1, 2, \ldots$

By Banach Alaoglu Theorem K is weak*-compact.

Again, by theorem 3.16, K is metrizable, w.r.t. relative weak*-topology.

Thus K is a compact metric space and hence it is sequentially compact and so the sequence $\{\wedge_n\}$ has a convergent subsequence $\{\wedge_{n_i}\}$ such that

$\wedge_{n_i} \xrightarrow{w} \wedge \in X^*$ as $i \to \infty$

i.e. $\lim_{i \to \alpha} \wedge n_i x = \wedge x \ (x \in X)$

Theorem :

In a **locally convex space** X, every **weakly bounded set** is **originally bounded** and conversely.

Proof :

Let τ be the original topology of X and τ_w be the weak-topology of X and E be an originally bounded subset of X. Since, every weak neighbourhood of o in X is an original neighbourhood of o it follows from definition that

E is originally bounded

$\Rightarrow$ E is absorbed by every τ-neighbourhood of o

$\Rightarrow$ E is absorbed by every τ_w-neighbourhood of o

$\Rightarrow$ E is weakly-bounded.

Conversely, suppose E is weakly-bounded. To show E is originally bounded. Let U be any τ-neighbourhood of o. Then since X is locally convex, there is a convex, balanced, original neighbourhood V of o in X, such that

$\overline{V} \subset U$.

Let $K = \{\wedge \in X^* : |\wedge x| \le 1, \ \forall \ x \in V\}$ ......(1)

By Banach Alaoglu Theorem K is weak*-compact.

We claim that $\overline{V} = \{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\}$ ......(2)

Clearly $V \subseteq \{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\}$

Since right hand side of (2) is closed,

$\overline{V} \subseteq \{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\}$

Next suppose that $x_0 \in X$and $x_0 \notin \overline{V}$, Since $\overline{V}$ is a convex, balanced and closed set in a locally convex space X, $\exists \wedge \in X^*$ such that

$\wedge x_0 > 1$ but $|\wedge x| \le 1, \ \forall \ x \in \overline{V}$

$\Rightarrow \wedge x_0 > 1$ but $\wedge \in K$ $[\because V \subseteq \overline{V}]$

$\Rightarrow x_0 \notin \{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\}$

So, $\{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\} \subseteq \overline{V}$.

Consequently,

$\overline{V} = \{x \in X : |\wedge x| \le 1, \ \forall \ \wedge \in K\}$.

Since E is weakly bounded there corresponds to each $\wedge \in X^*$ a number $\gamma(\wedge) < \infty$ such that

$$|\wedge x| \leq \gamma(\wedge), (x \in E) \qquad ......(3)$$

Since K is convex and weak*-compact and since $\wedge \rightarrow \wedge x$ are weak*-continuous,

We can apply Hahn-Banach theorem (with X^* in place of X and the scalar field in place of Y) to conclude from (3), that there is a constant $\gamma < \infty$ such that

$$|\wedge x| \leq \gamma(\wedge), (x \in E, \wedge \in K) \qquad ......(4)$$

Now, (2) and (4) show that

$$\gamma^{-1}x \in \overline{V} \subset U \ \forall \ x \in E$$

Since V is balanced

$$E \subset t\overline{V} \subset tU \qquad (t > \gamma)$$

Thus E is originally bounded.

Corollory :

Let X be a normal space. If $E \subset X$ and if

$$\sup_{x \in E} |\wedge x| < \infty \qquad (\wedge \in X^*)$$

then there exists $\gamma < \infty$ such that

$$\|x\| \leq \gamma \qquad (x \in E)$$

Proof :

Try yourself.

3.19. Definition :

(a) If X is a vector space and $E \subset X$, the convex hull of E will be denoted by $c_0(E)$ and it is the intersection of all convex subsets of X which contain E. Equivalently, $c_0(E)$ is the set of all finite convex combinations of members of E.

(b) If X is a TVS and $E \subset X$, then closed convex hull of E, written $\overline{c_0}(E)$ is the closure of E.

(c) Let K be a subset of a vector space X. A non empty set $S \subset K$ is called an extreme set of K if no point of S is an internal point of any line interval whose end points are in K, except when both end points are in S. Analytically, the condition can be expressed as follows :

If $x \in K$, $y \in K$, $0 < t < 1$, and $(1 - t)x + ty \in S$, then

$x \in S$ and $y \in S$

The extreme points of K are the extreme sets that consists of just one point. The set of all extreme points of K will be denoted by E(k).

Theorem :

Suppose X is a TVS on which X^* separates points. Suppose A and B are disjoint, non-empty, compact, convex sets in X. Then there exists $\wedge \in X^*$ such that

$$\sup_{x \in A} \operatorname{Re} \wedge x < \inf_{y \in B} \operatorname{Re} \wedge y$$

Proof :

Let X_w be X with its weak-topology. The sets A and B are evidently compact in X_w. They are also closed in X_w, because X_w is a Hausdorff space. Since X_w is locally convex, Hahn Banach can be applied to X_w in place of X; it provides $\wedge \in X_w^*$ that satisfies

$$\operatorname{Re}_{x \in A} \wedge x < \operatorname{Re}_{y \in B} \wedge y \qquad \text{......(1)}$$

But since X^* separates points $(X_w)^* = X^*$ and hence (1) holds for some $\wedge \in X^*$.

The Krein Milman Theorem :

Suppose X is a topological vector space on which X^* separates points. If K is a non-empty, compact, convex set in X, then K is the closed convex hull of the set of its extreme points.

In Symbols

$$K = \overline{co}\,(E(K)).$$

Proof :

Let P be the collections of all compact extreme sets of K. Then

$P \neq \phi$, since $K \in P$.

We shall prove the following two properties of P :

(a) The intersection S of non-empty subcollection of P is a member of P, unless $S \neq \phi$.

(b) If $S \in P$, $\wedge \in X^*$, μ the maximum of $\operatorname{Re}\wedge$ on S, and

$$S_\wedge = \{x \in S : \operatorname{Re}\wedge x = \mu\}$$

then $S_\wedge \in P$.

(a) **By definition :**

$$S = \bigcap_{\alpha \in \Delta} P_\alpha \text{, where } \{P_\alpha : \alpha \in \Delta\} \subset P$$

Since, every TVS is a Hausdroff space and since every subspace of a Hausdorff space is Hausdorff, it follows that K is a compact Hausdorff space.

Now since compact subspace of a Housedorff space is closed and any intersection of closed sets is closed, it follows that being the intersection of compact subsets of K, $S = \bigcap_{\alpha \in \Delta} P_\alpha$ is a closed subset of the compact set K and hence itself compact(because closed subset of a compact set is compact).

Hence S is compact subset of K. **S is an extreme subset of K.**

Let $x \in K$, $y \in K$, $0 < t < 1$ and

$$tx + (1 - t)y \in S$$

$$\Rightarrow tx + (1 - t)y \in P_\alpha, \quad \forall\ \alpha \in \Delta$$

$$\Rightarrow x \in P_\alpha, y \in P_\alpha, \quad \forall\ \alpha \in \Delta$$

[$\because$ P_α is an extreme point]

$$\Rightarrow x \in \bigcap_{\alpha\in\Delta} P_\alpha, y \in \bigcap_{\alpha\in\Delta} P_\alpha$$

$$\Rightarrow x, y \in S$$

$\Rightarrow$ S is an extreme subset of K.

This complets the proof of (a).

(b) For $\wedge \in X^*$

$S_\wedge = \{x \in S : \text{Re}\wedge x = \mu\}$ is a compact extreme subset of K where

$\mu = \max\{\text{Re}\wedge x : x \in S\}$.

Clearly, $S_\wedge \subseteq S$.

$S_\wedge$ is an extreme subset of K :

Suppose $x \in K$, $y \in K$, $0 < t < 1$ and

$$z = tx + (1 - t)y \in S_\wedge$$

We prove $x, y \in S_\wedge$ i.e. $\text{Re}\wedge x = \text{Re}\wedge y = \mu$.

Now, $z \in S_\wedge \Rightarrow \text{Re}\wedge z = \mu$.

Again $z \in S_\wedge \Rightarrow z \in S$ $\quad (\because\ S_\wedge \subseteq S)$

$$\Rightarrow x \in S, y \in S$$

$$\Rightarrow \text{Re}\wedge x \leq \mu, \text{Re}\wedge y \leq \mu.$$

We claim that

$$\text{Re}\wedge x = \text{Re}\wedge y$$

If possible, let $\text{Re}\wedge x \neq \text{Re}\wedge y$ and in particular

$$\text{Re}\wedge x < \text{Re}\wedge y \leq \mu \quad \text{......(1)}$$

Then since $\wedge$ is linear, and $z \in S_\wedge$

$$\mu = \text{Re}\wedge z = t\text{Re}\wedge x + (1 - t)\text{Re}\wedge y$$

$$< t\mu + (1 - t)\mu \quad [\text{using (1)}]$$

$\Rightarrow \mu \neq \mu$, a contradiction.

$\therefore$ $\text{Re}\wedge x = \text{Re}\wedge y = \mu$

$\Rightarrow x \in S_\wedge, y \in S_\wedge$

$\Rightarrow S_\wedge$ is an extreme subset of K.

$S_\wedge$ is compact :

By definition

$S_\wedge = (\mathrm{Re}\wedge)^{-1}|_S(\mu)$

Since $\mathrm{Re}\wedge$ is continuous and $\{\mu\}$ is closed

$\Rightarrow S_\wedge = (\mathrm{Re}\wedge)^{-1}(\{\mu\})$ is a closed subset of K.

$\Rightarrow S_\wedge$ is a compact subset of K.

This proves (b).

Now, we proceed to prove that

$K = \overline{co}(E(K))$.

By definition,

$E(K) \subset K$

$\Rightarrow coE(K) \subset K$ [$\because$ K is convex]

$\Rightarrow \overline{co}\,E(K) \subset K$ [$\because$ K, being compact subset of a Hausedorff space, is closed]

Thus $\overline{co}(E(K)) \subset K$ ......(A)

This shows that $\overline{co}(E(K))$ is compact.

To eastablish the other inclusion, we first show that, every compact extreme set of K contains an extreme point of K.

Choose some $S \in P$. Let P' be the collection of all members of P that are subsets of S. Since $S \in P'$, $P' \neq \phi$.

Then $(P', \subseteq)$ is a non-empty poset and hence by Hausdorff maximality theorem, P' contains a maximal totally ordered subset Ω of P'.

Let M be the intersection of all members of Ω.

Since Ω is a collection of compact sets with the finite intersection property, a topological space X is compact, iff every collection of closed subsets of X with the FIP is fixed i.e., has a non-empty intersection.

$M \neq \phi$

$\therefore$ by (a), $M \in P'$.

The maximality of Ω implies that no proper subset of M belongs to P.

[For of if $P \subsetneq M$ and $P \in P$, then $P \subsetneq S$ and $\{P\} \cup \Omega$ is a chain in P', which contains Ω and it is not true]

It follows from (b), that every $\wedge \in X^*$ is constant on M.

[For if $\wedge$ is not constant on M, then $\exists$ at least one $x \in M$ such that $\mu = \max \mathrm{Re}\wedge \neq \mathrm{Re}\wedge x$ and hence $S_\wedge$ is a proper subset of M belongs to P, which is not possible]

Since X^* separates points on X, M has only one point.

[For if $x \in M$, $y \in M$, $x \neq y$, then $\exists \wedge \in X^*$ such that $\wedge x \neq \wedge y$

$\Rightarrow S_\wedge \subsetneq M$ and is not true $\quad (\because \quad S_\wedge \in P)$]

Therefore M is an extreme point of K contained in S.

We have now proved that

$$E(K) \cap S \neq \phi, \qquad \forall\ S \in P \qquad ......(B)$$

In other words, every compact extreme set of K contains an extreme point of K.

We are left to show that $K \subset \overline{co}(E(K))$.

Assume to reach a contradiction, that some $x_0 \in K$ is not in $\overline{co}(E(K))$.

i.e. $x_0 \in K$ but $x_0 \notin \overline{co}(E(K))$.

Then with $A = \overline{co}(E(K))$ and $B = \{x_0\}$

$$\text{Re}\wedge x < \text{Re}\wedge x_0. \qquad \forall\ x \in \overline{co}(E(K)).$$

If we define

$$K_\wedge = \{x \in K : \text{Re}\wedge x = \mu\} \qquad \text{where } \mu = \max_{x \in K} \text{Re} \wedge x.$$

Then $K_\wedge \in P$ and $\qquad K_\wedge \cap \overline{co}(E(K)) = \phi$.

which contradict (B), because $E(K) \subset \overline{co}(E(K))$.

Hence $\quad K \subset \overline{co}(E(K)) \qquad ......(C).$

From (A) and (C), it follows that

$$K = \overline{co}\, E(K)$$

Hence the theorem has been completly establish.

●●●

Duality in Banach Space

4.1. Suppose X and Y are normal spaces. Associate to each $\wedge \in B(X, Y)$ the number

$$\| \wedge \| = \sup\{\| \wedge x \| : x \in X, \| x \| \leq 1\}.$$

Then B(X, Y) is a normal space w.r.to the above normed and B(X, Y) is a Banach space if Y is a Banach space.

Proof :

B(X, Y) is a subspace of the vector space L(X, Y) of all linear mappings from X into Y.

For $\wedge_1, \wedge_2 \in B(X, Y)$ and $\alpha, \beta \in K$,

$$\| (\alpha\wedge_1 + \beta\wedge_2)(x) \| = \| \alpha\wedge_1(x) + \beta\wedge_2(x) \|$$
$$\leq | \alpha | \| \wedge_1(x) \| + | \beta | \| \wedge_2(x) \|$$
$$\leq (| \alpha | \| \wedge_1 \| + | \beta | \| \wedge_2 \|) \| x \|$$

So, $\alpha\wedge_1 + \beta\wedge_2' \in B(X, Y)$ and $B(X, Y)$ is a vector subspace.

N_1) $\|\wedge\| = \text{Sup}\{\|\wedge x\| : x \in X, \|x\| \leq 1\}$

$\geq 0.$

and is finite, because $\wedge$ is bounded in the closed unit ball.

If $\wedge = 0$, then $\|\wedge\| = 0$.

and if $\|\wedge\| = 0$, then for any x,

$\|\wedge x\| \leq \|\wedge\| \|x\| = 0.$

$\therefore \quad \|\wedge x\| = 0$

$\Rightarrow \wedge x = 0, \quad \forall x$

$\Rightarrow \wedge = 0.$

N_2) For $\alpha \in K$, $\|\alpha\wedge\| = \sup\{\|(\alpha\wedge)(x)\| : \|x\| \leq 1\}$

$= \sup\{|\alpha| \|\wedge(x)\| : \|x\| \leq 1\}$

$= |\alpha| \sup\{\|\wedge x\| : \|x\| \leq 1\}$

$= |\alpha| \|\wedge\|.$

N_3) For $\wedge_1 \in B(X, Y)$, $\wedge_2 \in B(X, Y)$

$\|(\wedge_1 + \wedge_2)(x)\| \leq \|\wedge_1(x)\| + \|\wedge_2(x)\|$

$\leq (\|\wedge_1\| + \|\wedge_2\|) \|x\|$

$\leq \|\wedge_1\| + \|\wedge_2\| \quad \forall x, \|x\| \leq 1$

So, $\sup_{\|x\|\leq 1} \|\wedge_1(x) + \wedge_2(x)\| \leq \|\wedge_1\| + \|\wedge_2\|$

and $\|\wedge_1 + \wedge_2\| \leq \|\wedge_1\| + \|\wedge_2\|$

2nd Part :

Assume y is a Banach space. To show B(X, Y) is a Banach space.

Let $\{f_n\}$ be a cauchy seq in B(X, Y).

For $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$\|f_n - f_m\| < \varepsilon \qquad \forall m, n \geq n_0.$

Now $\|f_n(n) - f_m(n)\| = \|(f_n - f_m)(x)\|$

$\leq \|f_n - f_m\| \|x\|$

$< \varepsilon \|x\| \qquad \text{......}(*)$

Hence $\{f_n(x)\}$ is a cauchy sequence in Y for each $x \in X$.

Since Y is a Banach space $\exists y \in Y$ such that $\lim_{n\to\infty} f_n(x) = y$.

We define $f : X \to Y$ by $f(x) = y$.

We have to show that (i) $f \in B(X, Y)$

(ii) $f_n \to f$.

(1) f is linear :

$$f(\alpha x + \beta y) = \lim_{n\to\infty} f_n(\alpha x + \beta y)$$
$$= \alpha \lim_{n\to\infty} f_n(x) + \beta \lim_{n\to\infty} f_n(y)$$
$$= \alpha f(x) + \beta f(y).$$

f is bounded :

$f(x) - f_n(x) = \lim_{m\to\infty} f_m(x) - f_n(x)$, for each $x \in X$ and each $n \in N$.

Then

$$\| f(x) - f_n(x) \| = \left\| \lim_{m\to\infty} f_m(x) - f_n(x) \right\|$$
$$= \lim_{m\to\infty} \| f_m(x) - f_n(x) \|$$
$$< \varepsilon \| x \| \qquad \forall n \geq n_0.$$

Hence

$$\| f(x) \| = \| f(x) - f_{n_0}(x) + f_{n_0}(x) \|$$
$$\leq \| f(x) - f_{n_0}(x) \| + \| f_{n_0}(x) \|$$
$$< \varepsilon \| x \| + \| f_{n_0} \| \| x \|$$
$$= (\varepsilon + \| f_{n_0} \|) \| x \|$$

So, f is a bounded and $f \in B(X, Y)$.

(ii) To show $f_n \to f$.

$$\| f - f_n \| = \text{Sup}\{\| f(x) - f_n(x) \| : \| x \| \leq 1\}$$
$$< \text{Sup}\{\varepsilon \| x \| : \| x \| \leq 1\}$$
$$< \varepsilon \qquad \forall n \geq n_0.$$

Hence, $f_n \to f$ and $B(X, Y)$ is a Banace space.

Note : keeping $Y = K$, $B(X, Y) = B(X, K) = X^*$ is a Banach space.

Theorem :

Suppose B is the closed unit ball of a normal linear space X. Define

$$\| x^* \| = \text{Sup}\{|\langle x, x^*\rangle| : x \in B\}, \text{ for every } x^* \in X^*.$$

$\langle x, x^*\rangle$ stands for $x^*(x)$.

(a) This norm makes X^* into a Banach space.

(b) If B^* is the closed unit ball in X^*, then for every $x \in X$,

$$\| x \| = \text{Sup}\{|<x, x^*>| : x^* \in B^*\}$$

Consequently $x^* \to <x, x^*>$ is a bounded linear functional on X^* with norm $\| x \|$.

(c) B^* is weak* compact.

Proof :

(a) We know B(X, Y) is a Banach space if Y is a Banach space. Since K is a Banach space, $X^* = B(X, K)$ is a Banach space with

$$\| x^* \| = \text{Sup}\{| x^*(x) | : \| x \| \leq 1\}$$
$$= \text{Sup}\{| <x, x^*> | : x \in B\}.$$

(b) $\quad B^* = \{x^* \in X^* \mid \| x^* \| \leq 1\}.$

To show $\| x \| = \text{Sup}\{| <x, x^*> | : x^* \in B^*\}$.

$$| <x, x^*> | = | x^*(x) | \leq \| x^* \| \| x \| \leq \| x \| \qquad \forall x^* \in B^*.$$

So, $\text{Sup}\{| <x, x^*> | : x^* \in B^*\} \leq \| x \| \qquad \forall x^* \in B^*$.

If possible let,

$$\text{Sup}\{| <x, x^*> | : x^* \in B^*\} < \| x \|$$
$$| <x, x^*> | < \| x \| \qquad \forall x^* \in B^*.$$

But by Hahn-Banach Theorem, for $x \neq 0$, $\exists\, x^* \in X^*$ such that

$$| <x, x^*> | = \| x \| \text{ with } \| x^* \| = 1.$$

This is a contracdiction.

Thus, $\text{Sup}\{| <x, x^*> | : x^* \in B^*\} = \| x \|$.

Next we have to show $x^* \to <x, x^*>$ is a bounded linear functional on X^* norm $\| x \|$.

We define a map $\phi_x : X^* \to K$ by,

$$\phi_x(x^*) = x^*(x).$$

ϕ_x is linear :

$$\phi_x(\alpha x^* + \beta y^*) = (\alpha x^* + \beta y^*)(x)$$
$$= \alpha x^*(x) + \beta y^*(x)$$
$$= \alpha\phi_x(x^*) + \beta\phi_x(y^*)$$

ϕ_x is bounded :

$$| \phi_x(x^*) | = | x^*(x) |$$
$$\leq \| x^* \| \| x \|$$

$= M \| x^* \| \qquad (M = \| x \|, \text{ fixed})$

So, ϕ_x is bounded.

To show,

$\| \phi_x \| = \| x \|$

$\| \phi_x \| = \sup\{| \phi_x(x^*) | : \| x^* \| \leq 1\}.$

$= \text{Sup}\{| x^*(x) | : x^* \in B^*\}$

$= \| x \|.$

(c) B^* is weak* compact Hausdorff.

B^* is weak* Hausdorff :

Let $x^*, y^* \in B^*$ such that $x^* \neq y^*$.

$\therefore \quad \| x^* \| \leq 1$ and $\| y^* \| \leq 1$

and $\exists\, x \in X$ such that $x^*(x) \neq y^*(x)$.

Now we put, $3\varepsilon = \| x^*(x) - y^*(x) \|$.

Take weak* neighbourhood N_1 and N_2 as,

$N_1(x^*, x, \varepsilon) = \{z^* \mid \| z^*(x) - x^*(x) \| < \varepsilon\}$

$N_2(y^*, x, \varepsilon) = \{z^* \mid \| z^*(x) - y^*(x) \| < \varepsilon\}$

We have to show $N_1 \cap N_2 = \phi$.

If not, $\exists\, z_0^* \in N_1 \cap N_2$.

$\| z_0^*(x) - x^*(x) \| < \varepsilon$ and $\| z_0^*(x) - y^*(x) \| < \varepsilon$.

$3\varepsilon = \| x^*(x) - y^*(x) \|$

$= \| x^*(x) - z_0^*(x) + z_0^*(x) - y^*(x) \|$

$\leq \| x^*(x) - z_0^*(x) \| + \| z_0^*(x) - y^*(x) \|$

$< \varepsilon + \varepsilon$

$= 2\varepsilon.$

This shows that $3 < 2$, which is absurd.

So $N_1 \cap N_2 = \phi$.

Hence B^* is weak* Hausdorff.

$B^* = \{x^* : \| x^* \| \leq 1\}$ is weak* compact :

Let $C_x = [-\| x \|, \| x \|]$ when x is real.

$= \{z \mid | z | \leq \| x \|\}$ when x is complex.

C_x being closed and bounded subset of K is compact. By Tychnoff's theorem,

$C = \prod_{x \in X} C_x$ is also compact space.

For $x \in X$, $\{f(x) \mid f \in B^* \subset C_x\}$.

So $f \in B^*$

$$\Rightarrow \{f(x) \mid x \in X\} \in \prod_{x \in X} C_x = C.$$

So, $B^* \subset C$.

The weak* topology on B^* is same as relative product topology of C on B^*.

Since C is w^* compact it is enough to show that B^* is weak* closed subset of C.

To show $\overline{B^*} = B^*$, the closure w.r.t. weak* topology.

Let $g \in \overline{B^*}$, we have to show $g \in B^*$.

$$g \in C \Rightarrow g(x) \in C_x.$$

$$\Rightarrow |g(x)| \le \|x\| \le 1 \quad \text{if} \quad \|x\| \le 1$$

$$\Rightarrow \sup_{\|x\| \le 1} |g(x)| \le 1$$

$$\Rightarrow \|g\| \le 1.$$

So g is bounded and $\|g\| \le 1$.

We are left to show g is linear.

Let $x, y \in X$ and $\alpha, \beta \in K$.

We shall show (a) $g(x+y) = g(x) + g(y)$

(b) $g(\alpha x) = \alpha g(x)$.

$g \in \overline{B^*} \Rightarrow$ every w^*-neighbourhood of g intersect B^*.

Choose weak* neighbourhood N_1, N_2, N_3 of g such that

$$N_1(g, x, \varepsilon) = \{h : |h(x) - g(x)| < \frac{\varepsilon}{3}\}$$

$$N_2(g, y, \varepsilon) = \{h : |h(y) - g(y)| < \frac{\varepsilon}{3}\}$$

$$N_3(g, x+y, \varepsilon) = \{h : |h(x+y) - g(x+y)| < \frac{\varepsilon}{3}\}.$$

Then $N_1 \cap N_2 \cap N_3$ is w^* neighbourhood of g, which contains a member f of B^* ($\because g \in \overline{B^*}$).

$$\therefore \quad |f(x) - g(x)| < \frac{\varepsilon}{3}, |f(y) - g(y)| < \frac{\varepsilon}{3}, |f(x+y) - g(x+y)| < \frac{\varepsilon}{3}.$$

$$|g(x+y) - g(x) - g(y)| = |g(x+y) - f(x+y) - g(x) + f(x) - g(y) + f(y)|$$

$$[\because \quad f(x+y) = f(x) + f(y)]$$

$$\leq | f(x+y) - g(x+y) | + | f(x) - g(x) | + | f(y) - g(y) |$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So, $g(x+y) = g(x) + g(y)$.

Choose w^*-neighbourhood N_4 and N_5.

$$N_4(g, \alpha x, \varepsilon) = \{h : | h(\alpha x) - g(\alpha x) | < \frac{\varepsilon}{2}\}$$

$$N_5(g, x, \epsilon) = \{h : | h(y) - g(x) | < \frac{\varepsilon}{2|\alpha|}\}$$

Since $g \in \overline{B^*}$, $N_4 \cap N_5$, a weak*-neighbourhood of g contains a member f .of B^*.

So, $| f(\alpha x) - g(\alpha x) | < \frac{\varepsilon}{2}$ and $| f(x) - g(x) | < \frac{\varepsilon}{2|\alpha|}$

$$| g(\alpha x) - \alpha g(x) | = | g(\alpha x) - f(\alpha x) - \alpha g(x) + \alpha f(x) | \qquad (\because f(\alpha x) = \alpha f(x))$$

$$\leq | g(\alpha x) - f(\alpha x) | + | \alpha | | g(x) - f(x) |$$

$$< \frac{\varepsilon}{2} + |\alpha| \frac{\varepsilon}{2|\alpha|}$$

$$= \varepsilon.$$

Hence, $g(\alpha x) = \alpha g(x)$ and $g \in B^*$.

So, $\overline{B^*} = B^*$.

Thus B^* is w^*-closed and hence w^*-compact.

Note : 4.4.Comparison of sup norm topology and weak* topology on X^* :

Weak* topology on X^* is the coarset topology w.r.t. which ϕ_x define by $\phi_x(x^*) = x^*(x)$ are continuous.

Also $\| \phi_x(x^*) \| = \| x^*(x) \| \leq \| x \| \| x^* \|$

So, ϕ_x are continuous w.r.t. sup norm top of X^*. Hence weak* topology is coarset r than the sup norm top of X^*.

Alternatively $\| f \|$ for $f \in B(x, y)$ can be defined as

$$\| f \| = \sup\{\| f(x) \| : \| x \| \leq 1\} \qquad \text{......(1)}$$

Also $\| x \| = \sup\{\| x^*(x) \| : \| x^* \| \leq 1\}$

Replacing x by $f(x) \in Y$,

$$\| f(x) \| = \sup\{\| y^*(f(x)) \| : \| y^* \| \leq 1\}$$

So from (1), $\| f \| = \sup_{\|x\| \leq 1} \sup_{\|y^*\| \leq 1} \|y^*(f(x))\|$

$$= \sup\{\| \langle f(x), y^* \rangle \| : \| x \| \leq 1, \| y^* \| \leq 1\}.$$

4.6. Annihilator :

Let X be a Banach space and X^* be its dual space. For subspace $M \subset X$, annihilator of M is defined by,

$$M^{\perp} = \{x^* \in X^* : x^*(M) = \{0\}\}$$
$$= \{x^* \in X^* : \langle x, x^* \rangle = 0, \ \forall x \in M\}.$$

For subspace N of X^* annihilator of N is defined by ${}^{\perp}N = \{x \in X \mid \langle x, x^* \rangle = 0 \ \forall x^* \in N\}$.

Note 1 :

$$M^{\perp} = \bigcap_{x \in M} \ker \phi_x \text{ , where } \phi_x(x^*) = x^*(x).$$

$$x^* \in M^{\perp} \Leftrightarrow x^*(x) = 0 \qquad \forall x \in M$$
$$\Leftrightarrow \phi_x(x^*) = 0 \qquad \forall x \in M$$
$$\Leftrightarrow x^* \in \bigcap_{x \in M} \ker \phi_x$$

$\therefore \quad M^{\perp} = \bigcap_{x \in M} \ker \phi_x$.

Note 2 :

$M^{\perp}$ is a weak* closed subspace of X^*.

weak*topology is the coarset topology on X^* w.r.t. which all ϕ_x defined by $\phi_x(x^*) = x^*(x)$ are continuous.

Hence, ϕ_x is weak*continuous.

$\therefore \quad \ker\phi_x = \phi_x^{-1}(\{0\})$ is weak* closed in X^*.

$\Rightarrow M^{\perp} = \bigcap_{x \in X} \ker \phi_x$ is weak* closed in X^*.

Note 3 :

${}^{\perp}N$ is norm closed subspace of X.

Let $\{x_n\}$ be a sequence in ${}^{\perp}N$ such that $x_n \to x$ in X.

To show $x \in {}^{\perp}N$.

Also, $x_n \in {}^{\perp}N \langle x_n, x^* \rangle = 0 \ \forall x^* \in N \ \forall n \in \mathbb{N}$.

Now, $x^*(x) = x^*\left(\lim_{n \to \infty} x_n\right)$

$$= \left(\lim_{n \to \infty} x^*(x_n)\right) \qquad (\because \quad x^* \text{ is continuous})$$

$$= 0 \qquad \forall x^* \in N$$

Hence $x \in {}^{\perp}N$ and ${}^{\perp}N$ is norm **closed**.

Theorem :

Suppose X is a Banach space, M is a subspace of X and N a subspace of X^*. Then

(a) $^{\perp}(M^{\perp})$ is the norm closure of M in X.

(b) $(^{\perp}N)^{\perp}$ is the weak* closure of N in X^*.

Proof :

(a) We observe that $M \subset {}^{\perp}(M^{\perp})$.

Let $x \in M \Rightarrow x^*(x) = 0 \ \forall x^* \in M^{\perp}$.

$\Rightarrow x \in {}^{\perp}(M^{\perp})$

So, $M \subset {}^{\perp}(M^{\perp})$.

Also $^{\perp}(M^{\perp})$ is normed closed in X.

And $M \subset \overline{M} \subset {}^{\perp}(M^{\perp})$.

Next to show, $^{\perp}(M^{\perp}) \subset \overline{M}$.

Suppose $x_0 \notin \overline{M}$. The $\exists\, x^* \in X^*$ such that

$x^*(x) = 0 \quad \forall x \in M$

and $x^*(x_0) \neq 0$.

Then $x^* \in M^{\perp}$ but $x_0 \notin {}^{\perp}(M^{\perp})$.

Hence $^{\perp}(M^{\perp}) \subset \overline{M}$ and $\overline{M} = {}^{\perp}(M^{\perp})$.

(b) $N \subset (^{\perp}N)^{\perp}$

Let $x^* \in N \Rightarrow x^*(x) = 0 \ x \ \forall \in {}^{\perp}N$

$\Rightarrow x^* \in (^{\perp}N)^{\perp}$.

So, $N \subset (^{\perp}N)^{\perp}$ and $(^{\perp}N)^{\perp}$ is weak* closed.

But weak* closure $\overline{N_{w^*}}$ is the smallest weak* closed set containing N_0. Consequently,

$\overline{N_{w^*}} \subset (^{\perp}N)^{\perp}$

For reverse inclusion, for $x_0^* \notin \overline{N_{w^*}}$.

Applying Hahn-Banach Theorem in (X^*, T_{w^*}), $\exists\, F_{x_0}$ in weak* dual of X^* such that

$x^*(x_0) = F_{x_0}(x^*) = 0 \qquad \forall x^* \in N$

and $F_{x_0}(x_0^*) \neq 0$

So $x_0 \in {}^{\perp}N$ but $x_0^* \notin (^{\perp}N)^{\perp}$

Hence $(^{\perp}N)^{\perp} \subset \overline{N_{w^*}}$ and $\overline{N_{w^*}} = (^{\perp}N)^{\perp}$.

Dual space of a subspace and quotient space :

Theorem :

Let M be a closed subspace of a Banacn space X.

Define $\sigma : M^* \to \frac{X^*}{M^\perp}$ by

$$\sigma(m^*) = x^* + M^\perp, \text{ when } x^* \text{ is H.B. extension of } x^*.$$

Then σ is an isometric isomorphism.

Proof :

σ is we defined :

Let x_1^* and x_2^* be two extensions of $m^* \in M$.

To show $x_1^* + M^\perp = x_2^* + M^\perp$.

$$x_1^*(m) = x_2^*(m) = m^*(m), \forall\, m \in M.$$

$$\Rightarrow (x_1^* - x_2^*)(m) = 0 \qquad \forall\, m \in M.$$

$$\Rightarrow x_1^* - x_2^* \in M^\perp$$

$$\Rightarrow x_1^* + M^\perp = x_2^* + M^\perp.$$

(ii) σ is linear :

Let x_1^* and x_2^* be extensions of m_1^* and m_2^* and $\alpha, \beta \in K$.

Then $\alpha x_1^* + \beta x_2^*$ is an extension of $\alpha m_1^* + \beta m_2^*$.

So $\sigma(\alpha m_1^* + \beta m_2^*) = \alpha x_1^* + \beta x_2^* + M^\perp$

$$= \alpha(x_1^* + M^\perp) + \beta(x_2^* + M^\perp)$$

$$= \alpha\sigma(m_1^*) + \beta\sigma(m_2^*).$$

(iii) σ in onto :

Let $x^* + M^\perp \in \frac{X^*}{M^\perp}$.

Put $m^* = x^*/_M$. Then $m^* \in M^*$ and $\sigma(m^*) = x^* + M^\perp$.

(iv) σ is isometric :

To show that $\| m^* \| = \| x^* + M^\perp \|$.

If x^* be an extension of $m^* \in M^*$. then

$$\| x^* \| \geq \| m^* \| \qquad \text{......(1)}$$

$$\| x^* + y^* \| = \sup_{\substack{\|x\|\le 1 \\ x\in X}} \| (x^* + y^*)(x) \|$$

$$\ge \sup_{\substack{\|x\|\le 1 \\ x\in M}} \| x^*(x) \|$$

$$= \sup_{\|x\|\le 1} \| m^*(x) \|$$

$$= \| m^* \|$$

Now $\text{glb}\, \| x^* + y^* \| \ge \| m^* \|$

$y^* \in M^\perp$

$\Rightarrow \| x^* + M^\perp \| = \text{glb}\, \| x^* + y^* \|$

$y^* \in M^\perp$

$\le \| x^* \| \qquad (\because \quad 0 \in M^\perp)$

$\therefore \quad \| m^* \| \le \| x^* + M^\perp \| \le \| x^* \|$.

This is true for all extension x^* of m^*.

By H.B. theorem $\exists\, x^*$ such that $\| x^* \| = \| m^* \|$

$\therefore \quad \| m^* \| \le \| x^* + M^\perp \| \le \| m^* \|$

$\therefore \quad \| m^* \| = \| x^* + M^\perp \|$

i.e., $\| m^* \| = \| \sigma(m^*) \|$. So, σ is an isometry.

Thus M^* is isometrically isomorphic to $\dfrac{X^*}{M^\perp}$.

Theorem :

Let M be a closed subspace of a Banach space X.

Define $\tau : \left(\dfrac{X}{M}\right)^* \to M^\perp$.

by $[\tau(z^*)](x) = z^*(x + M)$

Then τ is a isometric isomorphism.

Proof :

First we show that τ is well defined.

i.e. for $z^* \in \left(\dfrac{X}{M}\right)^*$,

$\tau(z^*) \in M^\perp \subset x^*$

To show $\tau(z^*)$ is linear bounded on X and $\tau(z^*)(m) = 0 \ \forall\, m \in M$.

$$\therefore \quad [\tau(z^*)](\alpha x + \beta y) = z^*(\alpha x + \beta y + M)$$
$$= z^*\{\alpha(x + M) + \beta(y + M)\}$$
$$= \alpha z^*(x + M) + \beta z^*(y + M)$$
$$= \alpha[\tau(z^*)](x) + \beta[\tau(z^*)](y)$$

This shows that $\tau(z^*)$ is linear.

And from $|[\tau(z^*)](x)| = |z^*(x + M)|$

$$\le \|z^*\| \|x + M\| \quad \because \ z^* \in \left(\frac{X}{M}\right)^*$$
$$\le \|z^*\| \|x\| \qquad (\because \ \|x + M\| \le \|x + m\| \ \forall\, m \in M)$$

it follows that $\tau(z^*)$ is bounded.

So, $\tau(z^*) \in X^*$.

And for $m \in M$, $[\tau(z^*)](m) = z^*(m + M)$

$$= z^*(M)$$
$$= 0.$$

Hence $\quad \tau(z^*) \in M^\perp$ and the mapping is well defined.

. τ is a linear on $\left(\frac{X}{M}\right)^*$:

For $z_1^*, z_2^* \in \left(\frac{X}{M}\right)^*$

$$[\tau(\alpha z_1^* + \beta z_2^*)](x) = (\alpha z_1^* + \beta z_2^*)(x + M)$$
$$= \alpha z_1^*(x + M) + \beta z_2^*(x + M)$$
$$= \alpha[\tau(z_1^*)](x) + \beta[\tau(z_2^*)](x)$$
$$= [\alpha\tau(z_1^*) + \beta\tau(z_2^*)](x)$$

So $\tau(\alpha z_1^* + \beta z_2^*) = \alpha\tau(z_1^*) + \beta\tau(z_2^*)$ and linearly of τ follows.

τ is onto :

Let $x^* \in M^\perp$. We define

$$z^* : \frac{X}{M} \to k \text{ by } z^*(x + M) = x^*(x).$$

To show z^* is well defined and $z^* \in \left(\frac{X}{M}\right)^*$.

For x, y $\in$ M

$$x + M = y + M$$
$$\Rightarrow x - y \in M$$
$$\Rightarrow x^*(x - y) = 0 \qquad \because \quad x^* \in M^\perp$$
$$\Rightarrow x^*(x) = x^*(y)$$
$$\Rightarrow z^*(x + M) = z^*(y + M)$$

z^* is linear on $\left(\frac{X}{M}\right)$:

For $x + M, y + M \in \frac{X}{M}$, $\alpha, \beta \in k$

$$\begin{aligned} z^*[\alpha(x+M) + \beta(y+M)] &= z^*((\alpha x + \beta y) + M) \\ &= x^*(\alpha x + \beta y) \\ &= \alpha x^*(x) + \beta x^*(y) \\ &= \alpha z^*(x+M) + \beta z^*(y+M) \end{aligned}$$

Hence z^* is linear.

z^* is bounded on $\frac{X}{M}$:

$$\begin{aligned} |z^*(x+M)| &= |z^*(x+m+M)| && \forall m \in M \\ &= |x^*(x+m)| && \text{(definition of } z^*) \\ &\le \|x^*\| \, \|x+m\| && (\because \ x^* \text{ is bounded}) \end{aligned}$$

$$\therefore \quad \frac{1}{\|x^*\|} |z^*(x+M)| \le \|x+m\| \qquad \forall m \in M.$$

$$\therefore \quad \frac{1}{\|x^*\|} |z^*(x+M)| \le \text{glb} \|x+m\| = \|x+M\|$$

or $\quad |z^*(x+M)| \le \|x^*\| \, \|x+M\| \qquad$ ......(*)

So, z^* is bounded and $z^* \in \left(\frac{X}{M}\right)^*$.

Then $[\tau(z^*)](x) = z^*(x+M) = x^*(x)$.

So, $\tau(z^*) = x^*$. Henec τ is onto and $R(\tau) = M^\perp$.

τ is isometric :

To show $\| z^* \| = \| \tau(z^*) \|$

$\tau(z^*) = x^* \in M^\perp$

$\| \tau(z^*) \| = \| x^* \| \geq \| z^* \|$ (from (*)) ......(**)

Also $|(\tau(z^*))(x)| = |z^*(x+M)|$

$\leq \| z^* \| \| x+m \|$

$\leq \| z^* \| \| x+m \| \qquad \forall\, m \in M$

$\therefore \quad |(\tau(z^*))(x)| \leq \| z^* \| \| x \| \leq \| z^* \| \qquad \forall\, x, \| x \| \leq 1.$

$\therefore \quad \sup_{\|x\|\leq 1} |(\tau(z^*))(x)| \leq \| z^* \|$

$\therefore \quad \| \tau(z^*) \| \leq \| z^* \|$ ......(***)

From (**) and (***)

$\| z^* \| = \| \tau(z^*) \|.$

So τ is isometric isomorphism.

Adjoint

Theorem :

Suppose X and Y are normed spaces.

To each $T \in B(X, Y)$, $\exists\, T^* \in B(Y^*, X^*)$ satisfying,

(i) $\langle Tx, y^* \rangle = \langle x, T^*y^* \rangle$ i.e., $y^*(Tx) = (T^*y^*)(x)$

(ii) $\| T \| = \| T^* \|$.

Proof :

Define $T^* : Y^* \to X^*$ by

$[T^*(y^*)](x) = y^*(Tx).$

T^* is linear :

$$\begin{aligned}[T^*(\alpha y_1^* + \beta y_2^*)](x) &= (\alpha y_1^* + \beta y_2^*)(Tx)\\ &= \alpha y_1^*(Tx) + \beta y_2^*(Tx)\\ &= \alpha(T^*y_1^*)(x) + \beta(T^*y_2^*)(x)\\ &= [\alpha T^*(y_1^*) + \beta T^*(y_2^*)](x)\end{aligned}$$

$\therefore \quad T^*(\alpha y_1^* + \beta y_2^*) = \alpha T^*(y_1^*) + \beta T^*(y_2^*).$

Bounded :

$|T^*(y^*)| = \sup\{|(T^*y^*)(x)| : \| x \| \leq 1\}$

$$= \text{Sup}\{| y^*(Tx) | : \| x \| \le 1\}$$
$$\le \text{Sup}\{\| y^* \| \| Tx \| : \| x \| \le 1\}$$
$$= \| y^* \| \| T \|$$

So, T^* is bounded.

Thus $| T^*(y^*) | \le \| y^* \| \| T \|$ for all $\| y^* \| \le 1$.

Consequently, $\| T^* \| \le \| T \|$.

Again by 4.4. alt definition of $\| T \|$ is

$$\| T \| = \text{Sup}\{| y^*(Tx) | : \| x \| \le 1, \| y^* \| \le 1\}$$
$$= \text{Sup}\{| (T^*y^*)(x) | : \| x \| \le 1, \| y^* \| \le 1\}$$
$$= \text{Sup}\{\| T^*y^* \| : \| y^* \| \le 1\}$$
$$= \| T^* \|$$

$\therefore \quad \| T^* \| = \| T \|$.

Uniqueness of T^* :

If $S^* \in B(Y^*, X^*)$ such that

$$(S^*(y^*))(x) = y^*(Tx)$$

Also $(T^*y^*)(x) = y^*(Tx)$

$\therefore \quad (S^*y^*)(x) = (T^*y^*)(x)$

$\therefore \quad S^*y^* = T^*y^*$

$\Rightarrow S^* = T^*$.

If $T : X \to Y$ be a (bounded) linear operator then null space of T, $N(T) = \{x \in X \mid Tx = 0\}$

Range space of T, $R(T) = \{Tx \mid x \in X\}$.

Theorem :

Suppose X, Y are Banach spaces and $T \in B(X, Y)$ then

(a) $N(T^*) = R(T)^{\perp}$

(b) $N(T) = {}^{\perp}R(T^*)$.

Proof :

(a) $N(T^*) = \{y^* \in Y^* \mid T^*y^* = 0\}$

$$= \{y^* \in Y^* \mid (T^*y^*)(x) = 0, x \in X\}$$
$$= \{y^* \in Y^* \mid y^*(Tx) = 0, x \in X\}$$

$= \{y^* \in Y^* \mid y^*(R(T)) = 0\}$

$= R(T)^{\perp}$.

(b) $N(T) = \{x \in X \mid Tx = 0\}$

$= \{x \in X \mid y^*(Tx) = 0 \quad \forall y^* \in Y^*\}$ $\quad$ ($\because$ y^* separates points of Y)

$= \{x \in X \mid (T^*y^*)(x) = 0 \ \forall y^* \in Y^*\}$

$= {}^{\perp}R(T^*)$

Corollary (a) :

$N(T^*)$ is weak* closed in Y^*.

Proof :

$N(T^*) = R(T)^{\perp}$

But $M^{\perp}$ is weak* closed for every subspace M of X.

$\therefore$ $N(T^*) = R(T)^{\perp}$ is weak* closed.

Corollary (b) :

R(T) is dense in Y iff T^* is one-one.

Proof :

Suppose R(T) is dense in Y. i.e. $\overline{R(T)} = Y$.

To show $T^* : Y^* \to X^*$ is 1 - 1. It is enough to show $N(T^*) = R(T)^{\perp} = \{0\}$.

Let $f \in R(T)^{\perp}$

$\Rightarrow f(R(T)) = \{0\}$,

For $y \in Y \Rightarrow y = \lim\limits_{n\to\infty} Tx_n$ $\quad$ ($\because$ $\overline{R(T)} = Y$)

$\Rightarrow f(y) = \lim\limits_{n\to\infty} f(Tx_n) = 0$ $\quad \because$ $T(x_n) \in R(T)$

$\Rightarrow f = 0$.

$\therefore$ $R(T)^{\perp} = N(T^*) = 0$

$\therefore$ T^* is one-one.

Conversely suppose T^* is one-one.

i.e. $R(T)^{\perp} = \{0\}$.

We have to show R(T) is dense in Y.

Suppose $\overline{R(T)} \neq Y$.

Then $\exists y_0 \in Y$ such that $y_0 \notin \overline{R(T)}$.

By H.B. Theorem $\exists\, y^* \in Y^*$ such that

$y^*(R(T)) = 0$ and $y^*(y_0) \neq 0$.

$\Rightarrow y^* \in R(T)^{\perp}$ and $y^* \neq 0$.

This contradicts $R(T)^{\perp} = \{0\}$.

$\therefore$ R(T) is dense in Y.

Corollary (c) :

T is one-one iff $R(T^*)$ is weak* dense in X^*.

Proof :

Let $R(T^*)$ be weak*-dense in X^*.

To show T is one-one it is enough to show that

$$N(T) = {}^{\perp}R(T^*) = \{0\}.$$

Let $x \in {}^{\perp}R(T^*)$.

Then $(T^*y^*)(x) = 0 \qquad \forall y^* \in Y^*$

$\Rightarrow x^*(x) = 0 \qquad \forall x^* \in R(T^*) \qquad \text{......(*)}$

$\Rightarrow x^*(x) = 0 \qquad \forall x^* \in X^*.$

[$x^* \in X^* = \overline{R(T^*)}$, weak* closure.

So every weak* neighbourhood of x^* intersects $R(T^*)$.

$\{f : |f(x) - x^*(x)| < \varepsilon\}$ is weak* neighbourhood of x^*.

So, $\exists f_0 \in R(T^*)$ such that

$|f_0(x) - x^*(x)| < \varepsilon$

$\Rightarrow |x^*(x)| < \varepsilon \qquad$ (Sinec $f_0 \in R(T^*) \Rightarrow f_0(x) = 0$ by (*))

$\Rightarrow x^*(x) = 0 \qquad \forall x^* \in X^*.$]

$\Rightarrow x = 0 \qquad$ (Since X^* separates points of X).

Conversely let T be one-one.

So, $N(T) = {}^{\perp}R(T^*) = \{0\}$.

To show $R(T^*)$ is weak* dense in X^*.

Suppose $\overline{R(T^*)} \neq X^*$, the closure being weak* closure.

Hence $\exists x_0^* \in X^*$ such that $x_0^* \notin \overline{R(T^*)}$.

Applying H.B.Theorem in (X^*, T_{w^*}), $\exists \phi_x \in (X^*, T_{w^*})^*$ such that

$\phi_x(x^*) = 0 \quad \forall x^* \in R(T^*)$.

But $\phi_x(x_0^*) \neq 0$ i.e. $x_0^*(x) \neq 0$ or $x \neq 0$.

i.e. $x^*(x) = 0 \quad \forall x^* \in R(T^*)$.

$\Rightarrow x \in {}^{\perp}R(T^*)$ and $x \neq 0$

$\therefore$ ${}^{\perp}R(T^*) \neq \{0\}$, a contradiction.

$\therefore$ $R(T^*)$ is weak* dense in X^*.

Theorem :

If X and Y are Banach spaces and if $T \in B(X, Y)$ then each of the three conditions implies the other three.

(a) $R(T)$ is closed in Y.

(b) $R(T^*)$ is weak*-closed in X^*.

(c) $R(T^*)$ is norm closed in X^*.

Proof :

(a) $\Rightarrow$ (b).

Let $R(T)$ be norm closed in Y.

We have

$N(T) = {}^{\perp}R(T^*)$ (Theorem 4.12)

$(N(T))^{\perp} = ({}^{\perp}R(T^*))^{\perp}$ is weak* closure of $R(T^*)$ in X^* ......(1)

$\supseteq R(T^*)$ $(\because$ $({}^{\perp}N)^{\perp} \supseteq N)$

We can show that,

$(N(T))^{\perp} \subseteq R(T^*)$ ......(2)

Then $R(T^*) = (N(T))^{\perp}$ which is weak* closed by (1).

To prove (2), let $x^* \in (N(T))^{\perp}$.

We have to show $x^* \in R(T^*)$.

For this we have to show $\exists y^* \in Y^*$ such that $x^* = T^* y^*$.

Define $\wedge : TX \rightarrow k$ by $\wedge(T_x) = x^*(x)$.

$\wedge$ is well define :

If $Tx = Tx'$

$\Rightarrow T(x - x') = 0$

$\Rightarrow x - x' \in N(T)$

$\Rightarrow x^*(x) = x^*(x')$ (Since $x^* \in N(T)^{\perp}$)

$\Rightarrow \wedge(Tx) = \wedge(Tx')$

Hence $\wedge$ is well defined.

$\wedge$ is linear :

$$\begin{aligned}\wedge[\alpha T(x_1) + \beta T(x_2)] &= \wedge[T(\alpha x_1) + T(\beta x_2)] \\ &= \wedge[T(\alpha x_1 + \beta x_2)] \\ &= x^*(\alpha x_1 + \beta x_2) \\ &= \alpha x^*(x_1) + \beta x^*(x_2) \\ &= \alpha\wedge(Tx_1) + \beta\wedge(Tx_2).\end{aligned}$$

$\wedge$ is continuous or bounded :

Since $R(T)$ is close in Y, $R(T)$ is a Banach space.

Applying open mapping theorem to $T : X \to R(T)$

$\therefore \quad \|x\| < k\|Tx\| = k\|y\|$ for each $y \in R(T)$.

Thus $|\wedge y| = |\wedge(Tx)|$

$$\begin{aligned} &= |x^*(x)| \\ &\leq \|x^*\|\|x\| \\ &\leq k\|x^*\|\|y\| \qquad \forall y \in R(T).\end{aligned}$$

Thus $\wedge$ is a bounded linear on $R(T)$.

By H.B. Theorem $\wedge$ can be extended by $y^* \in Y^*$.

$y^*(Tx) = \wedge(Tx) \qquad (\because \quad \wedge = y^* \text{ on } R(T)).$

$= x^*(x).$

$(T^*y^*)(x) = x^*(x) \qquad \forall x \in X.$

Hence $x^* = T^*y^*$ and $x^* \in R(T^*)$.

$R(T^*) = [N(T)]^{\perp}$, weak* closure of $R(T^*)$.

So, $R(T^*)$ is weak* closed.

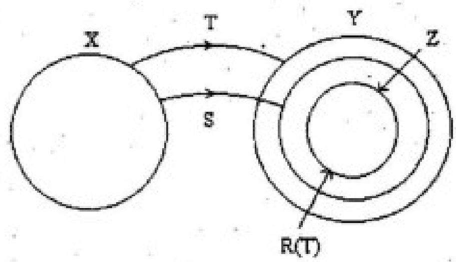
(b) $\Rightarrow$ (c)

It is obvious as weak* topology is coarser then the norm topology on X^*.

(c) $\Rightarrow$ (a)

Let $R(T^*)$ be norm closed in X^*.

To show $\overline{R(T)} \subseteq R(T)$



Then $Z = \overline{R(T)}$ is a Banach space.

Consider $Z \in B(X, Z)$ such that

$Sx = Tx \qquad \forall x \in X$

So, $R(S) = R(T)$.

Since $\overline{R(S)} = Z$, $R(S)$ is dense in Z.

$\therefore$ By theorem 4.12 we know that $T^* : Y^* \to X^*$ is 1 - 1 $\Leftrightarrow \overline{R(T)} = Y$.

So, $S^* : Z^* \to X^*$ is one-one.

If $z^* \in Z^*$, by Hahn-Banach Theorem z^* can be extended by $y^* \in Y^*$.

We can show that $T^*y^* = S^*z^*$.

For $x \in X$,

$$(T^*y^*)x = y^*(Tx) = y^*(Sx) \qquad (\because \quad Sx = Tx)$$
$$= z^*(Sx)$$
$$= (S^*z^*)(x)$$

Since $R(T^*) = R(S^*)$ is norm closed by (c), $R(S^*)$ is complete.

By the Open Mapping Theorem to $S^* : Z^* \to R(S^*)$ $\exists$ $c > 0$ such that

$$c \| z^* \| \leq \| S^*(z^*) \| \qquad \text{for every } z^* \in Z^*.$$

Then by lemma 4.13(b) :

"Suppose U and V are the open unit balls in the Banach spaces X and Y, respectively. Suppose $T \in B(X, Y)$ and $c > 0$, if $c \| y^* \| \leq \| T^*y^* \|$ for every $y^* \in Y^*$ then $T(U) \supset cV$."

We get,

$S(U) \supset cV$, where U and V are unit balls in X and Z.

So, $S : X \to Z$ is an open mapping and $Z = S(X)$.

Hence, $Z = R(S) = R(T)$ and $\overline{R(T)} = Z$.

Consequently, $R(T) = \overline{R(T)}$ and $R(T)$ is closed.

Compact Operators :

Definition :

Suppose X and Y are two Banach spaces and U is the open unit ball in X. A linear map $T : X U Y$ is said to be compact if $\overline{T(U)}$ is compact in Y.

In the study of compact operator we need the followings.

Theorem A :

If X is a metric space then the following are equivalent

(1) X is compact.

(2) every sequence in X has a convergent subsequence.

(3) X is complete and totally bounded.

Theorem B :

Let X and Y be normed linear spaces and $F : X \to Y$ be a linear map.

(a) F is compact iff for every bounded sequence $\{x_n\}$ in X, $\{F(x_n)\}$ has a convergent subsequence in Y.

(b) If F is compact then F(U) is totally bounded.

The converse holds if Y is a Banach-space.

Note 1 :

A compact operator $F : X \to Y$ is bounded.

F is compact $\Rightarrow$ F(U) is totally bounded [Theorem B(b)]

$\Rightarrow$ F(U) is bounded.

$\Rightarrow$ F is continuous (Boundedness of a linear map in a unit ball implies map is continuous)

$\Rightarrow$ F is bounded.

Note 2 :

A continuous operator may not be compact.

Consider $I : X \to X$ which is bounded linear.

Take $\{e_1, e_2, e_3, ...\} \subseteq \ell^2 = X$

where $e_n = \{0, 0, ..., 0, \underbrace{1}_{n^{th}\ place}, 0, ...\}$.

Hence $\| e_n \| = 1 \ \forall n$ and $\{e_n\}$ is bounded.

But $\{e_n\} = \{I(e_n)\}$ has no convergent subsequence.

Since $\| e_i - e_j \| = \sqrt{2}$, no subsequence of $\{e_n\}$ is cauchy and hence not convergent. Hence I is not a compact operator.

Theorem :

Let X and Y be Banach spaces.

(a) If $T \in B(X, Y)$ and $\dim R(T) < \infty$ then T is compact.

(b) If $T \in B(X, Y)$, T is compact and R(T) is closed and then $\dim R(T) < \infty$.

(c) The set of compact operators form a closed subspace of B(X, Y) in its norm topology.

(d) If $T \in B(X)$, T is compact and $\lambda \neq 0$, then $\dim N(T - \lambda I) < \infty$.

(e) If $\dim X = \infty$, $T \in B(X)$ and T is compact then

$0 \in \sigma(T)$ [Spectrum of T, $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I$ is not invertible$\}$]

(f) If $S \in B(X)$, $T \in B(X)$ and T is compact so are ST and TS.

Proof (a) :

Let U be unit ball in X.

To show $\overline{T(U)}$ is compact.

Since $T(U) \subset R(T)$ and $R(T)$ is closed as $R(T)$ is finite diminished.

We have $\overline{T(U)} \subset R(T)$.

Again T is bounded $\Rightarrow T(U)$ is bounded $\Rightarrow \overline{T(U)}$ is bounded.

So, $\overline{T(U)}$ is closed and bounded subset of f.d. space R(T). So $\overline{T(U)}$ is compact and hence T is compact.

Proof (b) :

We have (Theorem 1.22) every locally compact topological vector space is finite dimensional.

The closed subspace R(T) is a Banach space.

Since $T : X \to R(T)$ is linear bounded onto map from X onto R(T), by the open mapping theorem T(U) is an open neighbourhood of 0 in R(T) and $\overline{T(U)}$ is compact as T is compact. So R(T) is a locally compact space and hence it is finite dimensional.

Proof (c) :

Let Σ be the set of all compact operators from X to Y and $S, T \in \Sigma$. To show $\alpha S + \beta T$ is compact.

Let $\{x_n\}$ be a bounded sequence in X. To show $\exists$ a convergent subsequence of $\{(\alpha S + \beta T)(x_n)\}$.

By the compactness of T, $\{T(x_n)\}$ has a convergent subsequence $\left\{T_{x_{n_K}}\right\}$. By compactness of S, $\left\{S_{x_{n_K}}\right\}$ has a convergent sequence $\left\{S_{x_{n_{K_\ell}}}\right\}$. Subsequence of a convergent sequence is convergent.

Hence $\{(\alpha S + \beta T)\left(x_{n_{K_\ell}}\right)\}$ is convergent subsequence.

Thus $\alpha S + \beta T \in \Sigma$ and Σ is a subspace.

To show $\overline{\Sigma} \subset \Sigma$.

Let $T \in \overline{\Sigma}$. For $\varepsilon > 0, \exists\ S \in \Sigma$ such that

$$\| T - S \| < \frac{\varepsilon}{3}.$$

If U be open unit ball in X, thus S(U) is totally bounded. Hence $\exists$ finite points $x_1, x_2, ..., x_n$ in U such that S(U) is covered by balls of radius $\frac{\varepsilon}{3}$ centered at $Sx_1, Sx_2, ..., Sx_n$.

$$\| S - T \| < \frac{\varepsilon}{3} \Rightarrow \| Sx - Tx \| < \frac{\varepsilon}{3} \quad \forall\ x \in U.$$

$$x \in \cup \Rightarrow \exists\ x_k (1 \le k \le n) \text{ such that } \| Sx - Sx_k \| < \frac{\varepsilon}{3}$$

$$\| Tx - Tx_k \| \le \| Tx - Sx \| + \| Sx - Sx_k \| + \| Sx_k - Tx_k \|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So, $x \in U \Rightarrow Tx \in S_\varepsilon(Tx_k)$.

Hence $T(U) \subseteq \bigcup_{k=1}^{n} S_\varepsilon(Tx_k)$.

So, $\{Tx_1, Tx_2, ..., Tx_n\}$ is an ε-net for T(U) and T(U) is totally bounded. So $T \in \Sigma$.

Thus $\overline{\Sigma} \subset \Sigma$ and Σ is closed in B(X, Y).

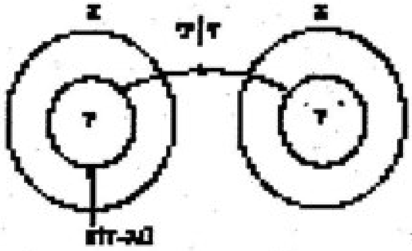
Proof (d) :

We put $Y = N(T - \lambda I)$.

Then $T \mid Y : Y \to Y$ is linear, bounded and onto since we shall see that $R(T \mid Y) = N(T - \lambda I)$.

$$y \in R(T \mid Y) \Rightarrow y = Tx$$

where $x \in Y = N(T - \lambda I)$ and hence $Tx = \lambda x$.



$$\begin{aligned}(T - \lambda I)y &= Ty - \lambda y \\ &= T(\lambda x) - \lambda y \qquad (\because y = Tx = \lambda x) \\ &= \lambda y - \lambda y = 0.\end{aligned}$$

Hence $y \in N(T - \lambda I)$ and $R(T \mid Y) \subseteq N(T - \lambda I)$.

Also $N(T - \lambda I) \subseteq R(T \mid Y)$.

Hence $R(T \mid Y) = N(T - \lambda I)$.

$T \mid Y \in B(Y)$, $T \mid Y$ is compact and $R(T \mid Y) = N(T - \lambda I)$ is closed.

Hence by (b) $\dim R(T \mid Y) < \infty$ i.e., $\dim N(T - \lambda I) < \infty$.

Proof (e) :

Suppose $0 \notin \sigma(T)$

$\Rightarrow$ T is invertible

$\Rightarrow$ T is onto

$\Rightarrow R(T) = X$

Also by (b) $\dim R(T) \dim X < \infty$

which is a contradiction.

So, $0 \in \sigma(T)$.

Proof (f) :

Let $\{x_n\}$ be a bounded sequence in X.

$$\| S(x_n) \| \le \| S \| \| x_n \| \le \| S \| k < \infty$$

So $\{S(x_n)\}$ is a bounded sequence in X.

By compactness of T, $\{T(Sx_n)\}$ has a convergent subsequence. TS is compact.

$\{Tx_n\}$ has a convergent subsequence $\left\{ Tx_{n_i} \right\}$.

Then $Tx_{n_i} \xrightarrow{i} x_0 \Rightarrow S\left(Tx_{n_i}\right) \xrightarrow{i} Sx_0 \Rightarrow (ST)\left(x_{n_i}\right) \rightarrow Sx_0$.

So, $(ST)(x_n)$ has a convergent subsequence. Hence ST is compact.

Theorem :

Suppose X and Y are Banach spaces and $T \in B(X, Y)$. Then T is compact iff T^* is compact.

Proof :

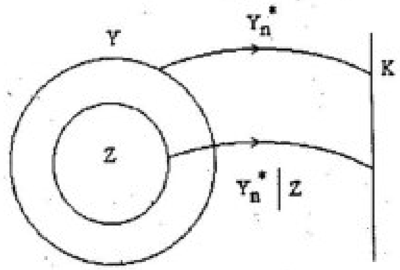
Suppose T is compact. To show $T^* : Y^* \rightarrow X^*$ is compact.

Let $\{y_n^*\}$ be a sequence in unit ball of Y^*.

Let U be unit ball in X and put $Z = \overline{T(U)} \subset Y$.

Consider $A = \{y_n^* \mid Z\} \subset B(Z, K)$.

We can show that A is relatively compact.



T is compact $\Rightarrow \overline{T(U)}$ is compact.

$\Rightarrow \overline{T(U)}$ is bounded.

$\Rightarrow \| y \| \leq M < \infty \qquad \forall y \in z = \overline{T(U)}$.

Then for such $y \in Z = \overline{T(U)}$

$$| y_n^*(y) | \leq \| y_n^* \| \| y \| < M < \infty \qquad (\because \quad y_n^* \in \text{unit ball})$$

$\{y_n^*\}$ is uniformaly bounded on Z.

For $y_1, y_2 \in Z$,

$$| y_n^*(y_1) - y_n^*(y_2) | = | y_n^*(y_1 - y_2) |$$
$$\leq \| y_n^* \| \| y_1 - y_2 \| < | y_1 - y_2 |$$

This shows that $\{y_n^* \mid Z\}$ is eqicontinuous.

By Ascoli's Theorem $\exists$ a subsequence $\left\{y_{n_k}^*\right\}$ of $\{y_n^*\}$ such that $yn_k^* | Z$ converges in $B(Z, K)$.

We can show that $\{T^* y_{n_k}^*\}$ is a convergent subsequence of $\{T^* y_n^*\}$.

For $\varepsilon > 0$,

$$| y_{n_j}^*(y) - y_{n_k}^*(y) | \leq \| y_{n_j}^* - y_{n_k}^* \| \| y \|$$

$$< \frac{\varepsilon}{M} M = \varepsilon \quad \forall j, k \geq n_0$$

$$(\because \quad yn_k^* | Z \text{ is equi-continuous, } \forall \ y \in Z)$$

If $x \in U$ then $T_x \in T(U) \subseteq \overline{T(U)} = Z$.

$$\left| y_{n_j}^*(Tx) - y_{n_k}^*(Tx) \right| < \varepsilon \text{ for every } x \in U.$$

or $\left|\left(T^{*}y_{n_j}^{*}\right)(x)-\left(T^{*}y_{n_k}^{*}\right)(x)\right|<\varepsilon$ for $x\in U$.

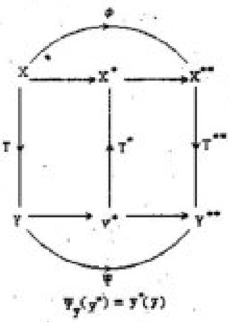
Taking supremum over all $x \in U$.

$$\left\|T^{*}y_{n_j}^{*}-T^{*}y_{n_k}^{*}\right\|<\varepsilon$$

$\left\{T^{*}y_{n_k}^{*}\right\}$ is a cauchy seq in the Banach space X^{*} and hence it is convergent. Thus $\{T^{*}y_n^{*}\}$ has a convergent subsequence. So T^{*} is compact.

Conversely suppose $T^{*}: Y^{*} \to X^{*}$ is compact.

To show $T: X \to Y$ is compact.



Consider $\phi : X \to X^{**}$

and $\Psi : Y \to Y^{**}$

We observe that

$$\Psi T = T^{**}\phi$$

i.e., $\Psi(Tx) = T^{**}(\phi(x))$.

Sine $(\Psi(Tx))y^{*} = y^{*}(Tx) = (T^{*}y^{*})(x)$

$= [\phi(x)](T^{*}y^{*})$ (definition of ϕ and taking $T^{*}y^{*} = x_1^{*} \in X^{*}$)

$= [T^{**}\phi(x)](y^{*})$.

$\therefore \quad \Psi T = T^{**}\phi$.

Also we observe that

$\phi(U) \subseteq U^{**}$ where U^{**} is unit ball in X^{**}

For $u \in U$, $\| \phi(u) \| = \sup\limits_{\|x^{*}\|\le 1} |\phi_u(x^{*})|$

$= \sup\limits_{\|x^{*}\|\le 1} |x^{*}(u)|$

$\le \sup\limits_{\|x^{*}\|\le 1} \|x^{*}\| \, \|u\|$

$< 1.$

Hence $\phi(u) \in U^{**}$ and $\phi(U) \subseteq U^{**}$.

$(\Psi T)(U) = [T^{**}\phi](U)$

$= T^{**}\phi(U)$

$\subseteq T^{**}U^{**}$

T^{*} is compact $\Rightarrow T^{**} : X^{**} \to Y^{**}$ is compact

$\Rightarrow T^{**}(U^{**})$ is totally bounded.

$\Rightarrow [\Psi(T)](U)$ is totally bounded.

$\Rightarrow T(U)$ is totally bounded ($\because \Psi$ is isometric isomorphism)

So, T is compact.

Complement of subspace :

Definition :

Suppose M is a closed subspace of a topological space X. If $\exists$ a closed subspace N of X such that

$$X = M + N \text{ and } M \cap N = \{0\}.$$

Then M is said to be complement in X.

4.21. Lemma :

Let M be a **closed** linear subspace of a tvs X.

(a) If X is locally convex and $\dim M < \infty$, then M is complemented in X.

(b) If $\dim\left(\frac{X}{M}\right)$ (i.e., codomain of M) is finite then M is complemented.

Proof (a) :

We supply the proof for nls. Let $\{e_1, e_2, ..., e_n\}$ be a basis for M.

For $x \in M$, $x = \alpha_1(x)e_1 + \alpha_2(x)e_2 + ... + \alpha_n(x)e_n$.

$\alpha_i : M \to C$ is continuous linear.

For $x, y \in M$ and scalars a and b,

$$ax + by = \sum_{i=1}^{n} \alpha_i\,(ax + by)e_i$$

Also $$ax + by = a\sum_{i=1}^{n} \alpha_i(x)e_i + b\sum_{i=1}^{n} \alpha_i(y)e_i$$

$$= \sum_{i=1}^{n} \left(a\alpha_i(x) + b\alpha_i(y)\right)e_i$$

By the uniqueness of representation of ax + by, we have

$$\alpha_i(ax + by) = a\alpha_i(x) + b\alpha_i(y).$$

In M any two norms are equivalent.

In $(X, \|\cdot\|)$ define another norm, $\|x\|_1 = \sum_{i=1}^{n} |\alpha_i(x)|$

Since $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent.

$$\|x\|_1 \le M_1 \|x\| \quad \forall x \in M$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i(x)| \le M\|x\|$$

$\therefore \quad |\alpha_i(x)| \le M\|x\|.$

$\therefore \quad \alpha_i$ is a bounded linear functional on M.

By H.B. Theorem $\exists x_i^* \in X^*$ such that

$$x_i^*(x) = \alpha_i(x) \qquad \forall x \in M \text{ with } \|x_i^*\| = \|\alpha_i\|.$$

Take $N = \bigcap_{i=1}^{n} \ker(x_i^*)$, which is a closed subspace of X.

We claim $X = M \oplus N$.

For $x \in X$,

$$x = \left(\sum_{i=1}^{n} x_i^*(x)e_i\right) + \left(x - \sum_{i=1}^{n} x_i^*(x)e_i\right)$$

where $\sum_{i=1}^{n} x_i^*(x)e_i \in M$.

and $$x_i^*\left(x - \sum_{k=1}^{n} x_k^*(x)e_k\right) = x_i^*(x) - \sum_{k=1}^{n} x_k^*(x)x_i^*(e_k)$$

$$= x_i^*(x) - \sum_{k=1}^{n} x_k^*(x)\alpha_i(e_k)$$

$$= x_i^*(x) - x_i^*(x)$$

$$= 0.$$

$\therefore \quad x - \sum_{k=1}^{n} x_k^*(x)e_k \in \bigcap_{i=1}^{n} \ker x_i^* = N$

$\therefore \quad X = M + N.$

Let $x \in M \cap N \Rightarrow x \in M$

So, $x = \alpha_1(x)e_1 + \ldots + \alpha_n(x)e_n$

and $x \in N = \bigcap_{i=1}^{n} \ker(x_i^*)$

$$\Rightarrow x_i^*(x) = \alpha_i(x) = 0 \qquad \forall i \in \{1, 2, \ldots, n\}$$

So, $x = 0$ and $M \cap N = \{0\}$.

Hence $X = M \oplus N$.

So M is complemented.

Proof (b) :

Let $\dim\left(\frac{X}{M}\right) < \infty$

Let $\prod : X \to \frac{X}{M}$, by $\prod(x) = x + M$.

Let $\{e_1, e_2, ..., e_n\}$ be basis for $\frac{X}{M}$.

Let $x_i \in X$ such that $\prod(x_i) = e_i$.

We put $N = \text{span}\{x_1, x_2, ..., x_n\}$

It can be shown that $X = M \oplus N$.

Lemma :

If M is a subspace of a nls X, if M is not dense in X, and if $r > 1$, then $\exists\, x \in X$ such that $\| x \| < r$ but $\| x - y \| \geq 1$ for all $y \in M$.

Proof :

Since $\overline{M} \neq X$, $\exists\, x_0 \in X$ such that $x_0 \notin \overline{M}$.

So $\exists\, k > 0$ such that $\| x_0 - m \| > k \quad \forall m \in M$.

$$\operatorname*{glb}_{m \in M} \|x_0 - m\| = k_0 \geq k$$

$$\Rightarrow \operatorname*{glb}_{m \in M} \left\| \frac{x_0}{k_0} - m \right\| = 1$$

So $\exists\, x_1 = \frac{x_0}{k_0}$ such that $d(x, M) = 1$

$\therefore \quad r > 1 = \operatorname*{glb}_{m \in M} \|x_1 - m\|$

$\Rightarrow \exists\, m_1 \in M$ such that $r > \| x_1 - m_1 \| = \| x \|$ where $x = x_1 - m_1$

and $\quad 1 = \operatorname*{glb}_{m \in M} \|x_1 - m\|$

$= \operatorname*{glb}_{m \in M} \|x_1 + m_1 + m\|$

$= \operatorname*{glb}_{m \in M} \|x - y\|$

$\therefore \quad 1 \leq \| x - y \| \qquad \forall y \in M$. Hence Proved.

Theorem :

If X is a Banach space, $T \in B(X)$, T is compact and $\lambda \neq 0$, then $T - \lambda I$ has closed range.

Proof :

We have $\dim N(T - \lambda I) < \infty$ (Theorem 4.18)

and $N(T - \lambda I)$ is complement (Theorem 4.21(a)) .

This means $\exists$ a closed subspace M of X such that

$$X = N(T - \lambda I) \oplus M.$$

Define $\quad S : M \to X$ by $S(x) = Tx - \lambda x$.

S is one-one : $\quad S(x_1) = S(x_2)$

$$\Rightarrow Tx_1 - \lambda x_1 = Tx_2 - \lambda x_2$$

$$\Rightarrow T(x_1 - x_2) - \lambda(x_1 - x_2) = 0$$

$$\Rightarrow (T - \lambda I)(x_1 - x_2) = 0$$

$$\Rightarrow x_1 - x_2 \in N(T - \lambda I).$$

Also $x_1 - x_2 \in M$ So, $x_1 - x_2 \in N(T - \lambda I) \cap M = \{0\}$.

Hence, $x_1 = x_2$ and S is one-one.

R(S) = R(T - λI) :

Let $\quad y \in R(S) \Rightarrow y = Sx$ where $x \in M$

$$\Rightarrow y = (T - \lambda I)(x) \qquad \text{(definition of S)}$$

$$\Rightarrow y \in R(T - \lambda I).$$

$\therefore \quad R(S) \subseteq R(T - \lambda I)$.

Conversely let

$$y \in R(T - \lambda I)$$

$$\Rightarrow y = (T - \lambda I)(x)$$

For $x \in X$,

$$x = n + m \quad (\because \quad X = N(T - \lambda I) \oplus M)$$

where $n \in N(T - \lambda I)$ and $m \in M$.

$$(T - \lambda I)(x) = (T - \lambda I)(n) + (T - \lambda I)(m)$$

$$\Rightarrow y = (T - \lambda I)(m)$$

$$\Rightarrow y = Sm$$

$$\Rightarrow y \in R(S)$$

Hence $R(T - \lambda I) \subseteq R(S)$ and $R(S) = R(T - \lambda I)$.

Finally to show, $R(T - \lambda I) = R(S)$ is closed.

We use the following result proved is chapter 1.

If (X, d_1) and (Y, d_2) are metric spaces and (X, d_1) is complete. If E is closed in X, $f : X \to Y$ is continuous and $d_2(f(x'), f(x'')) \geq d_1(x', x'')$, $\forall x', x'' \in E$ then f(E) is closed.

The equivalent form of which is nls is

$$\| Sx \| \geq r \| x \| \text{ for } r > 0 \qquad ......(*)$$

Now we prove (*).

Suppose (*) is not true. Then for $n \in \mathbb{N}$, $\exists\, x_n \in M$ such that $\| x_n \| = 1$ and $\| Sx_n \| < \frac{1}{n} \| x_n \| = \frac{1}{n}$.

So, $Sx_n \to 0$.

$\{x_n\}$ is bounded $\Rightarrow \exists$ a subsequence $\{x_{n_k}\}$ such that

$$Tx_{n_k} \to x_0 \in X \qquad \text{(compactness of T).}$$

$$Sx_{n_k} \to 0$$

$$\Rightarrow Tx_{n_k} - \lambda x_{n_k} \to 0$$

$$\Rightarrow \lambda x_{n_k} \to x_0 \in M \qquad (\because \ \text{M is closed})$$

and $$Sx_0 = S\left(\lim_{n\to\infty} \lambda x_{n_k}\right)$$

$$= 0 = S(0)$$

$$\Rightarrow x_0 = 0 \qquad (\because \ \text{S is one-one})$$

But $\|x_{n_k}\| = 1 \ \ \forall\, n_k$ and $x_0 = \lim_{n\to\infty} \lambda x_{n_k}$.

So, $\| x_0 \| = | \lambda | > 0$. This is a contradiction.

So (*) is true. Hence $R(S) = R(T - \lambda I)$ is closed.

Theorem :

Suppose X is a Banach space, $T \in B(X)$, T is compact, $r > 0$ and E is the set of eigen values λ of T such that $| \lambda | > r$, then

(a) for each $\lambda \in E$ $R(T - \lambda I) \neq X$.

(b) E is finite set.

Proof :

We can show that if either (a) or (b) fails then $\exists$ closed subspaces M_n of X and scalars $\lambda_n \in E$ such that

(1) $M_1 \subset M_2 \subset M_3$ ... and $M_n \neq M_{n+1}$

(2) $T(M_n) \subset M_n$ for $n \geq 1$.

(3) $(T - \lambda_n I)(M_n) \subsetneq M_{n-1}$ for $n \geq 2$.

We can complete the proof showing that (1), (2) and (3) contradicts compactness of T.

Suppose (a) fails then $\exists\, \lambda_0 \in E$ such that

$$R(T - \lambda_0 I) = X.$$

Put $S = T - \lambda_0 I$ and $M_1 = N(T - \lambda_0 I) = N(S)$.

Define $M_n = N(S^n)$ which is a closed subspace.

(1) Let $\quad x \in M_n \Rightarrow x \in N(S^n)$

$\Rightarrow S^n x = 0$

$\Rightarrow S^{n+1} x = 0$

$\Rightarrow x \in N(S^{n+1}) = M_{n+1}$

$\therefore \quad M_n \subseteq M_{n+1}$.

$\lambda_0 \in E \Rightarrow \exists x_1 \neq 0 \, Tx_1 = \lambda_0 x_1$

$\Rightarrow (T - \lambda_0 I)(x_1) = 0$

$\Rightarrow x_1 \in N(T - \lambda_0 I) = N(S)$

$\Rightarrow x_1 \in M_1$

$R(S) = X, \Rightarrow \exists\, x_2 \in X$ such that $Sx_2 = x_1$

In general $\exists\, x_{n+1} \in X$ such that $S(x_{n+1}) = x_n$.

Thus we have a sequence $\{x_n\}$ such that

$S^n x_{n+1} = S^{n-1} x_n = S^{n-2} x_{n-1} = \ldots = Sx_2 = x_1$

$S^{n+1}(x_{n+1}) = Sx_1 = (T - \lambda_0 I)(x_1) = 0 \quad (\because \quad Tx_1 = \lambda_0 x_1)$

$\therefore \quad x_{n+1} \in N(S^{n+1}) = M_{n+1}$

But $x_{n+1} \notin M_n$ since $S^n(x_{n+1}) = x_1 \neq 0$.

$\therefore \quad M_n \neq M_{n+1}$ i.e. $M_n \subset M_{n+1}$.

(2) $T(M_n) \subset M_n$ for $n \geq 1$

Let $Tx \in T(M_n) \Rightarrow x \in M_n = N(S^n)$

$\Rightarrow S^n(x) = 0.$

$S^n(Tx) = T(S^n x) \quad [\because \quad TS = T(T - \lambda_0 I)$

$= T^2 - \lambda_0 T$

$= T(0) \qquad ST = (T - \lambda_0 I)T$

$= 0 \qquad = T^2 - \lambda_0 T$

$\therefore \quad ST = TS]$

$\therefore \quad Tx \in N(S^n) = M_n$

$\therefore \quad Tx \in T(M_n) \Rightarrow Tx \in M_n$

$\therefore \quad T(M_n) \subseteq M_n$.

(3) $(T - \lambda_0 I)(M_n) \subseteq M_{n-1} \quad n \geq 2$

$y \in (T - \lambda_0 I)(M_n)$

$\Rightarrow y = (T - \lambda_0 I)x, \; x \in M_n$

$\Rightarrow y = Sx.$

$S^{n-1} y = S^{n-1}(Sx) = S^n x = 0 \quad (\because \quad x \in M_n = N(S^n))$

So, $y \in M_{n-1}$

and $(T - \lambda_0 I)M_n \subseteq M_{n-1}$.

Suppose (b) is not satisfied. then E contains a seq(λ_n) of distinct eigen values of T.

Let $M_n = \text{span}\{e_1, e_2, ..., e_n\}$ where $Te_n = \lambda_n e_n$.

Thus M_n being finite dimensional is closed.

(1) $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct $\Rightarrow M_{n-1} \subsetneq M_n$.

(2) $\mathbf{T(M_n) \subset M_n}$

Suppose $\quad Tx \in T(M_n)$

$$\Rightarrow x \in M_n$$

$$\Rightarrow x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

$$\Rightarrow Tx = \alpha_1 Te_1 + \alpha_2 Te_2 + ... + \alpha_n Te_n \in M_n$$

$\therefore \quad T(M_n) \subseteq M_n$

(3) $(T - \lambda_n I)M_n \subseteq M_{n-1}$.

Let $\quad (T - \lambda_n I)x \in (T - \lambda_n I)M_n$

$$\Rightarrow x \in M_n$$

$$\Rightarrow x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$$

Now $(T - \lambda_n I)x = Tx - \lambda_n x$

$$= T(\alpha_1 e_1 + ... + \alpha_n e_n) - \lambda_n(\alpha_1 e_1 + ... + \alpha_n e_n)$$

$$= \alpha_1(\lambda_1 - \lambda_n)e_1 + ... + \alpha_n(\lambda_{n-1} - \lambda_n)e_{n-1} \in M_{n-1}$$

$\therefore \quad (T - \lambda_n I)M_n \subseteq M_{n-1}$.

Thus if (a) or (b) fails then (1), (2) and (3) hold.

$$\overline{M_{n-1}} = M_{n-1} \subsetneq M_n.$$

$\therefore \quad \overline{M_{n-1}} \neq M_n$.

So M_{n-1} is not dense in M_n.

By lemma **4.22** $M \subset X$, $\overline{M} \neq X$, $r > 1$,

Then $\exists\, x \in X$ such that $\| x \| < r$ and $\| x - y \| > 1 \ \forall y \in M$.

For $r = 2 > 1 \ \exists\, y_n \in M_n$ such that

$$\| y_n \| < 2 \text{ and } \| y_n - x \| \geq 1 \ \forall x \in M_{n-1}.$$

If $2 \leq m < n$, define

$$z = Ty_m - (T - \lambda_n I)y_n$$

$$y_m \in M_m \subseteq M_{n-1} \qquad \text{(by (2))}$$

$$\Rightarrow y_m \in M_{n-1}$$

$\Rightarrow Ty_m \in M_{n-1}$ $\qquad (\because \quad T(M_{n-1}) \subset M_{n-1})$

$y_n \in M_n \Rightarrow (T - \lambda_n I)y_n \in M_{n-1}$ (by (3))

$\therefore \quad z = Ty_m - (T - \lambda_n I)y_n \in M_{n-1}$

Hence $\quad \| Ty_m - Ty_n \| = \| Z - \lambda_n y_n \|$

$= |\lambda_n| \, \| y_n - \lambda_n^{-1} Z \| \geq |\lambda_n|$

$\{y_n\}$ is a bounded seq in X but $\{Ty_n\}$ has no convergent subsequence. Thus T is not compact. This is a contradiction.

$\therefore$ (a) and (b) must hold. Proved.

●●●

Unit 4
Banach Algebra

4.1 Algebra : An algebra is a linear space whose vectors can be multiplied in such a way that

(i) $x(yz) = (xy)z$

(ii) $x(y + z) = xy + xz$

$(x + y)z = xz + yz$

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y), \ \forall \ \alpha \in K$

4.2 Banach algebra :

Let A be an algebra. If A is also a Banach space w.r.t a norm that satisfies the multiplicative inequality

$\|xy\| \leq \|x\|. \|y\|, \ \forall \ x, y \in A.$

then A is called a **Banach algebra.**

If $\exists$ a unit element 'e' in A s.t.

$ex = xe = x \ \forall \ \in A$, then $\|e\| = 1$ and A is called unital Banach Algebra.

If $xy = yx \ \forall \ x, y \in A$, then A is called commutative Banach algebra.

Examples :

1. The complex plane C is a Banach algebra.

(i) C is an algebra

(ii) C is a normed algebra, where

$\|z\| = |z|, z \in C$

(iii) C is complete.

$\|z_1z_2\| = \|z_1\| \ \|z_2\| \ \forall \ z_1, z_2 \in C$

$\therefore$ C is a Banach algebra.

Also, 1 is the unit element ($\because \ \|1\| = 1$) of C and C is commutative. Hence C is unital and commutative Banach algebra.

2. Let K be a compact Hausdorff space. C(K) is the collection of all continuous complex valued functions defined on the set K.

Then C(K) is a Banach algebra w.r.t usual operations.

Proof : Let f, g $\in$ C(K) and α be any scalar.

We define

(i) $(f + g)(x) = f(x) + g(x), x \in K.$

(ii) $(\alpha f)(x) = \alpha f(x), x \in K.$

Then C(K) is a linear space.

The multiplication is defined as

$$(fg)(x) = f(x)\, g(x), x \in K.$$

Let f, g, h $\in$ C(K) and x $\in$ K.

(i)
$$\begin{aligned}(f(gh))(x) &= f(x)\,(gh)(x) \\ &= f(x)\,[g(x)\,h(x)] \\ &= [f(x)\,g(x)]\,h(x) \\ &= [(fg)(x)]\,h(x) \\ &= ((fg)h)(x) \quad \forall\ x \in K.\end{aligned}$$

$\therefore$ f(gh) = (fg)h.

(ii)
$$\begin{aligned}[f(g+h)](x) &= f(x)\,[(g+h)(x)] \\ &= f(x)\,[g(x) + h(x)] \\ &= f(x)\,g(x) + f(x)\,h(x) \\ &= (fg)(x) + (fh)(x) \\ &= (fg + fh)(x) \quad \forall\ x \in K.\end{aligned}$$

$\therefore$ f(g + h) = fg + fh

similarly, (f + g)h = fh + gh

(iii)
$$\begin{aligned}[\alpha(fg)](x) &= \alpha(fg)(x) \\ &= \alpha[f(x)\,g(x)] \\ &= [\alpha f(x)]\,g(x) \\ &= (\alpha f)(x)\,g(x) \\ &= [(\alpha f)g](x) \quad \forall\ x \in K\end{aligned}$$

$\therefore \alpha(fg) = (\alpha f)g$

similarly, $\alpha(fg) = f(\alpha g)$

$\therefore \alpha(fg) = (\alpha f)g = f(\alpha g)$

$\therefore$ C(K) is an algebra.

We have, $\|f\| = \sup \{|f(x)| : x \in K\}$

Then C(K) is a Banach space w.r.t the above norm.

Let f, g $\in$ C(K).

$$
\begin{aligned}
\text{Now, } \|fg\| &= \sup \{|(fg)(x)| : x \in K\} \\
&= \sup \{|f(x)\, g(x)| : x \in K\} \\
&\leq \sup \{|f(x)|\, |g(x)| : x \in K\} \\
&= \sup \{|f(x)| : x \in K\} \\
&\quad \sup \{|g(x)|\} : x \in K\} \\
&= \|f\|\, \|g\|
\end{aligned}
$$

$\Rightarrow \|fg\| \leq \|f\|\, \|g\| \ \forall\ f, g \in C(K)$

$\therefore$ C(K) is a Banach algebra.

Let us consider the mapping

$I : K \to C$ by $I(x) = 1 \ \forall\ x \in K$.

Then $\|I\| = 1$

$\therefore$ C(K) is unital Banach algebra.

and also C(K) is commutative Banach algebra.

3. Let X be a Banach space, $\beta(X)$ be the collection of all bounded linear operations on X. Then $\beta(X)$ is a Banach algebra w.r.t the usual operation.

Proof : We know that $\beta(X)$ is a Banach space w.r.t the norm $\|T\| = \sup \{\|Tx\| : \|x\| \leq 1\}$

For S, T $\beta(X)$, we define,

$(ST)(x) = S(Tx)$, $\forall\ x \in X$.

Let $T_1, T_2, T_3 \in \beta(X)$ and $x \in X$.

$$
\begin{aligned}
\textbf{(i) } [T_1 (T_2 T_3)](x) &= T_1((T_2T_3)(x)). \\
&= T_1 (T_2(T_3(x))) \\
&= (T_1T_2)(T_3(x)). \\
&= [(T_1T_2)T_3](x) \ \forall\ x \in X.
\end{aligned}
$$

$\therefore T_1(T_{12} T_3) = (T_1 T_2) T_3$

(ii) $(T_1(T_2+T_3))(x) = T_1((T_2+T_3)(x)).$

$= T_1(T_2(x) + T_3(x))$

$= T_1(T_2(x)) + T_1(T_3(x))$

$= (T_1T_2)(x) + (T_1T_3)(x)$

$= (T_1T_2 + T_1T_3)(x) \ \forall \ x \in X.$

$\therefore T_1(T_2+T_3) = T_1T_2 + T_2T_3.$

Similarly, $(T_1 + T_2)T_3 = T_1T_3 + T_2T_3$

(iii) Let α be any scalar. ($\in K$)

$((\alpha T_1)T_2(x) = (\alpha T_1)(T_2(x))$

$= \alpha T_1(T_2(x))$

$= \alpha(T_1T_2)(x)$

$= (\alpha(T_1T_2))(x) \ \forall \ x \in X.$

$\therefore \alpha(T_1T_2) = (\alpha T_1)T_2.$

Again,

$(T_1(\alpha T_2))(x) = T_1((\alpha T_2)(x))$

$= T_1(\alpha T_2(x))$

$= \alpha T_1(T_2(x))$

$= \alpha(T_1T_2)(x)$

$= (\alpha(T_1T_2))(x) \ \forall \ x \in X.$

$\therefore T_1(\alpha T_2) = \alpha(T_1T_2)$

$\therefore T_1(\alpha T_2) = \alpha(T_1T_2) = (\alpha T_1)T_2.$

$\therefore \beta(X)$ is an algebra.

Let $S, T \in \beta(K)$

Now,

$||ST|| = \sup \{||ST(x)|| : ||x|| \leq 1, x \in X\}$

$= \sup \{||S(T(x))|| : ||x|| \leq 1, x \in X\}$

$\leq \sup \{||S|| \, ||T(x)|| : ||x|| \leq 1, x \in X\}$

$= ||S|| \sup \{||T(x)|| : ||x|| \leq 1, x \in X\}$

$= ||S|| \, ||T||$

$\Rightarrow ||ST|| \leq ||S|| \, ||T||$

$\therefore \beta(X)$ is a Banach algebra w.r.t the defined norm.

Let us consider the identity mapping

$I : X \to X$ s.t $I(x) = x$, $\forall$ $x \in X$.

Now for all $S \in \beta(X)$,

$$SI = IS = S$$

and $\|I\| = \sup \{\|Ix\| : \|x\| \le 1\}$

$\quad = \sup \{\|x\| : \|x\| \le 1\}$

$\quad = 1$

$\therefore$ I is the unit element in $\beta(X)$.

Also, $ST(x) = S(Tx)$

$\quad TS(x) = T(Sx)$ $\forall$ $x \in X$

In general, $\beta(X)$ is not commutative.

4. Let $\mathcal{A}$ be the collection of all matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ $\alpha, \beta \in C$. Then $\mathcal{A}$ is a Banach algebra.

Proof : $\mathcal{A}$ is a linear space with ordinary matrix addition and scalar multiplication as below :

Let

$$A = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_1 \end{pmatrix} \in \mathcal{A}$$

$$B = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} \in \mathcal{A}$$

$$A + B = \begin{pmatrix} \alpha_1 + \alpha_2 & \beta_1 + \beta_2 \\ 0 & \alpha_1 + \alpha_2 \end{pmatrix} \in \mathcal{A}$$

Let $\delta \in K$,

$$\text{then, } \delta A = \begin{pmatrix} \delta\alpha_1 & \delta\beta_1 \\ 0 & \delta\alpha_1 \end{pmatrix} \in \mathcal{A}$$

If

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}, \text{ then}$$

$\|A\| = |\alpha| + |\beta|$

Now,

(i) $\|A\| = |\alpha| + |\beta| \geq 0$

(ii) $\|A\| = 0 \Leftrightarrow |\alpha| + |\beta| = 0$

$\Leftrightarrow |\alpha| = 0, \ |\beta| = 0$

$\Leftrightarrow \alpha = 0, \ \beta = 0$

$\Leftrightarrow A = 0$

(iii) Let δ be a scalar

$$\therefore \ \|\delta A\| = \left\| \begin{pmatrix} \delta\alpha & \delta\beta \\ 0 & \delta\alpha \end{pmatrix} \right\|$$

$$= |\delta\alpha| + |\beta\delta|$$

$$= |\delta| \ |\alpha| + |\delta| \ |\beta|$$

$$= |\delta| \ (|\alpha| + |\beta|)$$

$$= |\delta| \ \|A\|$$

(iv) Let

$$A = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_1 \end{pmatrix} \in \mathcal{A}$$

$$B = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} \in \mathcal{A}$$

$$\text{Then, } \|A + B\| = \left\| \begin{pmatrix} \alpha_1 + \alpha_2 & \beta_1 + \beta_2 \\ 0 & \alpha_1 + \alpha_2 \end{pmatrix} \right\|$$

$$= |\alpha_1 + \alpha_2| + |\beta_1 + \beta_2|$$

$$\leq |\alpha_1| + |\alpha_2| + |\beta_1| + |\beta_2|$$

$$= (|\alpha_1| + |\beta_1|) + (|\alpha_1| + |\beta_2|)$$

$$= \|A\| + \|B\|$$

$\Rightarrow \|A + B\| \leq \|A\| + \|B\|$

$\therefore \ \mathcal{A}$ is a normed linear space.

Let $\{A_n\}_n$ be a carchy sequence in $\mathcal{A}$, where

$$A_n = \begin{pmatrix} \alpha_n & \beta_n \\ 0 & \alpha_n \end{pmatrix}$$

$\therefore$ For each $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$\|A_m - A_n\| < \varepsilon \ \forall \ m, n \geq n_0$

$$\Rightarrow \left\| \begin{pmatrix} \alpha_m & \beta_m \\ 0 & \alpha_m \end{pmatrix} - \begin{pmatrix} \alpha_n & \beta_n \\ 0 & \alpha_n \end{pmatrix} \right\| < \varepsilon \ \forall \ m, n \geq n_0$$

$$\Rightarrow \left\| \begin{pmatrix} \alpha_m - \alpha_n, & \beta_m - \beta_n \\ 0 & \alpha_m \alpha_n \end{pmatrix} \right\| < \varepsilon \ \forall \ m, n \geq n_0$$

$\Rightarrow |\alpha_m - \alpha_n| + |\beta_m - \beta_n| < \varepsilon \ \forall \ m, n \geq n_0$

$\Rightarrow |\alpha_m - \alpha_n| < \varepsilon, |\beta_m - \beta_n| < \varepsilon \ \forall \ m, n \geq n_0$.

$\Rightarrow \{\alpha_n\}$ and $\{\beta_n\}$ are Carchy sequences in C.

$\because$ C is complete, so these Carchy sequences must converge in C.

Let $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in C.

Let

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}. \text{ Then } A \in \mathcal{A}$$

$$\text{Then } \|A_n - A\| = \left\| \begin{pmatrix} \alpha_n - \alpha & \beta_n - \beta \\ 0 & \alpha_n - \alpha \end{pmatrix} \right\|$$

$= |\alpha_n - \alpha| + |\beta_n - \beta|$

$\to 0$ as $n \to \infty$

Therefore $\{A_n\}_n \to A$ in $\mathcal{A}$

$\therefore \mathcal{A}$ is a Banach space.

$$\text{For } A = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_1 \end{pmatrix}, B = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} \in \mathcal{A}, \text{ we}$$

$$\text{define } AB = \begin{pmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 + \alpha_2\beta_1 \\ 0 & \alpha_1\alpha_2 \end{pmatrix} \in \mathcal{A}$$

From the properties of matrices,we have,

(i) A(BC) = (AB)C

(ii) A(B + C) = AB + AC

(A + B)C = AC + BC

(iii) $\alpha(AB) = A(\alpha B) = (\alpha A)B$, where A, B, C $\in \mathscr{A}$ and α 'a' scalar.

$\therefore \mathscr{A}$ is an algebra.

Now, for A, B $\in \mathscr{A}$, we get

$$\|AB\| = \left\| \begin{pmatrix} \alpha_1\alpha_2 & \alpha_1\beta_2 + \alpha_2\beta_1 \\ 0 & \alpha_1\alpha_2 \end{pmatrix} \right\|$$

$= |\alpha_1|\,|\alpha_2| + |\alpha_1|\,|\beta_2 + |\alpha_2|\,|\beta_1| + |\beta_1|.|\beta_2|$

$= (|\alpha_1| + |\beta_1|)\,(|\alpha_2| + |\beta_2|)$

$= ||A||.||B||$

$\Rightarrow ||AB|| \leq ||A||.||B||$

$\therefore \mathscr{A}$ is a Banach algebra.

And, let I $\in \mathscr{A}$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \therefore \| I \| = |1| + |0| = 1$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit element of $\mathscr{A}$.

In general $\mathscr{A}$ is not commutative.

5. Let P_{n-1} denote the linear space of all polynomials with complex co-efficients of degree less than or equal to n.

If,

$x(t) = a_0 + a_1t + a_2t^2 + ... + a_nt^n$,

we define,

$$\| x(t) \| = \sum_{i=0}^{n} |a_i|$$

Then P_{n-1} is a normed space.

$\because$ Deimension of P_{n-1} is (n + 1) i.e., P_{n+1} is finite dimensional, so P_{n-1} is complete.

For,

$x(t) = a_0 + a_1t + a_2t^2 + \dots + a_nt^n$

$y(t) = b_0 + b_1t + b_2t^2 + \dots + b_nt^n$,

$$(xy)(t) = \sum_{k=0}^{n} \alpha_k t^k, \text{ where } \alpha_k = \sum_{j+i=k} a_j b_i$$

Then P_{n+1} is a Banach algebra.

6. Let $L^1(R)$ be the space of Lebesgue integrable complex valued functions on R.

Addition is pointwise addition

$$(f + g)(x) = f(x) + g(x)$$

Multiplication by scalar

$$(\lambda f)(x) = \lambda f(x)$$

Norm,

Norm, $\|f\| = \int_R |f(t)|\, dt$

Convolution,

$$(f * g)(s) = \int_R f(t)\, g(S-t)dt,\ s \in R$$

Then $L^1(R)$ is a Banach algebra. This is called Group algebra of R.

7. Let G be a group. Let $l^1(G)$ denote the set of mappings 'f' of G into C s.t

$$\sum_{S \in G} |f(s)| < \infty$$

$$(f * g)(s) = \sum_{t \in G} f(t)\, g(t^{-1}s),\ s \in G$$

$$\|f\| = \sum_{S \in G} |f(s)|$$

Then $l^1(G)$ is a Banach algebra, which is called discrete group algebra of G.

8. Let G be a locally compact group and μ be left invariant Haar measure on G. Let $L^1(G)$ be the corresponding Banach space of integrable functions.

Define,

$$(f * g)(s) = \int_G f(t)\, g(t^{-1}s)\, d\mu(t), s \in G.$$

Then $L^1(G)$ is a Group algebra of G.

Theorem Multiplication is jointly continuous in any Banach algebra. In particular, multiplication is left continuous and right continuous.

Proof : Let $\mathcal{A}$ be a Banach algebra.

Let the sequences $\{x_n\}$ and $\{y_n\}$ converges to x and y respectively in $\mathcal{A}$.

We have to show that $x_n y_n \to xy$.

Since, $x_n \to x$, so $\|x_n - x\| \to 0$ as $n \to \infty$

since, $y_n \to y$, so $\|y_n - y\| \to 0$ as $n \to \infty$

Now,

$$\begin{aligned}\|x_n y_n - xy\| &= \|x_n y_n - x_n y + x_n y - xy\| \\ &= \|x_n (y_n - y) + (x_n - x)y\| \\ &\le \|x_n(y_n - y)\| + \|(x_n - x)y\| \\ &\le \|x_n\|\, \|y_n - y\| + \|x_n - x\|\, \|y\| \\ &\to 0 \text{ as } n \to \infty\end{aligned}$$

Hence, $x_n y_n \to xy$.

$\therefore$ Multiplication is jointly continuous.

Next let x be any fixed point in $\mathcal{A}$, and $\{y_n\}$ is a sequence in $\mathcal{A}$ s.t $y_n \to y$ in $\mathcal{A}$.

Then $\|xy_n - xy\| = \|x(y_n - y)\|$

$$\le \|x\|\, \|y_n - y\| \to 0 \text{ as } n \to \infty$$

$\Rightarrow xy_n \to xy$

$\Rightarrow$ Multiplication is right continuous.

Again, for any sequence $x_n \to x$ in $\mathcal{A}$ and any fixed point $y \in \mathcal{A}$, we get

$\|x_n y - xy\| = \|(x_n - x)y\|$

$$\le \|x_n - x\|\, \|y\| \to 0 \text{ as } n \to \infty$$

$\Rightarrow x_n y \to xy \Rightarrow$ Multiplication is let continuous.

Exercise

Let A be a Banach algebra without unity.

Let A_1 consists of all ordered pairs (x, α), where $x \in A$, $\alpha \in C$.

We define

$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$

$\lambda(x, \alpha) = (\lambda x, \lambda\alpha)$, $\lambda \in K$.

$(x, \alpha)(y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$

Then A_1 is an algebra w.r.t these operations.

The norm on A_1 is defined by,

$\|(x, \alpha)\| = \|x\| + |\alpha|$

Then A_1 is a normed space

and $\|(x, \alpha)(y, \beta)\| \leq \|(x, \alpha)\|.\|(y, \beta)\|$

Let $\{x_n, \alpha_n)\}$ be a Cauchy sequence in A_1.

Then $\{x_n\}$ is a Cauchy sequence in A

and $\{\alpha_n\}$ is a Cauchy sequence in C.

∵ A and C are complete,so $\exists\ x \in A$, $\alpha \in C$

s.t

$\lim\limits_{n\to\infty} x_n = x, \lim\limits_{n\to\infty} x_n = \alpha$

and $\lim\limits_{n\to\infty} (x_n, \alpha_n) = (x, \alpha) \in A_1$

∴ A_1 is complete

Let $(x, \alpha) \in A_1$,

Let,

$e = (0, 1) \in A_1$, 0 is the zero element of A.

Then $(x, \alpha)(0, 1) = (x.0 + \alpha.0 + 1.x, \alpha.1)$

$= (x, \alpha)$

and $(0, 1)(x, \alpha) = (x, \alpha)$

∴ e is the unit element in A_1.

We define,

$g : A \to A_1$ s.t

$g(x) = (x, 0)$

Then $\|g(x)\| = \|(x, 0)\| = \|x\|$

g preserves the algebric operations.

g is an isometry.

$\therefore$ A is isometrically isomorphic to some subspace of A_1.

Theorem : Let A be a Banach space as well as complex algebra with unit element e ($\neq 0$), in which multiplication is left continuous and right continuous. Then there is a norm on A, which induces the same topology as the given one and makes A into a Banach algebra.

Proof : Clearly, $A \neq \{0\}$, as $e \neq 0$

For each $x \in A$, we define a left multiplication operator.

$M_x : A \to A$ s.t $M_x(z) = xz,\ z \in A$

Let $\widetilde{A} = \{M_x : x \in A\}$

Let $\langle z_n \rangle$ be a sequence in A s.t $z_n \to z$.

Since, the multiplication is right continuous in A, so

$xz_n \to xz\ \forall x \in A$

$\Rightarrow M_x(z_n) \to M_x(z),\ \forall x \in A$

$\Rightarrow M_x$ is continuous $\forall x \in A$

$\Rightarrow M_x$ is bounded $\forall x \in A$

i.e., $M_x \in \beta\ (A)\ \forall x \in A$

$\therefore\ \widetilde{A} \subseteq \beta(A)$.

We define

$\phi : A \to \widetilde{A}$ s.t $\phi(x) = M_x,\ x \in A$.

Now,

$$\begin{aligned} M_{x+y}(z) = (x + y)z = xz + yz &= M_x(z) + M_y(z) \\ &= (M_x + M_y)(z) \\ &\forall\ z \in A. \end{aligned}$$

$\Rightarrow M_{x+y} = M_x + M_y$

$\Rightarrow \phi(x + y) = \phi(x) + \phi(y)$

And,

$$\begin{aligned} M_{xy}(z) &= (xy)z \\ &= x\,(yz) \end{aligned}$$

$= M_x(M_y(z))$

$= (M_x M_y)(z) \ \forall \ z \in A$

$\Rightarrow M_{xy} = M_x \ M_y$

$\Rightarrow \phi(xy) = \phi(x) \ \phi(y)$

Let $\phi(x) = \phi(y)$

$\Rightarrow M_x = M_y$

$\Rightarrow M_x(e) = M_y(e)$

$\Rightarrow xe = ye$

$\Rightarrow x = y$

$\therefore \phi$ is one-one.

Clearly, ϕ is onto.

$\therefore \phi : A \to \tilde{A}$ is an isomorphism.

Now,

$\phi^{-1} : \tilde{A} \to A$ s.t

$\phi^{-1} (M_x) = x$

Now,

$||x|| = ||xe|| \ = ||M_x(e)||$

$\leq ||M_x||.||e||$

$= ||M_x||$

$\Rightarrow ||\phi^{-1}(M_x)|| \leq ||M_x||$

$\Rightarrow ||\phi^{-1}|| \leq 1$

$\Rightarrow \phi^{-1}$ is bounded.

$\Rightarrow \phi^{-1}$ is continuous.

Again,

$||M_x M_y|| \leq ||M_x||.||M_y||$

Also,

$||M_e|| = ||I|| = 1$

To show $\tilde{A}$ is a Banach algebra, we have to show that it is complete, i.e., to show that it is a closed subspace of $\beta(A)$, relative to the topology given by operator norm.

Let $T \in \beta(A)$ s.t the sequence $< T_i >$ in $\widetilde{A}$ converges to T.

$\because T_i \in \widetilde{A} = \{M_x : x \in A\}$, so let $T_i = M_{x_i}$ for some $x_i \in A$.

$$T_i(y) = M_{x_i}(y) = x_i y$$
$$= (x_i e)y$$
$$= T_i(e)y, \ \forall \ y \in A \ .....(1)$$

Now,

$$\|T_i(y) - T(y)\| \leq \|T_i - T\| \, \|y\|$$
$$\to 0 \text{ as } i \to \infty$$

$\therefore T_i(y) \to T(y)$ as $i \to \infty, \ \forall \ y \in A$.

In particular,

$T_i(e) \to T(e)$ as $i \to \infty$

$\because$ Multiplication is left continuous in A, so

$T_i(e)y \to T(e)y$ as $i \to \infty$

In (1),

$$\left.\begin{array}{l} T_i(y) \to T(y) \\ T_i(e)y \to T(e)y \end{array}\right\} \text{as } i \to \infty$$

$$\therefore T(y) = T(e)y$$
$$= xy, \text{ putting } T(e) = x$$
$$= M_x(y) \ \forall \ y \in A$$

$\Rightarrow T = Mx \in \widetilde{A}$

$\therefore \widetilde{A}$ contains all its limit points and so it is closed.

$\therefore \widetilde{A}$ is complete.

$\therefore \widetilde{A}$ is a Banach algebra.

$\therefore$ By open-mapping theorem,

$\phi^{-1} : \widetilde{A} \to A$ is open

$\therefore \phi : A \to \widetilde{A}$ is continuous

$\therefore \phi$ is bounded

$\Rightarrow \exists\ K > 0$ s.t $||\phi(x)|| \leq K||x||$

$\Rightarrow ||M_x|| \leq K\ ||x||$

Again, $||x|| \leq ||M_x|| \leq K||x||$

$\therefore$ $||x||$ and $||M_x||$ are equivalent norms in A.

$\therefore$ A is isomorphic to the Banach algebra $\tilde{A}$, (with the norm $||M_x||$) and $||M_x||$ induces the same topology as $||x||$.

4.3 Singular and non-singular elements :

Let A be a Banach algebra with unit element e. An element $r \in A$ is called left (right) regular, if $\exists$ $s \in A$ such that $sr = e$ $(rs = e)$

An element which is both and right regular is called regular or invertible, or non-singular element.

i.e. $\exists\ s \in A$ s.t $rs = sr = e$

Then $s = r^{-1}$ and

$$rr^{-1} = r^{-1}r = e$$

$\therefore$ $s = r^{-1}$ is called inverse of r.

Not regular $\Leftrightarrow$ singular.

Note :

No $r \in A$ has more than one inverse.

If possible, let $r \in A$ has more than one inverse, say s and s_1.

Then, $rs = sr = e$

$rs_1 = s_1r = e$

Now

$s = se = s(rs_1) = (sr)s_1 = es_1 = s_1$.

$\Rightarrow s = s_1$

$\therefore$ No $r \in A$ has more than one inverse.

4.4 Complex homomorphism :

Let A be a complex algebra and ϕ is a linear functional on A which is not identically 0. If $\phi(xy) = \phi(x)\ \phi(y)$, $\forall$ x, y $\in$ A, then ϕ is called a complex homomorphism.

Proposition :

If ϕ is a complex homomorphism on a complex algebra A with unit e, then $\phi(e) = 1$ and $\phi(x) \neq 0$, for every invertible $x \in A$.

Proof : Since ϕ is not identically zero, so for some $y \in A$, $\phi(y) \neq 0$.

Now, $y = ye$

$\Rightarrow \phi(y) = \phi(ye) = \phi(y)\phi(e)$

$\Rightarrow \phi(e) = 1$

If x is invertible, then

$xx^{-1} = x^{-1}x = e$

$\Rightarrow \phi(xx^{-1}) = \phi(e)$

$\Rightarrow \phi(x)\ \phi(x^{-1}) = \phi(e) = 1$

$\Rightarrow \phi(x) \neq 0$

Theorem : Let A be a Banach algebra, $x \in A$, $||x|| < 1$. Then (a) $e - x$ is invertible.

Proof : Let $s_n = e + x + x^2 + ... + x^n$

$\therefore\ ||x^n|| \leq ||x||^n < 1.$

$\Rightarrow < s_n >$ forms a Cauchy sequence in A.

$\because$ A is complete, so $\exists\ s \in A$ s.t $s_n \to s$ as $n \to \infty$

$\because\ ||x|| < 1$, so $x^n \to 0$ as $n \to \infty$

Now,

$$s_n(e - x) = (e + x + x^2 + ... + x^n)\ (e - x)$$

$$= (e - x^{n+1})$$

$$= e - x)\ s_n$$

$\because$ Multiplication in A is continuous, so as $n \to \infty$,

$$s_n\ (e - x) \to s(e - x)$$

$$(e - x)s_n \to (e - x)s$$

Also, $e - x^{n+1} \to e - 0 = e$ as $n \to \infty$

$\therefore\ s(e - x) = e = (e - x)s$

$\therefore\ e - x$ is invertible.

Again, $(e - x)^{-1} = s$

$$= \lim_{n\to\infty} s_n$$

$$= \lim_{n\to\infty}\left(e + \sum_{j=1}^{n} x^j\right)$$

$$= e + \sum_{j=1}^{\infty} x^j$$

(b) $\left\|(e-x)^{-1} - e - x\right\| \le \dfrac{\|x\|^2}{1-\|x\|}$

Proof : $\|(e - x)^{-1} - e - x\| = \| s - e - x \|$

$$= \left\| e + \sum_{j=1}^{\infty} x^j - e - x \right\|$$

$$= \left\| \sum_{j=2}^{\infty} x^j \right\| \le \sum_{j=2}^{\infty} \|x\|^j$$

$$= \frac{\|x\|^2}{1-\|x\|}$$

$$\Rightarrow \left\|(e-x)^{-1} - e - x\right\| \le \frac{\|x\|^2}{1-\|x\|}$$

(c) $|\phi(x)| < 1$, for any complex homomorphism ϕ on A.

Proof :

Let $\lambda \in C$ s.t $|\lambda| \ge 1$

Then $\|\lambda^{-1}x\| = |\lambda^{-1}| \, \|x\|$

$\le 1. \, \|x\| < 1$

$\therefore$ By(a), $e - \lambda^{-1}x$ is invertible.

Then $\phi(e - \lambda^{-1}x) \ne 0$

$\Rightarrow \phi(e) - \phi(\lambda^{-1}x) \ne 0$

$\Rightarrow 1 - \lambda^{-1}\phi(x) \ne 0$

$\Rightarrow \lambda^{-1}\phi(x) \ne 1$

$\Rightarrow \phi(x) \ne \lambda$

So, $|\phi(x)| \ne |\lambda| \ge 1$.

$\Rightarrow |\phi(x)| < 1$, for every complex homomorphism ϕ on A.

Proposition :

Let A be a unital Banach algebra and G = G(A), be the set of all invertible elements of A. Then G is a group under multiplication.

Proof :

Clearly, $G \neq \phi$, as $e \in G$

Let $x, y \in G$, $\therefore$ $xx^{-1} = x^{-1}x = e$

$yy^{-1} = y^{-1}y = e$

$\Rightarrow y^{-1}\, x^{-1} \in A$

Now, $(xy)\,(y^{-1}x^{-1}) = x(yy^{-1})x^{-1}$

$= xex^{-1}$

$= xx^{-1}$

$= e$

Similarly $(y^{-1}x^{-1})\,(xy) = e$

$\therefore$ xy is invertible and $(xy)^{-1} = y^{-1}x^{-1}$

$\therefore$ $xy \in G$.

Let $x, y, z \in G$

Then $x(yz) = (xy)z$

Again, $xe = ex = x$

$\Rightarrow$ e is the identity element of G.

Let $x \in G$

$\therefore$ $xx^{-1} = x^{-1}x = e$

$\Rightarrow x^{-1}$ is invertible and $x^{-1} \in G$.

$\therefore$ G is a group under multiplication.

Theorem : Let A be a Banach algebra $x \in G(A)$, $h \in A$,

$\|h\| < \frac{1}{2}\ \|x^{-1}\|^{-1}$. Then $x + h \in G(A)$ and

$\|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \leq 2\ \|x^{-1}\|^{3}\ \|h\|^{2}$

Proof : Given that $\|h\| < \frac{1}{2} \|x^{-1}\|^{-1}$

$$\Rightarrow \|x^{-1}\| \,.\, \|h\| < \frac{1}{2}$$

Then

$\|x^{-1}h\| \leq \|x^{-1}\|.\|h\| < \frac{1}{2} < 1$

and $\|-x^{-1}h\| = \|x^{-1}h\| < 1$

$\therefore$ $e - (-x^{-1}h)$, i.e. $e + x^{-1}h$ is invertible.

Now, $x \in G$

$e + x^{-1}h \in G$

$\because$ G is a group, $x(e + x^{-1}h) \in G$

$$\Rightarrow xe + xx^{-1}h \in G$$

$$\Rightarrow x + h \in G.$$

Again,

$$(e + x^{-1}h)^{-1} = e + \sum_{n=1}^{\infty} (-1)^n (x^{-1}h)^n$$

$$= e - x^{-1}h + \sum_{n=2}^{\infty} (-1)^n (x^{-1}h)^n$$

$\Rightarrow (e + x^{-1}h)^{-1} - e + x^{-1}h$

$$= \sum_{n=2}^{\infty} (-1)^n (x^{-1}h)^n$$

$\therefore$ $\|(e + x^{-1}h)^{-1} - e + x^{-1}h\|$

$$= \left\| \sum_{n=2}^{\infty} (-1)^n (x^{-1}h)^n \right\|$$

$$\leq \sum_{n=2}^{\infty} \|x^{-1}h\|^n$$

$$= \|x^{-1}h\|^2 \sum_{n=0}^{\infty} \|x^{-1}h\|^n$$

$$< \|x^{-1}h\|^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$=\|x^{-1}h\|^2 \frac{1}{1-\frac{1}{2}} = 2\|x^{-1}h\|^2 \quad ......(1)$$

Now,

$$(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1} = [x(e + x^{-1}h)]^{-1} - x^{-1} + x^{-1}hx^{-1}$$
$$= (e + x^{-1}h)^{-1} x^{-1} - x^{-1} + x^{-1}hx^{-1}$$
$$= [(e + x^{-1}h)^{-1} - e + x^{-1}h]x^{-1}$$

$\therefore \|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\|$

$$\leq \|(e + x^{-1}h)^{-1} - e + x^{-1}h\|.\|x^{-1}\|$$
$$< 2\|x^{-1}h\|^2 . \|x^{-1}\|$$
$$\leq 2\|x^{-1}\|^3 \|h\|^2$$

Thm : G(A) is an open subset of A

Proof : Let $x_0 \in G(A)$.

We consider the open sphere $S\left(x_0, \frac{1}{\|x_0^{-1}\|}\right)$

with centre at x_0 and radius $\frac{1}{\|x_0^{-1}\|}$

If, $x \in S\left(x_0, \frac{1}{\|x_0^{-1}\|}\right)$, then $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$

Let $y = x_0^{-1}x$ and $z = e - y$

$\therefore \|z\| = \|-z\| = \|y - e\| = \|x_0^{-1}x - e\|$

$$= \|x_0^{-1}x - x_0^{-1}x_0\|$$
$$= \|x_0^{-1}(x - x_0\|$$
$$\leq \|x_0^{-1}\| \|x - x_0\|$$
$$< \|x_0^{-1}\| \cdot \frac{1}{\|x_0^{-1}\|} = 1$$
$$\Rightarrow \|z\| < 1$$

$\Rightarrow$ e – z is invertible.

$\therefore$ e – (e – y) is invertible.

$\Rightarrow$ y is invertible.

$\therefore$ $y \in G(A)$.

Thus $x_0 \in G(A)$, $y \in G(A) \Rightarrow x_0 y \in G(A)$

$\therefore$ $x_0 y = x_0(x_0^{-1}x) = x$

$\therefore$ $x \in G(A)$.

$$\therefore S\left(x_0, \frac{1}{\|x_0^{-1}\|}\right) \subseteq G(A)$$

$\Rightarrow$ G(A) is an open set.

COROLLARY :

Let S be the set of all non-invertible elements of A. Then S is a closed subset of A.

Proof : We know that G(A) is an open subset of A.

Now $S = A - G(A) = [G(A)]^C$

$\therefore$ G(A) is open subset of A

$\Rightarrow$ $[G(A)]^C$ is closed subset of A.

$\Rightarrow$ S is a closed subset of A.

Theorem : The mapping $x \to x^{-1}$ of G(A) into G(A) is continuous and is therefore a homeomorphism of G(A) onto itself.

Proof : Let $x_0 \in G(A)$. Let x be any other element of G(A) such that

$$\|x - x_0\| < \frac{1}{2\|x_0^{-1}\|}$$

$$\begin{aligned} \therefore \quad \|x_0^{-1}x - e\| &= \|x_0^{-1}(x - x_0)\| \\ &\leq \|x_0^{-1}\| \, \|x - x_0\| \\ &< \frac{1}{2} < 1 \end{aligned}$$

$\Rightarrow$ $e - (e - x_0^{-1}x)$ is invertible.

$\Rightarrow$ $x_0^{-1}x$ is invertible.

Now,

$$(x_0^{-1}x)^{-1} = x^{-1}x_0 = e + \sum_{n=1}^{\infty} (e - x_0^{-1}x)^n \quad ...(1)$$

Let,

$T : G(A) \to G(A)$ s.t $T(x) = x^{-1}$

$\therefore \|Tx - Tx_0\| = \|x^{-1} - x_0^{-1}\|$

$= \|x^{-1}x_0 - e)x_0^{-1}\|$

$\leq \|x^{-1}x_0 - e\|.\|x_0^{-1}\|$

$$= \left\{\left\|\sum_{n=1}^{\infty} (e - x_0^{-1}x)^n\right\|\right\} \left\|x_0^{-1}\right\|$$

$$\leq \left\|x_0^{-1}\right\| \sum_{n=1}^{\infty} \left\|(e - x_0^{-1}x)\right\|^n$$

$= \|x_0^{-1}\|.\|e - x_0^{-1}x\|$

$$\sum_{n=1}^{\infty} \left\|e - x_0^{-1}x\right\|^n$$

$$= \left\|x_0^{-1}\right\| . \left\|e - x_0^{-1}x\right\| . \frac{1}{1 - \left\|e - x_0^{-1}x\right\|}$$

$$< 2\left\|x_0^{-2}\right\| . \left\|e - x_0^{-1}x\right\| \left[\begin{array}{l} \because 1 - \left\|e - x_0^{-1}x\right\| > 1 - \frac{1}{2} \\ \Rightarrow \frac{1}{1 - \left\|e - x_0^{-1}x\right\|} < 2 \end{array}\right]$$

$= 2\|x_0^{-1}\|.\|x_0^{-1}(x_0 - x)\|$

$\leq 2\|x_0^{-1}\|^2 \|x - x_0\|$

$\therefore$ T is continuous at x_0

$\Rightarrow$ T is continuouson G(A)

Again T is one-one

T is onto.

Next, $T(Tx) = T(x^{-1}) = (x^{-1})^{-1} = x$

$\Rightarrow T^2 x = Ix. \ \forall \ x \in G(A)$

$\Rightarrow T^2 = I$

$\Rightarrow T = T^{-1}$

$\therefore$ T is a homeomorphism.

4.5 Spectrum :

Let A be a Banach algebra. If $x \in A$, the spectrum $\sigma(x)$ of x is the set of all complex numbers λ such that $\lambda e - x$ is not invertible.

$\therefore \sigma(x) = \{\lambda \in C : (\lambda e - x) \text{ is not invertible}\}$

Complement of $\sigma(x)$ is called the resolvent set of x.

$\Omega = C - \sigma(x)$

$= \{\lambda \in C : (\lambda e - x)^{-1} \text{ exists}\}$

Spectral radius of x :

$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$

$\therefore$ If $\lambda \in \Omega$, then $(\lambda e - x)^{-1}$ exists.

We define a vector valued function $x(\lambda)$ on Ω by $x(\lambda) = (x - \lambda e)^{-1}$

This is the resolvent function associated with x.

Let $\lambda_1, \lambda_2 \in \Omega$

$x(\lambda_1) = (x - \lambda_1 e)^{-1}$

$x(\lambda_2) = (x - \lambda_2 e)^{-1}$

$$\begin{aligned}\therefore (x(\lambda_1))^{-1} x(\lambda_2) &= (x - \lambda_1 e)\, x(\lambda_2) \\ &= [(x - \lambda_2 e) + (\lambda_2 e - \lambda_1 e)]\, x(\lambda_2) \\ &= [(x(\lambda_2))^{-1} + (\lambda_2 - \lambda_1)e]\, x(\lambda_2) \\ &= e + (\lambda_2 - \lambda_1)x(\lambda_2)\end{aligned}$$

$\Rightarrow x(\lambda_2) = x(\lambda_1) + (\lambda_2 - \lambda_1)x\, x(\lambda_1)\, x(\lambda_2)$

This is the resolvent equation.

Theorem : The resolvent function $x(\lambda)$ is analytic at every point of .

Proof : Let $\lambda, \lambda_0 \in \Omega$ and $\lambda \neq \lambda_0$

Then by resolvent equation,

$x(\lambda) = x(\lambda_0) + (\lambda - \lambda_0)\, x(\lambda_0)\, x(\lambda)$

$\Rightarrow x(\lambda) - x(\lambda_0) = (\lambda - \lambda_0)x(\lambda_0)\, x(\lambda)$

$$\Rightarrow \frac{x(\lambda) - x(\lambda_0)}{\lambda - \lambda_0} = x(\lambda_0)\, x(\lambda) \quad ....(1)$$

We have,

$x(\lambda) = (x - \lambda e)^{-1}$

The mapping $x \to x^{-1}$ of G(A) onto G(A) is continuous.

$$\therefore \lim_{\lambda \to \lambda_0} x(\lambda) = \lim_{\lambda \to \lambda_0} (x - \lambda e)^{-1}$$

$$= (x - \lambda_0 e)^{-1}$$

$$= x(\lambda_0)$$

Taking $\lambda \to \lambda_0$ in (1), we get

$$\lim_{\lambda \to \lambda_0} \frac{x(\lambda) - x(\lambda_0)}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} x(\lambda_0)\, x(\lambda)$$

$$= x(\lambda_0) \lim_{\lambda \to \lambda_0} x(\lambda)$$

$$= x(\lambda_0),\ x(\lambda)$$

$$= [x(\lambda_0)]^2$$

$\therefore x'(\lambda_0)$ exists.

$\Rightarrow x(\lambda)$ is analytic at every point of Ω.

Theorem : $\rho(x) \le \|x\|$

Proof : let $\lambda \in C$ s.t $|\lambda| > \|x\|$

$\therefore \|\lambda^{-1}x\| = |\lambda^{-1}|.\|x\| < 1$

$\therefore e - \lambda^{-1} x$ is invertible.

$\Rightarrow -\lambda(e - \lambda^{-1}x) = x - \lambda e$ is invertible.

$\therefore \lambda \in \Omega$

But $\rho(x) = \sup \{|\lambda| : \lambda \in \sigma(x)\}$

For $|\lambda| > \|x\|$, $\lambda \in \Omega$

i.e. $\lambda \notin \sigma(x)$

If $\lambda \in \sigma(x)$, then $|\lambda| \le \|x\|$

$\therefore \sup \{|\lambda| : \lambda \in \sigma(x)\} \le \|x\|$

$\Rightarrow \rho(x) \le \|x\|$

Theorem : Let $x \in A$. Then the spectrum $\sigma(x)$ is compact.

Proof : We have, if $\lambda \in \sigma(x)$, then $|\lambda| \le \|x\|$

$\therefore \sigma(x)$ is bounded.

To prove the theorem, we have to show that $\sigma(x)$ is closed, i.e., to show that its complement $C - \sigma(x) = \Omega$ is open.

We define $g : C \to A$ by $g(\lambda) = x - \lambda e$

Now, $\|g(\lambda_1) - g(\lambda_2)\| = \|x - \lambda_1 e) - (x - \lambda_2 e)\|$

$= \| - \lambda_1 e + \lambda_2 e)\|$

$= \|(\lambda_2 - \lambda_1)e\| = |\lambda_2 - \lambda_1|$

$\therefore$ g is continuous for every value $\lambda \in C$.

Let $\lambda_0 \in \Omega = (C - \sigma(x))$

$\therefore x - \lambda_0 e$ is invertible.

$\therefore (x - \lambda_0 e) \in G(A)$

Since, G(A) is open, so $\exists\ \varepsilon > 0$ s.t

$$S(x - \lambda_0 e, \varepsilon)G(A)$$

$\therefore S(x - \lambda_0 e,)$ contains only invertible elements.

$\because$ g is continuous at λ_0, is $\exists\ \delta > 0$ such that

$\|g(\lambda) - g(\lambda_0)\| < \varepsilon$ whenever $|\lambda - \lambda_0| < \delta$

i.e. $\|(x - \lambda e) - (x - \lambda_0 e)\| < \varepsilon$ whenever $|\lambda - \lambda_0| < \delta$.

$\therefore g(\lambda) = x - \lambda e \in S(x - \lambda_0 e, \varepsilon)$, for all values λ such that $|\lambda - \lambda_0| < \delta$

$\therefore \lambda \in \Omega$ whenever $|\lambda - \lambda_0| < \delta$

i.e $\lambda \in S(\lambda_0, \delta)$.

$\therefore S(\lambda_0, \delta) \subset \Omega$

$\therefore \Omega$ is an open set.

i.e. $C - \Omega = \sigma(x)$ is closed set

Hence $\sigma(x)$ is compact.

Theorem : For any $x \in A$, $\sigma(x)$ is non-empty.

Proof : Let f be a continuous linear functional defined on A.

For $\lambda \in \Omega$, let

$$f(\lambda) = f[(x - \lambda e)^{-1}]$$
$$= f[x(\lambda)]$$

where, $x(\lambda)$ is the resolvent function associated with x.

since $x \to x^{-1}$ of G(A) onto (A) is continuous, so $x(\lambda)$ is continuous. Also f is continuous.

$\therefore f(\lambda)$ is a continuous function of λ on the resolvent set of x.

From the resolvent equation, we get for any $\lambda, \mu \in \Omega$, $(\lambda \neq \mu)$,

$$\frac{x(\lambda)-x(\mu)}{\lambda-\mu} = x(\lambda)\, x(\mu)$$

since f is linear,

$$\frac{f(x(\lambda))-f(x(\mu))}{\lambda-\mu} = \frac{f(x(\lambda)-x(\mu))}{\lambda-\mu}$$

$$= f\left(\frac{x(\lambda)-x(\mu)}{\lambda-\mu}\right)$$

$= f(x(\lambda)\, x(\mu))$.

Taking limit as $\lambda \to \mu$, we get

$$\lim_{\lambda\to\mu} \frac{f(\lambda)-f(\mu)}{\lambda-\mu} = \lim_{\lambda\to\mu} f(x(\lambda))\, x(\mu))$$

$$= f[\{x(\mu)\}^2]$$

$\therefore$ f is analytic on Ω.

Next, $|f(\lambda)| = |f(x(\lambda))|$

$$\leq \|f\|.\|x(\lambda)\|$$

$$= \|f\|.\|(x - \lambda e)^{-1}\|$$

$$= \|f\|.\left\|\lambda^{-1}\left(\frac{1}{\lambda}x - e\right)^{-1}\right\|$$

$$= \|f\|.\frac{1}{|\lambda|}\left\|\left(e - \frac{1}{\lambda}x\right)^{-1}\right\| \text{....(A)}$$

For all large $|\lambda|$,

$$\left\|\frac{x}{\lambda}\right\| = \frac{1}{|\lambda|}\|x\| < 1$$

$\therefore \left(e - \frac{1}{\lambda}x\right)$ is invertible and

$$\left(e - \frac{1}{\lambda}x\right)^{-1} = e + \sum_{j=1}^{\infty}\left(\frac{1}{\lambda}x\right)^j$$

$$\therefore \left\| \left(e - \frac{1}{\lambda}x\right)^{-1} - e \right\| = \left\| \sum_{j=1}^{\infty} \left(\frac{x}{\lambda}\right)^{j} \right\|$$

$$\leq \sum_{j=1}^{\infty} \left\| \left(\frac{x}{\lambda}\right) \right\|^{j}$$

$$= \frac{\left\|\frac{x}{\lambda}\right\|}{1 - \left\|\frac{x}{\lambda}\right\|}$$

$$= \frac{\frac{1}{|\lambda|}\|x\|}{1 - \frac{1}{|\lambda|}\|x\|} \to 0 \text{ as } |\lambda| \to \infty$$

$$\Rightarrow \left(e - \frac{1}{\lambda}x\right)^{-1} \to e \text{ as } |\lambda| \to \infty$$

Now from (A),

$|f(\lambda)| \to 0$ as $|\lambda| \to \infty$ ......(B)

If, $\sigma(x) = \phi$, then $\Omega = C - \sigma(x) = C$

$\therefore$ f is analytic on the entire complex plane,

i.e. f is an entire function.

$\therefore$ By Lioville's Theorem, f is constant.

$\therefore$ $f(\lambda) = 0 \ \forall \ \lambda \in C = \Omega$

since f is an arbitrary continuous linear functional, so

$f[x(\lambda)] = 0 \ \forall \ \lambda \in C = \Omega \ [\forall \ f \in A^*]$

$\Rightarrow x(\lambda) = 0 \ \forall \ \lambda \in C = \Omega$

$\Rightarrow (x - \lambda e)^{-1} = 0, \ \forall \ \lambda \in C = \Omega$

$\therefore \|e\| = \|(x - \lambda e)\ (x - \lambda e)^{-1}\|$

$= \|(x - \lambda e)\ x(\lambda)\|$

$= \|0\| = 0$

But, this is a contradiction, as $\|e\| = 1$

$\therefore$ $\sigma(x)$ is non-empty i.e. $\sigma(x) \neq \phi$

Lemma :

If $x \in A$ and n is a positive integer, then

$$\sigma(x^n) = [\sigma(x)]^n$$
$$= \{\lambda^n : \lambda \in \sigma(x)\}$$

Proof : Let λ be a non-zero complex number and $\lambda_1, \lambda_2, ..., \lambda_n$ be its distinct n^{th} roots.

$\therefore\ x^n - \lambda e = (x - \lambda_1 e)(x - \lambda_2 e).....(x - \lambda_n e)$ ...(1)

Let $\lambda \in \sigma(x^n)$. Then $x^n - \lambda e$ is not-invertible. So, at least one of the factors on the RHS of (1) is say $x - \lambda_i e$ is not invertible.

$\therefore\ \lambda_i \in \sigma(x)$

$\Rightarrow \lambda_i^n \in [\sigma(x)]^n$

$\Rightarrow \lambda \in [\sigma(x)]^n$

$\therefore\ 6(x^n) \subseteq [\sigma(x)]^n$

Let $\lambda \in [\sigma(x)]^n$

$\therefore\ \lambda = \lambda_i^n,\ \lambda_i \in \sigma(x)$

$\therefore\ x - \lambda_i e$ is not invertible

$\therefore\ x^n - \lambda e = (x - \lambda_1 e)\ ....(x - \lambda_i e)\ ....\ (x - \lambda_n e)$ is not-invertible.

$\therefore\ \lambda \in \sigma(x^n)$.

$\therefore\ [\sigma(x)]^n \subseteq \sigma(x^n)$.

$$\therefore\ \sigma(x^n) = [\sigma(x)]^n$$
$$= \{\lambda^n : \lambda \in \sigma(x)\}.$$

Theorem : The spectral radius $\rho(x)$ of x satisfies

$$\rho(x) = \lim_{n \to \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n}$$

Proof : For any +ve integer n,

$$\rho(x^n) = \sup\ \{|\lambda| : \lambda \in \sigma(x^n)\}$$
$$= \sup\ \{|\lambda| : \lambda \in [\sigma(x)]^n\}$$
$$= \sup\ \{|\mu|^n : \mu \in \sigma(x)\}$$
$$= [\sup\ \{|\mu| : \mu \in \sigma(x)\}]^n$$
$$= [\rho(x)]^n$$

Now, $\rho(x^n) \le \|x^n\|$

$\Rightarrow [\rho(x)]^n \le \|x^n\|$

$\Rightarrow \rho(x) \le \|x^n\|^{1/n}$, for any positive number n ....(1)

$\therefore \rho(x) \le \inf_{n\ge 1} \|x^n\|^{1/n}$ ....(2)

$\Rightarrow \rho(x) \le \lim_{n\to\infty} \inf \|x^n\|^{1/n}$....(A)

Now, we show that if a is any real number such that $\rho(x) < a$, then for all sufficiently large values of n,

$\|x^n\|^{1/n} \le a$

If $|\lambda| > \|x\|$, then $\lambda \in \Omega$.

$\therefore x(\lambda) = (x - \lambda e)^{-1}$

$$= \lambda^{-1}\left(\frac{x}{\lambda} - e\right)^{-1}$$

$$= -\lambda^{-1}\left(e - \frac{x}{\lambda}\right)^{-1}$$

$$= -\lambda^{-1}\left[e + \sum_{n=1}^{\infty}\left(\frac{x}{\lambda}\right)^n\right]$$

$$= -\lambda^{-1}\left[e + \sum_{n=1}^{\infty}\frac{x^n}{\lambda^n}\right] \text{....(3)}$$

Let f be any continuous linear functional on A.

$$\therefore (3) \Rightarrow f(x(\lambda)) = -\lambda^{-1}\sum_{n=1}^{\infty} f\left(\frac{x^n}{\lambda^n}\right) \text{....(4)}$$

for all λ such that $|\lambda| > \|x\|$

We have, $\rho(x) \le \|x\|$

for $|\lambda| > \|x\|$, $\lambda \in \Omega$

i.e. for $|\lambda| > \|x\| \ge \rho(x)$, $\lambda \in \Omega$

i.e. for $|\lambda| > \rho(x)$, $\lambda \in \Omega$

f is analytic on Ω.

$\therefore f(x)(\lambda))$ is analytic in the region $|\lambda| > \rho(x)$

$\therefore$ The expansion (4) is valid for all λ such that $|\lambda| > \rho(x)$.

Let α be a real number such that

$$\rho(x) < \alpha < a \ , \ (\alpha, a > 0)$$

Since $\alpha > \rho(x)$, so (4) is valid for $\lambda = \alpha$

$\therefore$ (4) $\Rightarrow$ The infinite series

$\sum_{n=1}^{\infty} f\left(\frac{x^n}{\lambda^n}\right)$ converges and so its terms forms a bounded set of numbers.

Since this is true for all $f \in A^*$, so the sequence is itself bounded.

$$\left\|\frac{x^n}{\alpha^n}\right\| \leq K, \text{ for } n = 1, 2, \ldots\ldots$$

$$\Rightarrow \frac{1}{\|\alpha\|^n}\left\|x^n\right\| \leq K$$

$$\Rightarrow \left\|x^n\right\| \leq \alpha^n K$$

$$\Rightarrow \left\|x^n\right\|^{1/n} \leq \alpha^n K^{1/n}$$

Since $\alpha < a$, so for all sufficiently large values of n, $\alpha K^{1/n} \leq a$

$\therefore \ \|x^n\|^{1/n} \leq a$ ...(5)

$$\Rightarrow \lim_{n\to\infty} \sup \|x^n\|^{1/n} \leq a$$

Since a is arbitrary with $a > \rho(x)$

so $\lim_{n\to\infty} \sup \|x^n\|^{1/n} \leq \rho(x)$ ....(B)

$\therefore$ (A) and (B) $\Rightarrow$

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}$$

Theorem : The following conditions are equivalent.

(i) $\|x^2\| = \|x\|^2$ for every x.

(ii) $\rho(x) = \|x\|$, for every x.

Proof : (i) $\Rightarrow$ (ii)

Let $\|x^2\| = \|x\|^2$ for every x.

$\therefore \|x^4\| = \|(x^2)^2\| = \|x^2\|^2 = \|x\|^4$

$\|x^8\| = \|(x^4)^2\| = \|x^4\|^2 = \|(x^2)^2\|^2 = \|x^2\|^4 = \|x\|^8$

$\therefore \|x^{2^k}\| = \|x\|^{2^k}$ for any $k \in N$

$$\therefore \rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$$

$$= \lim_{k\to\infty} \left\|x^{2^k}\right\|^{\frac{1}{2^k}}$$

$$= \lim_{k\to\infty} \left(\|x\|^{2^k}\right)^{\frac{1}{2^k}}$$

$$= \|x\|$$

$\therefore$ **(i) $\Rightarrow$ (ii)**

Next (ii) $\Rightarrow$ (i)

Let $\rho(x) = \|x\|$ for every x.

We have

$$\rho(x^n) = [\rho(x)]^n$$

$\Rightarrow \|x^n\| = \|x\|^n$

For n = 2, $\|x^2\| = \|x\|^2$

$\therefore$ (ii) $\Rightarrow$ (i)

Gelfand-Mazur Theorem :

If A is a Banach algebra with unity in which every non-zero element is invertible, then A is isometrically isomorphicto complex field.

Proof : If $x \in A$ and $\lambda_1 \neq \lambda_2$, then at most one of the elements $x - \lambda_1 e$ and $x - \lambda_2 e$ is 0.

$\therefore$ At least one of them is invertible.

We know that $\sigma(x)$ is non-empty, so there must exist at least one λ s.t $x - \lambda e$ is not invertible.

$\therefore x - \lambda e = 0 \Rightarrow x = \lambda e$

Also, there cannot exist two different values of λ for which $x = \lambda e$.

$\therefore$ Every $x \in A$ can be written as a unique scalar multiple of unity.

We define $f : A \to \mathbb{C}$ s.t

$f(x) = \lambda$, if $x = \lambda e$

Then f is linear

f is one-one

and f is onto.

Let $x = \lambda e$.

$\therefore |f(x)| = |\lambda| = |\lambda|\ ||e||$ as $\lambda = \lambda e$

$= ||\lambda e||$

$= ||x||$

$\therefore$ f is an isometry.

$\therefore$ A is isometrically isomorphic to complex field.

Ex. B be a Banach algebra and A be a Banach algebra s.t A B. Then $\sigma_B(x) \subseteq \sigma_A(x)$

Soln. Let $\lambda \in \sigma_B(x)$

$\Rightarrow x - \lambda e$ is not invertible in B

$\Rightarrow x - \lambda e$ is not invertible in A.

$\Rightarrow \lambda \in \sigma_A(x)$

$\therefore \sigma_B(x) \subseteq \sigma_A(x)$

4.6 Component :

Let W be a topological space. Then a component of W is a maximal connected subset of W.

Lemma : Suppose V and W are open sets in some topological space X, $V \subset W$ and W contains no boundary point of V. The V is a union of components of W.

Proof : Let C be a component of W that intersects V.

Let $U = X - \bar{V}$. Then $U \cap V = (X - \bar{V}) \cap V = \phi$

$\therefore (C \cap U) \cap (C \cap V) = C \cap (U \cap V) = C \cap \phi = \phi$

$\therefore$ W contains no boundary point of V, so

$W \cap (\bar{V} - V) = \phi$

$\therefore C = C \cap (U \cup V)$

$= (C \cap U) \cup (C \cap V)$

$\therefore$ C is the union of two disjoint open sets $C \cap U$ and $C \cap V$.

But C is connected. Also $C \cap V \neq \phi$

So $C \cap U = \phi$

$\Rightarrow C \subset V$

Lemma : Let A be a Banach algebra. $x_n \in G(A)$ for $n = 1, 2, 3, \ldots$; x is a boundary point of G(A), and $x_n \to x$ as $n \to \infty$. Then $\|x_n^{-1}\| \not\to \infty$ as $n \to \infty$.

Proof : If possible let $\|x_n^{-1}\| \to \infty$ as $n \to \infty$.

Then $\exists$ M > 0 (< ∞) s.t $\|x_n^{-1}\| < M$ for infinitely many n's

Again, $x_n \to x$ as $n \to \infty$

$\therefore$ For one of these n's, say $n = n_0$

$$\|x_{n_0} - x\| < \frac{1}{M}$$

$$\begin{aligned} \text{Again, } \|e - x_{n_0}^{-1}x\| &= \|x_{n_0}^{-1}(x_{n_0} - x)\| \\ &\leq \|x_{n_0}^{-1}\| \, \|x_{n_0} - x\| \\ &< M \cdot \frac{1}{M} = 1 \end{aligned}$$

$\therefore e - (e - x_{n_0}^{-1}x)$ is invertible.

i.e. $x_{n_0}^{-1} x \in G(A)$

Again, $x_{n_0} \in G(A)$ and G(A) is a group.

$\therefore x_{n_0}(x_{n_0}^{-1} x) \in G(A)$

i.e. $x \in G(A)$

G(A) is open, so $\exists \, \varepsilon > 0$ s.t $S_\varepsilon(x) \subseteq G(A)$

$\therefore S_\varepsilon(x) \cap (A - G(A)) = \phi$

$\therefore$ x is not a boundary point of G(A), which is a contradiction.

$\therefore \|x_n^{-1}\| \to \infty$ as $n \to \infty$

Theorem If A is a closed subalgebra of a Banach algebra B and if A contains the unit element of B, then G(A) is a union of components of $A \cap G(B)$.

Proof : Let x be an element of A s.t x is invertible.

$\because$ $A \subseteq B$, so x is invertible in B.

$\therefore$ $G(A) \subseteq G(B)$.

Both G(A) and $A \cap G(B)$ are open subsets of A,

and $G(A) \subset A \cap G(B)$

Ley y be a boundary point of G(A). Then y is the limit point of a sequence $\{x_n\}$ in G(A).

$\therefore$ $\|x_n^{-1}\| \to \infty$ as $n \to \infty$

Let $y \in G(B)$.

$\because$ The mapping $x \to x^{-1}$ of G(B) onto G(B) is continuous, so $x_n^{-1} \to y^{-1}$

$\therefore$ $\{\|x_n^{-1}\|\}$ is bounded, which is impossible.

$\therefore$ $y \notin G(B)$.

i.e. G(B) contains no boundary point of G(A).

$\therefore$ $A \cap G(B)$ contains no boundary point of G(A).

$\therefore$ G(A) is a union of components of A G(B).

4.7 Topological divisors of zero :

Let A be a Banach algebra. An element $z \in A$ is called a **topological divisor of zero,** if there exists a sequence $\{z_n\}$, $z_n \in A$,

$\|z_n\| = 1$, for n = 1, 2, 3, ... and such that either $zz_n \to 0$

or $z_n z \to 0$

Let Z denote the set of all topological divisors of zero in A and $S = A - G(A)$.

Theorem : Z is a subset of S.

Proof : Let $z \in Z$. Then a sequence $\{z_n\}$ in A with $\|z_n\| = 1$, n = 1, 2, ... and either $zz_n \to 0$ or $z_n z \to 0$ as $n \to \infty$.

If possible, let $z \in G(A)$

Then z^{-1} exists.

Suppose, $zz_n \to 0$ as $n \to \infty$

$\because$ Multiplication is a continuous operation,

$z^{-1}(zz_n) \to z^{-1}\, 0$

$\Rightarrow z_n \to 0$ as $n \to \infty$

which is a contradiction to the fact that $||z_n|| = 1 \ \forall$ n.

$\therefore z \notin G(A)$

$\therefore z \in A - G(A) = S$

$\therefore Z \subseteq S$.

Theorem : The boundary of S is a subset of Z.

Proof : Since S is a closed subset of A, so any boundary point of S is also in S.

Again, for every boundary point of S, $\exists$ a sequence of elements from G(A) that converges to the boundary point.

Let x be any boundary point of S. Then $x \in S$ and $\exists$ a sequence of elements from G(A), say $\{s_n\}$, such that–

$s_n \rightarrow x$ as $n \rightarrow \infty$.

$s_n^{-1}x - e = s_n^{-1}(x - s_n)$ ....(1)

If $\{||s_n^{-1}||$ is a bounded sequence, then since $s_n \rightarrow x$, from (1), we get, for all large values of n,

$||s_n^{-1}x - e|| < 1$

$\therefore e - (e - s_n^{-1}x) = s_n^{-1}x \in G(A)$

Also, $s_n \in G(A)$ for all n.

$\therefore s_n(s_n^{-1}x) \in G(A)$

i.e. $x \in G(A)$, which contradicts the fact that $x \in S$.

$\therefore \{||s_n^{-1}||\}$ is not bounded. So, we can assume that $||s_n^{-1}|| \rightarrow \infty$ as $n \rightarrow \infty$.

Let $x_n = \dfrac{s_n^{-1}}{\|s_n^{-1}\|}$

$\therefore ||x_n|| = 1$

$$xx_n = x \cdot \frac{s_n^{-1}}{\|s_n^{-1}\|} = \frac{e + xs_n^{-1} - s_n s_n^{-1}}{\|s_n^{-1}\|}$$

$$= \frac{e + (x - s_n)\, s_n^{-1}}{\|s_n^{-1}\|}$$

$$= \frac{e}{\|s_n^{-1}\|} + (x - s_n)\, x_n$$

But $\|s_n^{-1}\| \to \infty$ as $n \to \infty$

$$\therefore \frac{e}{\|s_n^{-1}\|} \to 0 \text{ as } n \to \infty$$

Again, $s_n \to x$ as $n \to \infty$

$\Rightarrow x - s_n \to 0$ as $n \to \infty$

Also, $\|x_n\| = 1$

$\therefore xx_n \to 0$ as $n \to \infty$

$\therefore x \in Z.$

Theorem : If 0 is the only topological divisor of zero in A, then A is isometrically isomorphic to the complex field.

Proof : Let $x \in A$. We know that $\sigma(x)$ is non-empty and it is bounded.

So, $\sigma(x)$ has a boundary point.

Let λ be a boundary point of $\sigma(x)$.

Then $x - \lambda e$ is a boundary point of $S = A - G(A)$.

$\therefore x - \lambda e$ is a topological divisor of zero in A.

$\therefore x - \lambda e = 0$, by the given condition.

$\Rightarrow x = \lambda e.$

There cannot exists two different values of λ for which $x = \lambda e$.

$\therefore$ Every 'x' can be written as a unique scalar multiple of the unity.

We define $f : A \to C$ s.t

$$f(x) = \lambda, \text{ if } x = \lambda e.$$

Then $f(ax + by) = af(x) + bf(y)$

$\therefore$ f is linear.

Also f is one-one and onto.

Let $x = \lambda e$.

$$\begin{aligned}\therefore |f(x)| = |\lambda| &= |\lambda|\,\|e\|,\ \lambda = \lambda e \\ &= \|\lambda e\| \\ &= \|x\|\end{aligned}$$

$\therefore$ f is an isometry.

$\therefore$ A is isometrically isomorphic to complex field.

i.e. $A \cong C$.

Theorem If the norm in A satisfies the inequality,

$\|xy\| \geq K \|x\|.\|y\|$ for some positive constant K, then A is isometrically isomorphic to the complex field.

Proof : Let z be a topological divisor of zero in A. Then $\exists$ a asequence $\{z_n\}$ in A with $\|z_n\| = 1$ and $zz_n \to 0$ as $n \to \infty$, or $z_n z \to 0$ as $n \to \infty$

$\therefore \|zz_n\| \geq K \|z\|.\|z_n\| = K\|z\|$ ....(1)

If $zz_n \to 0$ as $n \to \infty$, then (1) $\Rightarrow$

$$\|z\| = 0 \Rightarrow z = 0$$

Again, $\|z_n z\| \geq K\|z\|$ ...(2)

If $z_n z \to 0$ as $n \to \infty$, then (2) $\Rightarrow$

$$\|z\| = 0 \Rightarrow z = 0$$

$\therefore$ 0 is the only topological divisor of zero in A.

$\therefore A \cong C$.

Theorem Let $x \in A$ and G is an open set in C such that $\sigma(x) \subset G$. Then $\exists\ \delta > 0$ such that $\sigma(x + y) \subset G$, whenever $\|y\| < \delta$, $y \in A$.

Proof : If $\lambda \in C - \sigma(x)$, then $(x - \lambda e)$ is invertible and $x(\lambda) = (x - \lambda e)^{-1}$ is a continuous function of λ.

$$\therefore \left\|(x-\lambda e)^{-1}\right\| = \left\|\lambda^{-1}\left(\frac{1}{\lambda}x - e\right)^{-1}\right\|$$

$$= \frac{1}{|\lambda|}\left\|\left(e - \frac{1}{\lambda}x\right)^{-1}\right\| \text{....(1)}$$

For all large values of λ, $\left\|\frac{x}{\lambda}\right\| < 1$

Also, $\left(e - \frac{1}{\lambda}x\right)^{-1} = e + \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}x\right)^j$

$$\left\|\left(e - \frac{1}{\lambda}x\right)^{-1} - e\right\| = \left\|\sum_{j=1}^{\infty} \left(\frac{1}{\lambda}x\right)^j\right\|$$

$$\le \sum_{j=1}^{\infty} \frac{\|x\|^j}{|\lambda|^j}$$

$$= \frac{\frac{\|x\|}{|\lambda|}}{1-\frac{\|x\|}{|\lambda|}} \to 0 \text{ as } |\lambda| \to \infty$$

$$\therefore \left(e - \frac{1}{\lambda}x\right)^{-1} \to e \text{ as } |\lambda| \to \infty$$

Thus (1) gives $\|(x - \lambda e)^{-1}\| \to 0$ as $|\lambda| \to \infty$

We may assume that $\exists\ 0 < M < \infty$, s.t

$$\|(x - \lambda e)^{-1}\| < M \text{ for all } \lambda \text{ outside } G.$$

Take $\delta = \frac{1}{M}$

Let $y \in A$, $\|y\| < \delta$ and $\lambda \notin G$.

$$\begin{aligned}\therefore\ x + y - \lambda e &= (x - \lambda e)\,(x - \lambda e)^{-1}\,y + (x - \lambda e)\\ &= (x - \lambda e)\,[(x - \lambda e)^{-1}\,y + e] \quad .....(2)\end{aligned}$$

Again, $\|(x - \lambda e)^{-1}y\| \le \|(x - \lambda e)^{-1}\|.\|y\|$

$$< M\,\delta = M\,.\frac{1}{M} = 1$$

$\therefore\ e + (x - \lambda e)^{-1}y$ is invertible.

$\therefore$ The R.H.S of (2) is invertible.

i.e. $(x - \lambda e)\,[(x - \lambda e)^{-1}\,y + e]$ is invertible.

$\therefore$ The L.H.S of (2) is invertible

i.e. $x + y - \lambda e$ is also invertible.

$\therefore\ \lambda \notin \sigma(x + y)$

$\therefore\ \lambda \notin G \Rightarrow \lambda \notin \sigma(x + y)$

$\therefore\ \sigma(x + y) \subset G$, whenever $\|y\| < \delta$, $y \in A$.

Lemma : Suppose that f is an entire function with $f(0) = 1$, $f'(0) = 0$ and $0 < |f(z)| \le e^{|z|}$ for all complex number 'z'. Then $f(z) = 1\ \forall\ z$.

Theorem : If ϕ is a linear functional on A such that $\phi(e) = 1$ and $\phi(x) \neq 0$ for every invertible $x \in A$, then

$\phi(xy) = \phi(x)\,\phi(y)$, $x, y \in A$.

i.e. ϕ is a complex homomorphism.

Proof : Let N denote the null space of ϕ

i.e. $N = \ker \phi$.

Let $x \in A$ and let $\beta = \phi(x)$

We consider the element $x - \beta e$.

$$\begin{aligned} \therefore\ \phi(x - \beta e) &= \phi(x) - \phi(\beta e) \\ &= \phi(x) - \beta\phi(e) \\ &= \phi(x) - \phi(x) \\ &= 0 \end{aligned}$$

$\Rightarrow x - \beta e \in N$.

Let $x - \beta e = a$, $a \in N$.

$\Rightarrow x = a + \beta e$

$\therefore$ Every element $x \in A$ canbe expressed as

$x = a + \beta e = a + \phi(x)e$, where $a \in N$.

If $y \in A$, then $y = b + \phi(y)e$, where $b \in N$.

$\therefore\ xy = (a + \phi(x)e)\,(b + \phi(y)e)$

$= ab + \phi(y)a + \phi(x)b + \phi(x)\,\phi(y)$

$\therefore\ \phi(xy) = \phi(ab) + \phi(y).\phi(a) + \phi(x).\phi(b) + \phi(x)\,\phi(y)$

$= \phi(ab) + \phi(x)\,\phi(y)$ ....(1)

$[\because\ a, b \in N]$

$\therefore$ The theorem will be proved if $\phi(ab) = 0$, i.e.

i.e. $ab \in N$, whenever $a,b \in N$ ....(2)

Suppose, we have proved a special case of (2) viz. $a^2 \in N$ if $a \in N$ ...(3)

In(1), we assume $x = y$

$$\begin{aligned} \text{Then } a + \phi(x)e &= b + \phi(y)e \\ &= b + \phi(x)e. \end{aligned}$$

$(1) \Rightarrow \phi(x^2) = \phi(a^2) + [\phi(x)]^2$

$\Rightarrow \phi(x^2) = [\phi(x)]^2$ .... (4), as $a^2 \in N$

Replace x by x + y in (4), we get

$\phi((x + y)^2) = [\phi(x + y)]^2$

$\Rightarrow \phi((x + y)(x + y) = [\phi(x + y)]^2$

$$= [\phi(x) + \phi(y)]^2$$

$\Rightarrow \phi(x^2 + xy + yx + y^2)$

$$= [\phi(x)]^2 + 2\phi(x)\,\phi(y) + [\phi(y)]^2$$

$\Rightarrow \phi(x^2) + \phi(xy) + \phi(yx) + \phi(y^2)$

$$= [\phi(x)]^2 + 2\phi(x)\,\phi(y) + [\phi(y)]^2$$

$\Rightarrow \phi(xy + yx) = 2\phi(x).\phi(y)$

$\therefore$ $xy + yx \in N$ if

$x \in N,\ y \in A$ ....(5)

We have,

$(xy - yx)^2 + (xy + yx)^2$

$= (xy - yx)(xy - yx) + (xy + yx)\,(xy + yx)$

$= (xy)(xy) - xy^2x - yx^2y + (yx)(yx) + (xy)(xy) + xy^2x + yx^2y + (yx)(yx)$

$= x\,(yx\,y) + (yx\,y)x + x(yx\,y) + (yx\,y)x$

$= 2[x(yx\,y) + (yxy)x]$ ...(6)

Replacing y by yxy in (5), we get

$x(yxy) + (yxy)x \in N$ if $x \in N$

$\therefore$ $(xy - yx)^2 + (xy + yx)^2 \in N$ if $x \in N$.

Again, if $x \in N$, by (5), $xy + yx \in N$

Applying (4), $(xy + yx)^2 \in N$

$(xy - yx)^2 \in N$

$\therefore$ $[\phi(xy - yx)^2] = 0$

$\Rightarrow [\phi(xy - yx)]^2 = 0$

$\Rightarrow \phi(xy - yx) = 0$

$\Rightarrow xy - yx \in N$.

Thus $xy - yx \in N$ if $x \in N,\ y \in A$ ....(7)

Adding (5) and (7), we get

$xy \in N$ if $x \in N,\ y \in A$.

$\therefore$ (2) is satisfied

$\therefore$ The theorem is proved, provided we establish (3).

Since by hypothesis, $\phi(x) \neq 0$ for every invertible element x,

So, N contains no invertible element.

If $a \in N$, then $||e - a|| \geq 1$

If $a \in N$, then $\frac{a}{\lambda} \in N$, for any complex number λ ($\neq 0$)

$\therefore$ $||e - \frac{a}{\lambda}\,|| \geq 1$

$\therefore$ $||\lambda e - a|| = |\lambda|\ ||e - \frac{a}{\lambda}\,||$

$\geq |\lambda|$

$= |\phi(\lambda e - a)|$ ....(8)

Now, every element $x \in A$ can be expressed as $x = \lambda e - a$, where $a \in N$, λ is a complex number.

$\therefore$ (8) $\Rightarrow |\phi(x)| \leq ||x||$

$\Rightarrow ||\phi|| \leq 1$

Now, ϕ is linear and bounded

$\therefore$ ϕ is a continuous linear functional.

Let $a \in N$, and without loss of generality we may assume that $||a|| = 1$.

We define

$$f(\lambda) = \sum_{n=0}^{\infty} \frac{\phi(a^n)}{n!} \lambda^n ; \lambda \in C$$

Now, $\leq |\phi(a^n)| \leq ||\phi||.||a^n||$

$\leq ||\phi||.||a||^n$

$\leq ||a||^n = 1$

$\therefore$ $f(\lambda)$ is analytic at λ.

$\therefore$ f is an analytic function on C.

$\therefore$ f is an entire function.

$$\therefore |f(\lambda)| = |\sum_{n=0}^{\infty} \frac{\phi(a^n)}{n!} \lambda^n|$$

$$\leq \sum_{n=0}^{\infty} \frac{|\phi(a^n)|}{n!} \lambda^n|$$

$$\leq \sum_{n=0}^{\infty} \frac{|\lambda|^n}{n!}$$

$= e^{|\lambda|}$

Again, $f(0) = \phi(e) = 1$

$f(0) = \phi(a) = 0$, since $a \in N$.

$\therefore$ If we can prove that $|f(\lambda)| > 0$ for every complex number λ, then $f(\lambda) = 1$, $\forall \in C$.

i.e. $f'(0) = 0$

$\phi(a^2) = 0$

$a^2 \in N$, if $a \in N$.

Consider the series

$$T(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a^n, \lambda \in C$$

This series converges in the norm of A.

$\therefore$ Since ϕ is continuous

$$\phi(T(\lambda)) = \phi\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \phi(a^n)$$

$= f(\lambda)$ ...(9)

Again, form the expression of $T(\lambda)$, we have

$T(\lambda + \mu) = T(\lambda)\, T(\mu)$; $\lambda, \mu \in C$

In particular, when $\mu = -\lambda$,

$T(\lambda)\, T(-\lambda) = T(0) = e$

Similarly, $T(-\lambda)\, T(\lambda) = e$

$\therefore$ $T(\lambda)$ is an invertible element in A.

$\therefore$ $\phi(T(\lambda)) \neq 0$

$\Rightarrow f(\lambda) \neq 0$ [by (9)]

$\therefore$ $|f(\lambda)| > 0$

Hence The theorem is proved.

The Theorem is known as G leason, Kahane Zalazko Theorem.

Unit 5

Spectrum of a a bounded operator

5.1 : Invertible operator :

An operator $T \in \beta(x)$ is a said to be invertible if there exists an operator

$S \in \mathcal{L}(X)$ s.t $TS = ST = 1$.

Then such operator S is unique and it is called the inverse of T and is denoted by T^{-1}

i.e. $S = T^{-1}$.

An element $\lambda \in C$ is caled a spectral value of $S \in \beta(X)$ if the operator

$S_\lambda = S - \lambda I$ is not invertible.

The set of all spectral values of S is called the spectrum of S and is denoted by $\sigma(S)$.

If $\lambda \in C - \sigma(S)$, then $S - \lambda I$ is invertible.

Let $R_\lambda(S) = S_\lambda^{-1} = (S - \lambda I)^{-1}$

$\therefore$ R_λ is called the resolvent operator of S and λ is called a regular value of S. The set of all regular values of S is denoted by Ω (S) or p(S), and is called the resolvent set of S.

If $\lambda \in \Omega(S)$, then the equation

$(S - \lambda I)x = y$ has a unique solution for all $y \in X$.

If λ is a spectral value of S, then the inverse of $S - \lambda I$ does not exists.

$\therefore$ $S - \lambda I$ is not 1–1 and onto.

An element $\lambda \in C$ is called an eigen value of $S \in \beta(X)$, if $S - \lambda I$ is not one-one, i.e.

$\exists$ if x λ X $(x \neq 0)$ s.t

$(S - \lambda I)(x) = 0$

$\Rightarrow Sx = \lambda x$.

The set of all elements $x \in X$ which satisfy $Sx = \lambda x$ is called the eigen space corresponding to the eigen value λ.

Theorem : If $x_1, x_2, ...x_n$ are eigen vectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2,..., \lambda_n$ of an operator $S \in \beta(X)$, then the set $\{x_1, x_2, ...x_n\}$ is L.I.

Proof : If possible let $\{x_1, x_2, ..., x_n\}$ be L.D.

Let x_m be the first element of the set which is a linear combination of all the preceding elements.

Let $x_m = \alpha_1 x_1 + \alpha_1 x_2 + \ldots + \alpha_{m-1} x_{m-1}$ ...(1)

Then the set $\{x_1, x_2, \ldots, x_{m-1}\}$ is L.I.

Operating on $(S - \lambda_m I)$ on both sides of (1), we get

$$(S - \lambda_m I)(x_m) = \sum_{i=1}^{m-1} \alpha_i (S - \lambda_m I)(x_i)$$

$$\Rightarrow Sx_m - \lambda_m x_m = \sum_{i=1}^{m-1} \alpha_i (\lambda_i x_i - \lambda_m)$$

$$\Rightarrow \lambda_m x_m - \lambda_m x_m = \sum_{i=1}^{m-1} \alpha_i (\lambda_i - \lambda_m)(x_i)$$

$$\Rightarrow \sum_{i=1}^{m-1} \alpha_i (\lambda_i - \lambda_m) x_i = 0$$

But $\{x_1, x_2, \ldots, x_{m-1}\}$ is L.I.

$\therefore \alpha_i (\lambda_i - \lambda_m) = 0 \ \forall \ i = 1, 2, \ldots, m-1$

$\Rightarrow \alpha_i = 0$ [$\because \lambda_i$'s are distinct]

$\forall i = 1, 2, \ldots, m-1.$

$\therefore$ (1) $\Rightarrow x_m = 0$, which is a contradiction as $x_m \neq 0$ ($\because$ it is an eigenvector).

$\therefore \{x_1, x_2, \ldots, x_n\}$ is L.I.

Lemma : Let X be a Banach algebra. If $x_i \in X$, $i = 1, 2, \ldots$ such that $\sum_{i=1}^{\infty} \|xi\| < \infty$, then the series $\sum_{i=1}^{\infty} x_i$ is convergent in X.

Proof : Let $\varepsilon > 0$ be arbitrary. Then there exists a positive integer N such that

$$\sum_{i=m+1}^{n} \|x_i\| < \varepsilon, \ n > m > N$$

Let, $S_n = x_1 + x_2 + \ldots + x_n$

$\therefore \|S_n - S_m\| = \|(x_1 + x_2 + \ldots + x_n) - (x_1 + x_2 + \ldots + x_m)\|$

$= \|x_{m+1} + x_{m+2} + \ldots + x_n\|$ ($\because n > m$)

$\leq \|x_{m+1}\| + \|x_{m+2}\| + \ldots + \|x_n\|$

$$= \sum_{i=m+1}^{n} \|x_i\| < \varepsilon \ \forall \ n > m > N$$

$\therefore$ $\{S_n\}$ is a Cauchy sequence in X, so it converges to some $x \in X$.

i.e. $\lim_{n\to\infty} S_n = x \Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} x_i = x$

$$\Rightarrow \sum_{i=1}^{\infty} x_i = x$$

$\therefore$ The series $\sum_{i=1}^{\infty} x_i$ is convergent in X.

Theorem : Let $S \in \beta$ (X) and $k \in C$ be s.t

$\|S\| < |k|$. Then $k \notin \sigma(S)$ and

$$(S - kI)^{-1} = -\sum_{n=0}^{\infty} \frac{S^n}{k^{n+1}}$$

Also,

$$\|(S - kI)^{-1}\| \leq \frac{1}{|k| - \|S\|}$$

Proof :

Let, $T = \frac{1}{K} S$

$\therefore \|T\| = \|\frac{1}{k} S\| < 1$

$$\therefore \sum_{n=0}^{\infty} \|T^n\| \leq \sum_{n=0}^{\infty} \|T^n\|^n < \infty$$

Since $\beta(X)$ is a Banach space, By the previous Lemma,

$\sum_{n=0}^{\infty} T^n$ Converges to some $A \in \beta(X)$

$$\sum_{n=0}^{\infty} T^n = A$$

Let $A_m = \sum_{n=0}^{m} T^n$

$\therefore (I-T)A_m = (I-T)\sum_{n=0}^{m} T^n$

$$= \sum_{n=0}^{m} T^n - \sum_{n=1}^{m+1} T^n$$

$$= I - T^{m+1}$$

$$= A_m (I - T)$$

$\Rightarrow (I - T)A_m = I - T^{m+1} = A_m (I - T)$ ...(1)

Again,

$\|T^{m+1}\| \to 0$ as $m \to \infty$ ($\because \|T\| < 1$)

$\therefore T^{m+1} \to 0$ as $m \to \infty$

Now, (1) $\Rightarrow$

$\lim_{m\to\infty} (I - T)A_m = \lim_{m\to\infty} (I - T^{m+1})$

$$= \lim_{m\to\infty} A_m(I - T)$$

$\Rightarrow (I - T)A = I = A(I - T)$

$\therefore (I - T)^{-1} = A = \lim_{m\to\infty} A_m$

$$= \lim_{m\to\infty} \sum_{n=0}^{m} T^n$$

$$= \sum_{n=0}^{\infty} T^n$$

$$\Rightarrow \left(1 - \frac{1}{k}S\right)^{-1} = \sum_{n=0}^{\infty} \left(\frac{1}{k}S\right)^n$$

$$\Rightarrow \left(\frac{1}{k}(kI - S)\right)^{-1} = \sum_{n=0}^{\infty} \frac{S^n}{k^n}$$

$$\Rightarrow k(kI - S)^{-1} = \sum_{n=0}^{\infty} \frac{S^n}{k^{n-1}}$$

Now,

$$\|(I - T)^{-1}\| = \|\sum_{n=0}^{\infty} T^n\|$$

$$\leq \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1-\|T\|}$$

$$= \frac{1}{1-\frac{1}{|k|}\|S\|} = \frac{|k|}{|k|-\|S\|}$$

$$\Rightarrow \|(S-kI)^{-1}\| \leq \frac{1}{|k|-\|S\|}$$

Theorem : If $S \in \beta(X)$ is invertible and $T \in \beta(X)$ such that $\|T - S\| < \frac{1}{\|S^{-1}\|}$, then T is invertible.

Further if $\|T - S\| < \frac{\eta}{\|S^{-1}\|}$, $0 < \eta < 1$,

then $\|T^{-1} - S^{-1}\| \leq \|S^{-1}\|^2 \frac{\|T-S\|}{\|1-\eta\|}$

Proof : If $\|T - S\| < \frac{1}{\|S^{-1}\|}$, then

$$\|I - S^{-1}T\| = \|S^{-1}S - S^{-1}T\|$$

$$\leq \|S^{-1}(S - T)\|$$

$$\leq \|S^{-1}\| \|S - T\|$$

$$< \|S^{-1}\| \frac{1}{\|S^{-1}\|} = 1$$

Replacing k by 1 and S by $I - S^{-1}T$ in the previous Theorem, we get

$S^{-1}T$ is invertible.

Let

$A = (S^{-1}T)^{-1}$. Then

$$A(S^{-1}T) = I = (S^{-1}T)A$$

$A(S^{-1}T) = I \Rightarrow (AS^{-1})T = I$

$(S^{-1}T)A = I \Rightarrow I = S(S^{-1}T)AS^{-1} \Rightarrow I = T(AS^{-1})$

$\therefore\ (AS^{-1})T = I = T(AS^{-1})$

$\therefore$ T is invertible and $T^{-1} = AS^{-1}$

Again,

$$\|T^{-1}\| - \|S^{-1}\| \le \|T^{-1} - S^{-1}\|$$

$$= \|T^{-1}SS^{-1} - S^{-1}\|$$

$$= \|T^{-1}SS^{-1} - T^{-1}TS^{-1}\|$$

$$= \|T^{-1}(S - T)S^{-1}\|$$

$$\le \|T^{-1}\|.\|S-T\|.\|S^{-1}\|$$

$$< \|T^{-1}\| \frac{\eta}{\|S^{-1}\|} \|S^{-1}\|$$

$$< \eta \|T^{-1}\|.$$

$$\Rightarrow \|T^{-1}\| - \eta\|T^{-1}\| < \|S^{-1}\|$$

$$\Rightarrow \|T^{-1}\| < \frac{\|S^{-1}\|}{(1-\eta)}$$

$$\therefore \|T^{-1} - S^{-1}\| \le \|T^{-1}\|\|S - T\|.\|S^{-1}\|$$

$$\le \frac{\|S^{-1}\|}{(1-\eta)} \|S - T\| \|S^{-1}\|$$

$$= \|S^{-1}\|^2 \frac{\|T-S\|}{(1-\eta)}$$

Spectral Mapping Theorem :

Let X be a Banach space and $S \in \beta(X)$ and $p(\lambda) = \lambda_n\lambda^n + \lambda_{n-1}\lambda^{n-1} + \ldots + \alpha_0$, where $\alpha_n \neq 0$

Then $p \to p(s) = \sigma \to (s))$, where

$p(S) = \alpha_n S^n + \alpha_{n-1}S^{n-1} + \ldots + \alpha_0 I.$

Proof : We know that $\sigma(S) \neq \phi$.

If n = 0, then $p \to (\sigma(S)) = \alpha_0$

$$= \sigma(p(s))$$

We now assume that $n > 0$

Let $T = p(s)$ and for $\mu \in C$

$T_\mu = p(s) - \mu I$, and

$t_\mu(\lambda) = p(\lambda) - \mu$

Let $t_\mu(\lambda) = p(\lambda) - \mu = \alpha_n (\lambda - \gamma_1)(\lambda - \gamma_2)...(\lambda - \gamma_n)$ ....(1)

Then,

$T_\mu = \alpha_n (S - \gamma_1 I)(S - \gamma_2 I) ...(S - \gamma_n I)$ ....(2)

If each $\gamma_j \notin \sigma(S)$, then each $(S - \gamma_j I)$ is invertible.

$\therefore S - \gamma_j I$ is 1_{-1} and onto.

$\therefore$ By inverse mapping Theorem.

$(S - \gamma_j I)^{-1}$ is bounded.

We know that $(ST)^{-1} = T^{-1}S^{-1}$

$$\therefore (2) \Rightarrow T_\mu^{-1} = \frac{1}{\alpha_n} (S - \gamma_n I)^{-1} (S - \gamma_{n-1})-1...(S - \gamma_1 I)^{-1}$$

$\Rightarrow p(S) - \mu I$ is invertible.

$\Rightarrow \mu \notin \sigma(p(s))$

$\Rightarrow \mu \in \Omega (p(s))$

i.e., if each $\gamma_j \in \Omega(S)$, then $\mu \in \Omega(p(S))$

i.e., if $\mu \notin \Omega(p(S))$, then $\gamma_j \notin \Omega(S)$, for some j.

i.e. if $\mu \in \sigma(p(S))$, then $\gamma_j \in \sigma(S)$, for some j.

$\therefore (1) \Rightarrow t_\mu(\gamma_j) = p(\gamma_j) - \mu = 0$

$\Rightarrow \mu = p(\gamma_j) \in p(\sigma(S))$

$\therefore \sigma(p(s)) \subseteq p(\sigma(S))$ ......(A)

Let $k \in p(\sigma(S))$. Then $k = p(\beta)$, for some $\beta \in \sigma(S)$.

$\therefore \beta$ is a zero of the polynomial

$t_k(\lambda)) = p(\lambda) - k$.

$\therefore$ We can write,

$t_k(\lambda) = (\lambda - \beta)g(\lambda)$,

where $g(\lambda)$ is the product of the remaining $(n - 1)$ linar factors and α_n.

Corresponding to this representation of t_k,

we can write T_k in the form

$T_k = p(S) - KI = (S - \beta I)g(S)$ ....(3)

$\therefore$ g(S) commutes with (S – βI)

$\therefore T_k = g(S)(S - \beta I)$ ....(4)

If T_k^{-1} exists, then (3), (4)

$I = (S - \beta I)\, g(S)\, T_k^{-1} = T_k^{-1}\, g(S)\, (S - \beta I)$

$\therefore$ S – βI is invertible, which is a contradiction, to the fact that

$\beta \in \sigma(S)$

$\therefore T_k$ is not invertible

$\Rightarrow$ p(S) – KI is not invertible.

$\Rightarrow K \in \sigma(p(S))$.

$\therefore p(\sigma(S))\ \sigma(p(S))$ ....(B)

From (A) and (B), we get

$$p(\sigma(S)) = \sigma(p(S))$$

5.2 Ideals :

Let A be an algebra. A non-empty subset I of A is called an ideal of A if

(i) I is a subspace of A.

(ii) If $x \in A$, $y \in I$, then $xy \in I$ and $yx \in I$.

If $I \neq A$, then I is called proper ideal.

Lemma : If I is a proper ideal and $x \in I$, then x is non-invertible.

Proof : If possible suppose x is invertible

$\Rightarrow x^{-1}$ exists and $x^{-1} \in A$

Now, $x \in I$ and x^{-1} A

$xx^{-1} \in I$

$\Rightarrow e \in I$.

$\therefore$ I = A, i.e. I is not proper ideal, a contradiction.

Hence $x \in I$ is not-invertible.

Theorem : Let I be a proper ideal of A. Then the closure $\bar{I}$ is also a proper ideal.

Proof : Let $x, y \in \bar{I}$. Then $\exists$ sequences $\{x_n\}$ and $\{y_n\}$ in I s.t $x_n \to x$ and $y_n \to y$ as $n \to \infty$

Let α, β be scalars. From the continuity of vector addition.

$\alpha x_n + \beta y_n \to \alpha x + \beta y$

$\therefore \alpha x_n + \beta y_n \in I$, so $\alpha x + \beta y \in \bar{I}$

Similarly, it can be shown that,

if $x \in \bar{I}$ and $y \in A$, then $xy \in \bar{I}$ and $yx \in \bar{I}$

$\therefore \bar{I}$ is an ideal.

$\because$ I is a proper ideal, I contains only non-invertible elements.

$\therefore I \subseteq A - G(A)$

Again, $A - G(A)$ is closed

$\therefore \bar{I} \subseteq \overline{A - G(A)} = A - G(A)$

$\because G(A) \neq \phi$ ($\because e \in G(A)$)

$\therefore \bar{I} \neq A$

$\therefore \bar{I}$ is a proper ideal.

Theorem : If A is a commutative complex algebra with unit, then every proper ideal of A is contained in a maximal ideal of A.

Proof : I be a proper ideal of A. Let P be the collection of all proper ideals of A which contain I. We define a relation $\leq$ on P by $M \leq N$ if $M \subseteq N$.

Then $\leq$ is a partial order on P.Let U be the maximal totally ordered subcollection of P. (Such existence of U is assured by Hausdorff's maximality Theorem).

Let $K = \bigcup_{M \in U} M$. Then K being the union of a totally ordered collection of ideals, is itself an ideal.

Clearly, $I \subset K$, and $K \neq A$, since no member of p contains the unit element of A.

Since, U is a maximal subcollection, so K is a maximal ideal containing I.

Theorem : If A is a commutative Banach algebra, then every maximal ideal of A is closed.

Proof : Let M be a maximal ideal of A. Since M contains no invertible element of A, so,

$M \subseteq A - G(A)$

$\Rightarrow \overline{M} \subseteq \overline{A - G(A)}$

$\because G(A)$ is open, so $A - G(A)$ is closed.

$\therefore \Rightarrow \overline{M} \subseteq \overline{A - G(A)} = A - G(A)$

i.e. $\overline{M}$ contains no invertible elements.

So, $\overline{M}$ is a proper ideal of A containg M.

But M is maximal, so $\overline{M} = M$

Hence M is closed.

Example : Let A, B be commutative Banach algebras.

$A \to B$ is a homomorphism.

Then $\ker\phi$ is an ideal of A, which is closed.

Soln. : Here $\ker\phi = \{x : \phi(x) = 0\}$

Clearly, $\ker \phi$ is a subspace of A.

Let $a \in A$ and $x \in \ker\phi \Rightarrow \phi(x) = 0$

Then

$$\therefore \phi(ax) = \phi(a)\phi(x)$$
$$= 0$$

$\Rightarrow ax \in \ker\phi$

similarly $xa \in \ker\phi$

Hence $\ker \phi$ is an ideal of A.

Let $\{x_n\}$ be a sequence in ker such that $x_n \to x$. To show that $x \in \ker\phi$

$\therefore \phi(x_n) = 0 \ \forall \ n.$

$$\therefore \phi(x) = \phi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \phi(x_n)$$
$$= 0$$

$\Rightarrow x \in \ker\phi$

$\therefore \ker\phi$ is a closed ideal of A.

5.3 Quotient Algebra :

Let J be a closed proper ideal of A, where A is a commutative Banach algebr.

Let $\pi : A \to A/J$ be the quotient map defined by

$\pi(x) = x + J, x \in A.$

Then A/J is a Banach algebra and π is a homomorphism.

Proof : In A/J, we define vector addition, scalar multiplication, vector multiplications by

$(x + J) + (y + J) = (x + y) + J$

$\alpha(x + J) = \alpha x + J$

$(x + J)(y + J) = xy + J$, $x, y \in A$, $\alpha \in F$.

Then A/J is a vector space w.r.t the first two operations.

We define the norm in A/J by,

$\|x + J\| = \inf \{\|x + j\| : j \in J\}$

With respect to this norm, A/J is a Banach space.

We know that $x, y \in A$ $xy \in A$.

$\therefore$ $xy + J \in A/J$. Then multipication is well-defined.

$\therefore$ $(x + J)(y + J) \in A/J$

$\therefore$ A/J is closed under multiplication.

Let $x + J$, $y + J$, $z + J \in A/J$.

$$
\begin{aligned}
\text{Then } (x + J)[(y + J)(z + J)] &= (x + J)(yz + J) \\
&= x(yz) + J \\
&= (xy)z + J \\
&= (xy + J)(z + J) \\
&= [(x + J)(y + J)](z + J)
\end{aligned}
$$

Again,

$$
\begin{aligned}
(x + J)[(y + J) + (z + J)] \\
&= (x + J)[(y + z) + J] \\
&= x(y + z) + J \\
&= (xy + xz) + J \\
&= (xy + J) + (xz + J) \\
&= (x + J)(y + J) + (x + J)(z + J)
\end{aligned}
$$

Next,

$$
\begin{aligned}
\alpha[(x + J)(y + J)] &= \alpha[xy + J] \\
&= \alpha(xy) + J \\
&= (\alpha x)y + J \\
&= (\alpha x + J)(y + J) \\
&= [\alpha(x + J)](y + J)
\end{aligned}
$$

Again,

$$
\begin{aligned}
(\alpha x)y + J &= x(\alpha y) + J \\
&= (x + J)(\alpha y + J)
\end{aligned}
$$

$$= (x + J)\,[\alpha(y + J)]$$

$\therefore \alpha[(x + J)(y + J)] = [\alpha(x + J)] = (x + J)\,[\alpha(y + J)]$

$\therefore$ A/J is an algebra.

Since A is commutative, so A/J is also commutative.

To show that π is a homomorphism.

Let $x, y \in A$, $\alpha \in F$.

Then $\pi(x + y) = (x + y) + J$

$$= (x + J) + (y + J)$$

$$= \pi(x) + \pi(y)$$

$\pi(\alpha x) = \alpha x + J = \alpha(x + J) = \alpha\pi(x)$

$\therefore \pi$ is linear.

Next,

$\pi(xy) = xy + J = (x + J)(y + J)$

$$= \pi(x)\,\pi(y)$$

$\therefore \pi$ is a homomorphism.

Again,

$\|\pi(x)\| = \|x + J\|$

$$= \inf \{\|x + j\| : j \in J\}$$

$$\leq \|x + 0\|$$

$$= \|x\|, \forall\ x \in A.$$

$\Rightarrow \|\pi\| \leq 1.$

To show that A/J is a Banach algebra :

Let $x_i \in A$ $(i = 1, 2)$

Then $\|\pi(x_1)\| = \|x_1 + J\|$

$= \inf \{\|x_1 + j\| : j \in J\}$

So, given $\delta > 0$, $\exists y_1 \in J$ such that

$\|\pi(x_1)\| + \delta \geq \|x_1 + y_1\|$

Similarly, considering x_2, we get given $\delta > 0$,

$\exists\ y_2 \in J$ such that $\|\pi(x_2)\| + \delta \geq \|x_2 + y_2\|$

Now,

$$(x_1 + y_1)(x_2 + y_2) = x_1x_2 + x_1y_2 + x_1y_2 + y_1x_2 + y_1y_2$$
$$= x_1x_2 + j,$$

where $j = x_1y_2 + y_1x_2 + y_1y_2 \in J$.

$\in x_1x_2 + J$.

Again,

$$\|\pi(x\, x_2)\| = \|x_1x_2 + J\|$$
$$= \inf \{\|x_1x_2 + z\| : z \in J\}$$
$$\leq \|x_1x_2 + j\|$$
$$= \|(x_1 + y_1)(x_2 + y_2)\|$$
$$\leq \|x_1 + y_1\|.\|x_2 + y_2\|$$
$$\leq (\|(x_1)\| + \delta)(\|\pi(x_2)\| + \delta)$$
$$= \|\pi(x_1)\|.\|\pi(x_2)\|$$
$$+ \delta\|\pi(x_1)\| + \delta\,\|\pi(x_2)\| + \delta^2$$
$$= \|\pi(x_1)\|.\|\pi(x_2)\| + \delta', \text{ say}$$

where $\delta' = \delta(\|\pi(x_1)\| + \|\pi(x_2)\|) + \delta^2$

Since δ' is arbitrarily small, we get

$\|\pi(x_1x_2)\| \leq \|\pi(x_1)\|.\|\pi(x_2)\|$.

$\Rightarrow \|x_1x_2 + J\| \leq \|x_1 + J\|.\|x_2 + J\|$

$\Rightarrow \|(x_1 + J)(x_2 + J)\| \leq \|x_1 + J\|.\|x_2 + J\|$

Let e be the unit element of A.

Then, $e + J \in A/J$

$\therefore (e + J)(x + J) = ex + J = x + J$

$= (x + J)(e + J)$

$\therefore (e + J)$ is the unit element in A/J.

Now,

$\|e + J\| = \inf \{\|e + j\| : j \in J\}$

$\leq \|e\| = 1.$

Let $x \notin J$. Then $x + J = J$

$\therefore \|x + J\| \neq 0$

Again,

$\|(e + J)(x + J)\| \leq \|e + J\|.\|x + J\|$

$\Rightarrow \|x + J\| \leq \|e + J\|.\|x + J\|$

$\Rightarrow \|e + J\| \geq 1$

$\therefore \|e + J\| = 1.$

$\therefore$ A/J is a commutative Banach algebra, with unit element.

Theorem : Let A be a commutative Banach algebra, and let Δ be the set of all complex homomorphisms of A.

(a) Every maximal ideal of A is the kernal of some $h \in \Delta$

Proof : Let M be a maximal ideal of A. Then M is closed and so A/M is a Banach algebra.

Let $x \in A, x \notin M$.

let $J = \{ax + y : a \in A, y \in M\}$

Then J is an ideal of A. Also, $M \subseteq J$.

Putting, $a = e, y = 0$, we get

$$ex + 0 = x \in J.$$

$\therefore M \subset J \ (\because x \notin M)$

But M is maximal, so $J = A$.

$\because e \in A$, so $e = ax + y$, for some

$$a \in A, y \in M.$$

Let, $\pi : A \rightarrow A/M$ be the quotient map.

$$\pi(e) = \pi(ax + y)$$

$$\Rightarrow \pi(ax) + \pi(y) = \pi(e)$$

$$\Rightarrow \pi(a)\ \pi(x) + \pi(y) = \pi(e)$$

$$\Rightarrow \pi(a)\ \pi(x) + M = \pi(e)$$

$$\Rightarrow \pi(a)\ \pi(x) = \pi(e)$$

$\therefore$ Every nonzero element $\pi(x)$ of the Banach algebra A/M is invertible in A/M.

$\therefore$ By Gelf and Mazur Theorem, A/M is isometrically isomorphic to C.

Let $j : A/M \rightarrow C$ be the isomorphism.

Let $h = j_0\ \pi : A \rightarrow C$

Then $h \in \Delta$

$\therefore \ker h = \{x \in A : h(x) = 0\}$

$= (x \in A : (j_0\pi)\ (x) = 0\}$

$= \{x \in A : j\ (\pi(x)) = 0\}$

$= \{x \in A : j(x + M) = 0\}$

$= \{x \in A : x + M = M\}$

$= \{x \in A : x \in M\} = M.$

$\therefore$ kerh = M.

(b) If $h \in \Delta$, then kernel of h is a maximal idela of a.

Proof : let $h : A \to C$

kerh = $\{x \in A : h(x) = 0\}$

Then kerh is an ideal of A and

kerh $\neq$ A ($\because$ h(e) = 1)

Let $Y \in A -$ kerh.

Let M = linear span of kerh $\cup \{y\}$.

For $a \in A$, we consider

$$\beta = a - \frac{h(a)}{h(y)} y$$

$$\therefore h(\beta) = h\left(a - \frac{h(a)}{h(y)} y\right)$$

$$= h(a) - h\left(\frac{h(a)}{h(y)} y\right)$$

$$= h(a) - \frac{h(a)}{h(y)} h(y)$$

$$= 0$$

$\Rightarrow \beta \in$ kerh.

$$\therefore a = \beta + \frac{h(a)}{h(y)} y \in M$$

$\therefore A \subseteq M \Rightarrow M = A$

$\therefore$ kerh is a maximal ideal of A.

(c) An element $x \in A$ is invertible in A $\Leftrightarrow h(x) \neq 0$ for every $h \in \Delta$.

Proof : Let x be invertible in A.

$\therefore xx^{-1} = x^{-1}x = e$

$\therefore h(xx - 1) = h(e) \ \forall \ h \in \Delta$

$\Rightarrow h(x)\, h(x^{-1}) = 1. \ \forall \ h \in \Delta$

$\Rightarrow h(x) \neq 0 \ \forall \ h \in \Delta$

Conversely, let $h(x) \neq 0 \ \forall \ h \in \Delta$

Let $J = \{ax : a \in A\}$

Then J is an ideal of A.

If x is not invertible, then $e \neq J$.

So, J is a proper ideal of A.

Let M be the maximal ideal containing J.

Then $M = \ker h_1$, for some $h_1 \in \Delta$

Now,

$x = ex \in J \subset M = \ker h_1$

$\therefore h_1(x) = 0$, which is a

contradiction as $h(x) \neq 0 \ \forall \ h \in \Delta$

$\therefore$ x is invertible.

(d) An element x A is invertible in A if and only if x lies in no proper ideal in A.

Proof : We know that no proper ideal of A contains any invertible elements of A.

Conversely, let x lies in no proper ideal in A.

We consider,

$J = \{ax : a \in A\}$

Then J is an ideal in A.

$\therefore x = ex \in J$.

If x is not invertible, then $e \notin J$.

$\therefore$ J is a proper ideal containing x, which is a contradiction.

$\therefore$ x must be invertible in A.

(e) $\lambda \in \sigma \rightarrow \sigma(x)$ if and only if $h(x) =$ for some $h \in \Delta$

Proof : Replacing x by $\lambda e - x$ in (c), we get

$\lambda e - x$ is invertible iff $h(\lambda e - x) \neq 0, \ \forall \ h \in \Delta$

$\therefore \lambda e - x$ is invertible iff $h(\lambda e) \neq h(x), \ \forall \ h \in \Delta$

$\lambda e - x$ is invertible iff $\lambda h(e) \neq h(x), \ \forall \ h \in \Delta$

$\lambda e - x$ is invertible iff $h(x) \neq \lambda, \ \forall \ h \in \Delta$

$\therefore \lambda e - x$ is not invertible in A.

$\Leftrightarrow h(x) = \lambda$ for some $h \in \Delta$

i.e. $\lambda \in 6(x)$ iff $h(x) = \lambda$ for some $h \in \Delta$.

5.4 Gelfand Transforms :

Definition :

Let A be a commutative Banach algebra and Δ is the set of all complex homomorphisms of A.

To each $x \in A$, we define a function

$\hat{x} : \Delta \to C$ by

$\hat{x}(h) = h(x), h \in \Delta$

Then $\hat{x}$ is called Gelfand transform of x.

Let, $\hat{A}$ be the collection of all such $\hat{x}$, where $x \in A$.

The Gelfand topology in Δ is the weak topology induce by $\hat{A}$, i.e. the weakest topology that makes every $\hat{x}$ continuous.

$\hat{A} \subset C(\Delta)$, the algebra of all complex continuous functions on Δ.

If M is a maximal ideal of A, then M = kerh, for some $h \in \Delta$.

Conversely, if $h \in \Delta$, then kerh is a maximal ideal of A.

So, $\exists$ a one-to-one correspondence between the maximal ideals of A and the members of Δ.

$\therefore \Delta$, equipped with Gelfand topology is called the maximal ideal space of A.

Radical of A, rad A is the intersections of all maximal ideals of A.

If rad A = $\{0\}$, then A is called semisimple.

Theorem :

Let Δ be the maximal ideal space of a commutative Banach algebra A.

(i) Gelfand transform is a homomorphism of A onto a sub-algebra $\hat{A}$ of $C(\Delta)$, whose kernel is rad A.

The Gelfand transform is therefore an isomorphism iff A is semi-simple.

Proof : Let $\phi : A \to \hat{A}$ such that $\phi(x) = \hat{x}$ is the Gelfand transformation.

Let $x, y \in A, \alpha \in C, h \in \Delta$

$\therefore \phi(\alpha x) = (\alpha x)^\wedge$

Now,

$(\alpha x)^\wedge (h) \quad = h(\alpha x)$

$= \alpha(h(x))$

$= \alpha\hat{x}(h)$

$= (\alpha\hat{x})h, \ \forall \ h \in \Delta$

$\Rightarrow (\alpha x)^\wedge = \alpha\hat{x}$

$\Rightarrow \phi(\alpha x) = \alpha\phi(x)$

Again,

$\phi(x + y) \quad = (x + y)^\wedge$

$\therefore \ (x + y)^\wedge(h) = h(x + y)$

$= h(x) + h(y)$

$= \hat{x}(h) + \hat{y}(h)$

$= (\hat{x} + \hat{y})(h)$

$\Rightarrow (x + y)^\wedge = \hat{x} + \hat{y}$

$\therefore \ \phi(x + y) = (x + y)^\wedge = \hat{x} + \hat{y}$

$= \phi(x) + \phi(y)$

$\therefore$ ϕ is linear.

Now,

$(xy)^\wedge(h) = h(xy)$

$= h(x)\,h(y)$ [$\because$ h is homomorphism]

$= ((\hat{x})(h))\,((\hat{y})(h)$

$= (\hat{x} \cdot \hat{y})(h)$

$\Rightarrow (xy)^\wedge = \hat{x} \cdot \hat{y}$

$\therefore \ \phi(xy) = (xy)^\wedge$

$= \hat{x} \cdot \hat{y}$

$= \phi(x) \cdot \phi(y)$

$\therefore$ ϕ is homomorphism

Clearly, ϕ is onto

Now, $\ker\phi = \{x \in A : \phi(x) = \hat{0}\}$

$= \{x \in A : \hat{x} = \hat{0}\}$

$\therefore \ \hat{x} = \hat{0} \Rightarrow \hat{x}(h) = \hat{0}(h), \ \forall \ h \in \Delta$

$\Rightarrow h(x) = 0, \ \forall \ h \in \Delta$

$\Rightarrow x \in \cap \{\ker h : h \in \Delta\}$

We know that if h $h \in \Delta$, then

kerh is a maximal ideal of A, and conversely every maximal ideal is kernel of some $h \in \Delta$

$\therefore\ x \in \cap$ {M : M is a maximal ideal of A}

$\therefore\ \ker\phi = \text{rad}A$.

If A is semi-simple, then rad A = {0}

$$\Rightarrow \ker\phi = \{0\}$$

$$\Rightarrow \phi \text{ is one-to-one.}$$

$\therefore\ \phi$ is isomorphism.

Conversely, if ϕ is isomorphism, then

ϕ is one-to-one

$\Rightarrow \ker \phi = \{0\}$

$\Rightarrow \text{rad } A = \{0\}$

$\therefore$ A is semi-simple.

(ii) For each $x \in A$, the range of $\hat{x}$ is the spectrum $\sigma(x)$.

Hence $\|\hat{x}\|_\infty = \rho(x) \leq \|x\|$

where $\|\hat{x}\|_\infty$ is the maximum of $|\hat{x}(h)|$ on Δ and $x \in \text{rad } A$ iff $\rho(x) = 0$

Proof : If $\lambda \in$ range $\hat{x}$, then $\lambda = \hat{x}(h)$ for some $h \in \Delta$.

Then $\lambda = \hat{x}(h)$ for some $h \in \Delta$

$= h(x)$ for some $h \in \Delta$

$\therefore\ \lambda \in \sigma(x) \Leftrightarrow h(x) = \lambda$ for some $h \in \Delta$.

$\therefore$ range $\hat{x} = \sigma(x)$.

Now,

$\|\hat{x}\|_\infty = \max\{|\hat{x}(h)| : h \in \Delta\}$

$= \max\{|\lambda| : \lambda \in \sigma(x)\}$

$= \rho(x) \leq \|x\|$

$\Rightarrow \|\hat{x}\|_\infty \leq \|x\|$

Again,

$x \in \text{rad } A \Leftrightarrow x \in \cap\{\ker h : h \in \Delta\}$

$\Leftrightarrow h(x) = 0\ \forall\ h \in \Delta$

$\Leftrightarrow \lambda = 0\ \forall\ \lambda \in \sigma(x)$

$\Leftrightarrow \rho(x) = 0$

$\therefore\ x \in \text{rad } A \Leftrightarrow \rho(x) = 0$

Theorem : If $\Psi : B \rightarrow A$ is a homomorphism of a commutative Banach algebra B into a semi simple commulative Banach algebra A, then Ψ is continuous.

Proof : Suppose $x_n \rightarrow x$ in B and

$\Psi(x_n) \rightarrow y$ in A

By closed Graph Theorem, we have to

show that $\Psi(x) = y$

Let Δ_A and Δ_B be the maximal ideal spaces of A and B respectively.

We fix $h \in \Delta_A$

Let $\phi = h \circ \Psi$. Then $\phi \in \Delta_B$

Then $\|x\| \leq 1 \Rightarrow |\phi(x)| \leq 1$

$\therefore \|\hat{x}\|_\infty \leq \|x\| \leq 1$

$\Rightarrow \max \{\|\hat{x}(h)| : h \in \Delta_B\} \leq 1$

$\Rightarrow |h(x)| \leq 1 \ \forall \ h \in \Delta_B$

$\therefore \phi$ is continuous and hence h is also continuous.

Now,

$$
\begin{aligned}
h(y) &= h(\lim \Psi(x_n)) \\
&= \lim h(\Psi(x_n)) \\
&= \lim \phi(x_n) \\
&= \phi(\lim x_n) \\
&= \phi(x) \\
&= h(\Psi(x))
\end{aligned}
$$

$\Rightarrow h(y - \Psi(x)) = 0 \ \forall \ h \in \Delta_A$

$\Rightarrow y - \Psi(x) \in \text{rad } A.$

$\because$ A is semisimple, so $\text{rad}A = \{0\}$

$\Rightarrow y - \Psi(x) = 0$

$\Rightarrow y = \Psi(x).$

$\therefore \Psi$ is continuous.

Corollary : Every isomorphism between two semisimple commulative Banach algebras is a homeomorphism.

Proof : Let A and B be two semi simple commulative Banach algebras and $\Psi : A \to B$ be an isomorphism.

Obviously Ψ is one-one and onto.

Given that B is semisimple so Ψ is continous.

Again $\Psi^{-1} : B \to A$ and A is semisimple, so Ψ^{-1} is continuous.

Hence Ψ is a homeomo rphism.

Lemma : If A is a commulative Banach algebra and

$$r = \inf \frac{\|x^2\|}{\|x\|^2}$$

$$s = \inf \frac{\|\hat{x}\|_\infty}{\|x\|}, (x \in A, x \neq 0)$$

then $s^2 \leq r \leq s$.

Proof :

$\|\hat{x}\|_\infty = \max \{|\hat{x}(h)| : h \in \Delta\}$

$\because\ s = \inf \frac{\|\hat{x}\|_\infty}{\|x\|}$, so

$$s \leq \frac{\|\hat{x}\|_\infty}{\|x\|}$$

$\Rightarrow \|x\|_\infty \geq s\|x\|$ .....(1)

We know that

$\|\hat{x}\|_\infty \leq \|x\|$

$\therefore\ \|x^2\| \geq \|\hat{x}^2\|_\infty$

$$\begin{aligned}\therefore\ \|\hat{x}^2\|_\infty &= \max \{|\hat{x}^2(h)| : h \in \Delta\} \\ &= \max \{|h(x^2)\} : h \in \Delta\} \\ &= \max \{|h(x)|.|h(x)| : h \in \Delta\} \\ &= \max \{|h(x)| : h \in \Delta\}^2 \\ &= \|\hat{x}\|^2_\infty\end{aligned}$$

$\therefore\ \|x^2\| \geq \|x^2\|_\infty = \|\hat{x}\|^2\infty$

$\geq s^2\|x\|^2 \ \forall \ x \in A$

$$\Rightarrow s^2 \leq \frac{\|x^2\|}{\|x\|^2};\, x \in A,\, x \neq 0$$

$$\therefore s^2 \leq \inf \frac{\|x^2\|}{\|x\|^2};\, x \in A,\, x \neq 0$$

$$\Rightarrow s^2 \leq r$$

Again,

$$r = \inf \ \frac{\|x^2\|}{\|x\|^2}$$

$$\Rightarrow r \leq \frac{\|x^2\|}{\|x\|^2};\, x \in A,\, x \neq 0$$

$\Rightarrow \|r\| \geq r.\|x\|^2 \ \forall \ x \in A$ ....(2)

We assume that $\|x^{2^n}\| \geq r^{2^n-1} \|x\|^{2^n}$ ....(3)

$\therefore \ \|x^{2^{n+1}}\| = \|x^{2^n})^2\|$

$\geq r \ \|x^{2^n}\|^2$ [using (2)]

$\geq r.(r^{2^n-1}.\|x\|2^n$ [by (3)]

$= r.r^{2^{n+1}} - 2 \ \|x\|^{2^{n+1}}$

$= r^{2^{n+1}} - 1 \ \|x\|^{2^{n+1}}$

$\therefore \ \|x^m\| \geq r^{m-1} \|x\|^m$

$(m = 2^n \ , \ n = 1, 2, ...)$ ....(4)

Taking m^{th} root in (4) and when $m \to \infty$, then

$$\lim_{m \leftarrow \infty} \|x^m\|^{1/m} \geq \lim_{m \leftarrow \infty} \ r^{1-\frac{1}{m}} \ \|x\|$$

$\Rightarrow \rho(x) \geq r \ \|x\|$

$\therefore \ \|\hat{x}\|_{\infty} = \rho(x) \geq r\|x\|$

$$\Rightarrow r \leq \ \frac{\|\hat{x}\|_{\infty}}{\|x\|};\, x \in A,\, x \neq 0$$

$\therefore r \leq \inf \dfrac{\|\hat{x}\|_{\alpha}}{\|x\|} = s$

Therefore,

$s^2 \leq r \leq s.$

Theorem : (a) The Gelfand Transform is an isometry

$\Leftrightarrow \|x^2\| = \|x\|^2 \;\; \forall \;\; x \in A$

Proof : we know that if $r = \inf \dfrac{\|x^2\|}{\|x\|^2}$,

$s = \inf \dfrac{\|\hat{x}\|_{\infty}}{\|x\|}$, then $s^2 \leq r \leq s$

The Gelfand transform is an isometry if and only if $\|\hat{x}\|_{\alpha} = \|x\|, \;\forall\; x \in A.$

$\Leftrightarrow \dfrac{\|\hat{x}\|_{\alpha}}{\|x\|} = 1, \forall x \in A, x \neq 0$

$\Leftrightarrow \inf \dfrac{\|\hat{x}\|_{\alpha}}{\|x\|} = 1$

$\Leftrightarrow s = 1$

$\Leftrightarrow r = 1$

$\Leftrightarrow \inf \dfrac{\|x^2\|}{\|x\|^2} = 1$

$\Leftrightarrow \inf \dfrac{\|x^2\|}{\|x\|^2} = 1, \forall x \in A, x \neq 0$

$\Leftrightarrow \|x^2\| = \|x\|^2, \forall x \in A$

Thus Gelfand transform is an isometry if and only if $\|\hat{x}\|_{\infty} = \|x\|, \;\forall\; x \in A.$

(b) A is semisimple and A is closed in $C(\Delta)$ iff $\exists K < \alpha$ such that $\|x\|^2 \leq k \|x^2\|$, $\forall\ x \in A$.

Proof : $\|x\|^2 \leq k \|x^2\|$

$$\Leftrightarrow \frac{\|x^2\|}{\|x\|^2} \geq \frac{1}{k} \forall x \in A$$

$$\Leftrightarrow \inf \frac{\|x^2\|}{\|x\|^2} \geq \frac{1}{k} > 0 \Leftrightarrow r > 0$$

$\therefore$ The existence of k is equivalent to $r > 0$

$\therefore\ s^2 \leq r \leq s$

$\therefore\ r > 0 \Leftrightarrow s > 0$

Now,

$$s > 0 \Leftrightarrow \inf \frac{\|\hat{x}\|_\infty}{\|x\|} > 0$$

$$\Rightarrow \frac{\|\hat{x}\|_\infty}{\|x\|} \geq k_1 (< \infty), \text{ say}$$

$$\forall\ x \in A$$

$$\Rightarrow \|\hat{x}\|_\infty \geq k_1 \|x\|,\ \forall\ x \in A$$

Let $T : A \rightarrow \hat{A}$ s.t $T(x) = \hat{x}$ be the Gelfand Transform.

Now,

$T_x = 0 \Rightarrow \hat{x} = 0$

$\Rightarrow \|\hat{x}\|_\infty = 0$

$\therefore\ k_1\|x\| \leq \|\hat{x}\|_\infty = 0$

$\Rightarrow \|x\| = 0\ [\because\ k_1 > 0]$

$\Rightarrow x = 0$

$\Rightarrow \ker T = \{0\}$

$\Rightarrow \text{rad } A = \{0\}$

$\Rightarrow$ A is semi-simple.

Also, T is one-one

Clearly, T is on-to

Now, $T^{-1} : \hat{A} \to A$ such that

$T^{-1}(\hat{x}) = x,\ \hat{x} \in \hat{A}$.

$\therefore \|T^{-1}\| (\hat{x})\| = \|x\| \leq \frac{1}{k_1} \|\hat{x}\|\ \|x\|_\infty$

$\Rightarrow \|T^{-1}\| \leq \frac{1}{k_1}$

$\therefore T^{-1}$ is bounded

$\therefore T^{-1}$ is continuous.

Let $< \hat{x}_n >$ be a Cauchy sequence in $\hat{A}$.

$\therefore \|\hat{x}_n - \hat{x}_m\| < \varepsilon\ \forall\ m,n \geq n_0$

$\therefore \|T^{-1}(\hat{x}_n) - T^{-1}(\hat{x}_m)\| \leq \|T^{-1}\|.\|\hat{x}_n - \hat{x}_m\|$

$\to 0$ as $m,n \to \infty$

$\therefore < T^{-1}(\hat{x}_n) >$ is a Cauchy sequence in

$T^{-1}(\hat{A}) = A$

and hence it must converge to $T^{-1}(\hat{x}) \in A$

$\therefore T^{-1}(\hat{x}_n) \to T^{-1}(\hat{x})$ in A.

$\Rightarrow T(T^{-1})(\hat{x}_n)) \to T(T^{-1}(\hat{x}))$ in $\hat{A}$, as T is continuous.

$\Rightarrow \hat{x}_n \to \hat{x}$ in $\hat{A}$

$\therefore \hat{A}$ is complete

$\Rightarrow \hat{A}$ is closed in $C(\Delta)$.

Lemma : Let X and Y be normed spaces and $T : X \to Y$ be linear. Then T is an open map iff $\exists$ some $\gamma > 0$ such that for every $y \in Y$, there in some $x \in X$ with $T(x) = y$ and $\|x\| \leq \gamma\ \|y\|$.

Proof : Let T be an open map. Let $S_1(0)$ denote the open sphere centred at 0 with radious '1' in X.

Since T is an open map, so, $T(S_1(0))$ is open in Y. $\because\ 0 = T(0) \in T(S_1(0))$, so $\exists$ some $\delta > 0$

such that $S_\delta(0) \subset T(S_1(0))$.

Let $y \in Y$, $y \neq 0$. Then $\frac{\delta y}{\|y\|} \in S_\delta(0) \subset T(S_1(0))$

$\therefore \exists$ some $x_1 \in S_1(0)$ such that $T(x_1) = \frac{\delta y}{\|y\|}$

Let $x = \frac{\|y\|}{\delta} x_1$. Then $T(x) = T\left(\frac{\|y\|}{\delta} x_1\right)$

$$= \frac{\|y\|}{\delta} T(x_1) = \frac{\|y\|}{\delta} \frac{\delta y}{\|y\|}$$

$$\Rightarrow T(x) = y$$

$\because x_1 \in S_1(0)$, so $\|x_1\| < 1$

$\therefore \|x\| = \|\frac{\|y\|}{\delta}\|T(x_1)\| = \frac{\|y\|}{\delta}\|x_1\| < \frac{\|y\|}{\delta}$

Taking $\gamma = \frac{1}{\delta}$, we get $\|x\| < \gamma\,\|y\|$

Conversely, suppose that for every $y \in Y$, there is some $x \in X$ with $T(x) = y$ and $\|x\| \leq \gamma\,\|y\|$, for some fixed $\gamma > 0$.

We consider an open set E in X. Let $x_0 \in E$.

Then $\exists\ \delta > 0$ such that $S_\delta(x_0) \subset E$.

Let $y \in Y$ with $\|y - T(x_0)\| < \frac{\delta}{\gamma}$ ....(1)

By, hypothesis, there is some $x \in X$ with

$T(x) = y - T(x_0)$ and $\|x\| \leq \gamma\,\|y - T(x_0)\|$ ...(2)

$\therefore \|x\| < \gamma \frac{\delta}{\gamma}$ (by (1))

$= \delta$

$\therefore \|(x + x_0) - x_0\| = \|x\| < \delta \Rightarrow x + x_0 \in S_\delta(x_0) \subset E$

$\therefore y = T(x) + T(x_0) = T(x + x_0) \in T(E)$

$\therefore S_{\frac{\delta}{\gamma}}(T(x_0)) \subset T(E)$.

$\therefore$ T(E) is open in Y.

$\Rightarrow$ T is an open map.

Next A is semisimple $\Rightarrow$ rad A = {0}

$\Rightarrow$ ker T = {0}

$\Rightarrow$ T is one-one.

Also T is onto.

$\hat{A}$ is closed $\Rightarrow$ $\hat{A}$ is complete.

By open mapping theoren, $T : A \to \hat{A}$ s.t

$T(x) = \hat{x}$ is an open map.

$\therefore$ There exists some $\gamma > 0$ s.t for each $\hat{x} \in \hat{A}$,

$\exists$ some $x \in A$ with $T(x) = \hat{x}$ and

$\|x\| \le \gamma \|\hat{x}\|_\infty$

$$\Rightarrow \frac{\|\hat{x}\|_\alpha}{\|x\|} \ge \frac{1}{\gamma} \Rightarrow \inf \frac{\|\hat{x}\|_\alpha}{\|x\|} > 0 \Rightarrow s > 0$$

Since $s^2 \le r \le s$, so $r > 0$

$$\Rightarrow \inf \frac{\|x^2\|}{\|x\|^2} > 0$$

$\therefore$ There exists some $\delta > 0$ such that

$$\frac{\|x^2\|}{\|x\|^2} \ge \delta \forall x \in A$$

$$\Rightarrow \|x^2\| \le \frac{1}{\delta}\|x^2\|, \forall x \in A,$$

$$\Rightarrow \|x^2\| \le k \|x^2\|; \; K = \frac{1}{\delta}$$

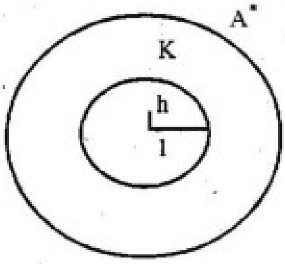
Theorem :

The maximal ideal space Δ of a commutative Banach algebra A is compect Hansdorff space.

Proof : Lat A^* be the dual space of A, regarding as a Banach space.

Let K be the norm-closed unit ball in A^*.

So, by Banach-Alaoglu's Theorem, K is w^* compect.



$\|h\| \leq 1$

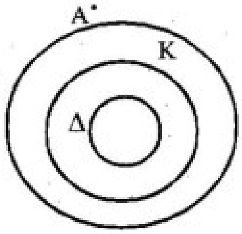
We have, for $x \in A$ with $\|x\| \leq 1$, $|\phi(x)| \leq 1$, for every complex homomorphism ϕ on A

$\therefore$ If $\phi \in \Delta$, then $\phi \in K$

$\therefore \Delta \subset K$

The Gelfand topology on Δ is the restriction to Δ of the weak *– topology on A^*.

So, we have to show that Δ is a w^* closed subset of A^*.



$\left[\because \text{K is } w^* \text{ compact and every closed subspace of a compact space is compact}\right]$

Let g_0 be in the w^* closure of Δ. We have to show that $g_0(xy) = g_0(x)g_0(y); x, y \in A$

$g_0(e) = 1$

We fix, $x \in A, y \in A$. Let $\varepsilon > 0$

Let $W = \{g \in A^* : |g(z_i) - g_0(z_i)| < \varepsilon \text{ for } 1 \leq i \leq 4\}$ (1)

where $z_1 = e$, $z_2 = x$, $z_3 = y$, $z_4 = xy$

Then W is a w^*-nbhd of g_0. So, W contains an $h \in \Delta$

For this h, $|1-g_0(e)| = |h(e)-g_0(e)| < \varepsilon$ [by (1)]

Since ε is arbitrarity small, so

$g_0(e) = 1$

Now,

$g_0(xy) - g_0(x)\, g_0(y)$

$= g_0(xy) - h(xy) + h(xy) - g_0(x)\, g_0(y)$

$= [g_0(xy) - h(xy)] + h(x)h(y) - g_0(x)\, g_0(y)$

$= [g_0(xy) - h(xy)] + [h(y) - g_0(y)]\, h(x) + [h(x) - g_0(x)]\, g_0(y)$

$\therefore |g_0(xy) - g_0(x)g_0(y)|$

$\leq |g_0(xy) - h(xy)| + |h(y) - g_0(y)|.|h(x)| + |h(x) - g_0(x)|.|g_0(y)|$

$< \varepsilon + \varepsilon.|h(x)| + \varepsilon.|g_0(y)|$ (2)

Since $h \in \Delta \subset K$, so, $|h(x)| \leq \|x\|$

$\therefore$ From (2), we get $|g_0(xy) - g_0(x)g_0(y)|$

$< \varepsilon + \varepsilon\|x\| + \varepsilon|g_0(y)|$

$= \varepsilon(1 + \|x\| + |g_0(y)|)$

Since ε is arbitrarity small, so we get

$g_0(xy) = g_0(x)g_0(y)$

ie, $g_0 \in \Delta$

Therefore, Δ is w*-closed and hence Δ is compact.

Problem :

Let X be a compect Housdorff space. C(X) is the collection of all complex valued continuous functions on X.

C(X) is a commutative Banach algebra with

$\|f\| = \sup\{|f(x)| : x \in X\}$ with unity e(x) = 1.

Fo find maximal ideal space of C(X).

Solution :

For each x $\in$ X, we consider the subset M_x of C(X), where

$M_x = \{f : f \in C(X) \text{ and } f(x) = 0\}$

To show that M_x is an ideal.

an involution on A if

1. $(x+y)^* = x^* + y^*$

2. $(\lambda x)^* = \bar{\lambda} x^*$

Clearly, $M_x \neq \varphi$, as $0 \in M_x$ $\quad (\because 0(x) = 0)$

Let, f,g $\in M_x$. Then f(x) = 0, g(x) = 0.

$\therefore (f+g)(x) = f(x) + g(x) = 0$

$\therefore f + g \in M_x$

If α is any scalar, then

$(\alpha f)(x) = \alpha f(x) = \alpha.0 = 0$

$\therefore \alpha f \in M_x$

Let $g \in C(X)$. Then

$(gf)(x) = g(x) f(x) = g(x).0 = 0$

$\therefore gf \in M_x$

$\therefore M_x$ is an ideal.

Suppose, U be an ideal in C(X) such that $M_x \subset U$

So, $\exists$ some $g \in u$ such that $g \notin Mx$

$\therefore g(x) \neq 0$. Let $g(x) = a (a \neq 0)$

Let $f = g - ae$

$\therefore f(x) = (g - ae)(x)$

$= g(x) - ae(x)$

$= a - a.1$

$= a - a$

$= 0$

$\Rightarrow f \in M_x$.

Since $M_x \subset U$. So $f \in U$

Now, $f, g \in U$

$\therefore f - g \in U$ $\quad [\because U \text{ is an ideal}]$

$\Rightarrow g - g + ae \in U$

$\Rightarrow ae \in U$

$\Rightarrow a^{-1}(ae) \in U$ $(\because a \neq 0 \text{ and } u \text{ is an ideal})$

$\Rightarrow e \in U$

$\therefore U = C(X)$

$\therefore M_x$ is a maximal ideal.

Therefore, each $x \in X$ gives rise to a maximal ideal M_x of C(X).

$\therefore \Delta(C(X)) \approx X$

Involutions :

Let A be an complex algebra. (not necessarily commulative) A mapping $x \to x^*$ of A into A is called an involution on A if

1. $(x+y)^* = x^* + y^*$
2. $(\lambda x)^* = \bar{\lambda} x^*$
3. $(xy)^* = y^* x^*$
4. $x^{**} = x$, for all $\therefore x, y \in A, \lambda \in C$

Any $x \in A$ for which $x^* = x$ is called hermition.

Example : Show that $f \to \bar{f}$ is an involution on C(X).

Solution : Here $f \to \bar{f}$, $\bar{f}(x) = \overline{f(x)}, f^* = \bar{f}$

Let $fg \in C(X)$ and $\lambda \in C$

1. $\overline{(f+g)}(x) = \overline{(f+g)(x)}$

$= \overline{f(x) + g(x)}$

$= \overline{f(x)} + \overline{g(x)}$

$= \bar{f}(x) + \bar{g}(x)$

$= (\bar{f} + \bar{g})(x) \forall x \in X$

$\Rightarrow \overline{(f+g)} = \bar{f} + \bar{g} \Rightarrow (f+g)^* = f^* + g^*$

2. $\Rightarrow \overline{(\lambda f)}(x) = \overline{(\lambda f)(x)}$

$= \bar{\lambda} \overline{f(x)}$

$= \bar{\lambda} \bar{f}(x) \ \forall x \in X$

$= (\bar{\lambda} \bar{f})(x), \forall x \in X$

$\Rightarrow (\overline{\lambda f}) = \bar{\lambda} \bar{f}$

$\Rightarrow (\lambda f)^* = \bar{\lambda} f^*$

3. $\overline{(fg)}(x) = \overline{(fg)(x)}$

$= \overline{f(x) g(x)}$

$= \overline{f(x)} \ \overline{g(x)}$

$= \overline{g(x)}\ \overline{f(x)}$

$= \overline{g}(x)\ \overline{f}(x)$

$= (\overline{g}\overline{f})(x)\ \forall x \in X$

$\Rightarrow \overline{(fg)} = \overline{g}\overline{f}$

$\Rightarrow (fg)^* = g^* f^*$

4. $\overline{\overline{f}}(x) = \overline{\overline{f}(x)}$

$= \overline{\overline{f(x)}}$

$= f(x) \forall x \in X$

$\Rightarrow \overline{\overline{f}} = f$

$\Rightarrow f^{**} = f$

Hence $f \to \overline{f}$ is an involution on C(X).

Theorem : If A is a Banach algebra with an involution and if $x \in A$, then

(a) $x + x^*, i(x - x^*)$ and xx^* are hermition.

Proof : Let A be a Banach algebra with an involution

$\therefore (x + x^*)^* = x^* + (x^*)^*$

$= x^* + x$

$= x + x^*$

and $[i(x - x^*)]^* = \overline{i}(x - x^*)^*$

$= \overline{i}(x^* - x)$

$= -i(x^* - x)$

$= i(x - x^*)$

Also, $(x.x^*)^* = (x^*)^* x^*$

$= xx^*$

$\therefore x + x^*, i(x - x^*)$ and xx^* are hermition.

(b) x has a unique representation $x = u + iv$, with $u \in A, v \in A$ and both u and v are hermition.

Proof : Let, $u = \frac{1}{2}(x + x^*)$

$v = \frac{1}{2}\left(x^* - x\right)$

$\therefore u^* = \frac{1}{2}\left(x^* + x^{**}\right) = \frac{1}{2}\left(x + x^*\right) = u$

$v^* = -\frac{1}{2}\left(x^{**} - x^*\right)$

$= -\frac{1}{2}\left(x^* - x\right) = v$

$\therefore u$ and v are hermition.

Now, $u + iv = \frac{1}{2}\left(x + x^* - x^* + x\right)$

$\Rightarrow u + iv = x$

Suppose x has another representation $u' + iv'$, where u' and v' are hermition.

Let $w = v' - v$

Then $w^* = v'^* - v^* = v' - v = w$

and $u + iv = u' = iv'$

$\Rightarrow \left(u - u'\right) = i\left(v' - v\right) = iw$ (1)

$\Rightarrow \left(iw\right)^* = \left(u - u'\right)^* = u^* - u'^*$

$= u - u'$

$= iw$

$\therefore$ Both w and iw are hermition.

$\therefore iw = \left(iw\right)^* = -iw^* = -iw$

$\Rightarrow w = 0$

$\Rightarrow v' - v = 0$

$\Rightarrow v' = v$

From (1), we get $\left(u - u'\right) = 0$

$\Rightarrow u' = u$

$\therefore$ The representation $x = u + iv$ is unique.

(c) The unit e is hermition.

Proof : Since $e^* = ee^*$, so e^* is hermition (by (a))

Then $e^* = (e^*)^* = e^{**} = e$

$\therefore$ e is hermition.

(d) x is invertible in A iff x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$

Proof : Let x be invertible. Then

$xx^{-1} = x^{-1}x = e$

$\therefore (xx^{-1})^* = e^*$

$\Rightarrow (x^{-1})^*x^* = e$

and $(x^{-1})^* = e^*$

$\Rightarrow x^* (x^{-1})^* = e$

$\therefore x^*$ is invertible and $(x^*)^{-1} = (x^{-1})^*$

Conversely, let x^* be invertible. Then

$x^*(x^{-1}) = (x^{-1})^*x^* = e$

$\therefore (x^*(x^{-1})^*)^* = e^*$

$\Rightarrow (x^{-1})^{**}x^{**} = e \Rightarrow x^{-1}.x = e$

and $((x^{-1})^*x^*)^* = e^*$

$\Rightarrow x^{**} (x^{-1})^{**} = e \Rightarrow xx^{-1} = e$

Thus $xx^{-1} = x^{-1} = e$

$\Rightarrow$ x is invertible.

(e) $\lambda \in \sigma(x)$ iff $\overline{\lambda} \in \sigma(x^*)$

Proof : We have, $\lambda e - x$ is invertible iff $\overline{\lambda} e - x^*$ is intertible

ie, $\lambda e - x$ is invertible iff $\overline{\lambda} e - x^*$ is intertible

$\Rightarrow \lambda \in \sigma(x) \Leftrightarrow \overline{\lambda} \in \sigma(x^*)$

Theorem : If a Banach algebra A is commulative and semisimple, then every involution on A is continuous.

Proof : Let h be a complex homomorphism on A

we define $\phi : A \to C$ by $\phi(x) = h(x^*)$

Then ϕ is also a complex homomorphism.

Now, if $\|x\| \leq 1$, then $|\phi(x)| \leq 1$

$\Rightarrow \phi$ is bounded

$\Rightarrow \phi$ is continuous.

Let $xn \to x$ and $xn^* \to y$ in A.

Now, $h(x^*) = \phi(x) = \phi(\lim xn)$

$= \lim \phi(xn)$

$= \lim h\left(x_n^*\right)$

$= h\left(\lim x_n^*\right)$

$= h(y)$

$\Rightarrow h\left(x^* - y\right) = 0$

$\Rightarrow x^* - y \in \ker h$

Since h is arbitrary, $x^* - y \in \ker h \forall h \in \Delta$

$\Rightarrow x^* - y \in \bigcap_{h \in \Delta} \ker h$

$\therefore x^* - y \in rad$ A

But rad A = $\{0\}$, as A is semisimple

$\therefore x^* - y = 0$

$\Rightarrow x^* = y$

$\Rightarrow y = x^*$

$\therefore$ By closed graph Theorem, the mapping $x \to x^*$ is continuous.

Unit 6
B*-Algebra and Its Properties

6.1 : B*- Algebra

A Banach algebra 'A' with an involution $x \to x^*$ that satisfies $\|xx^*\| = \|x\|^2 \ \forall x \in A$ is called a B*-algebra.

Here, $\|xx^*\| \le \|x\|.\|x\|$

$\Rightarrow \|x\|^2 \le \|x\|.\|x^*\|$

$\Rightarrow \|x\| \le \|x^*\| (\|x\| \ne 0), \forall x \in A$

Replacing x by x*, we get

$\|x^*\| \le \|(x^*)^*\| = \|x\|$

$\Rightarrow \|x^*\| \le \|x\|$

$\therefore \|x\| = \|x^*\|$

Theorem (Gelfand-Naimark) :

Let A be a commutative B*-algebra with maximal ideal space ...., The Gelfand transform is then an isometric isomorphism of A onto C(.....), which has the additional property that

$h(x^*) = \overline{h(x)} (x \in A, h \in \Delta)$ (1)

or, equivalenthy, that

$(x^*)^\wedge = \overline{\hat{x}}, (x \in A)$ (2)

In particular, x is hermition iff $\hat{x}$ is a real valued function. (3)

Proof : Let $u \in A, u = u^*$ *and* $h \in \Delta$. We prove that h(u) is real.

Let z = u + ite, for real 't'

If $h(u) = \alpha + i\beta$, where α and β are real,

then

$h(z) = h(u + ite)$

$= h(u) + i.t.h(e)$

$= \alpha + i\beta + it$

$= \alpha + i(\beta + t)$

$\therefore 33^* = (u + ite)(u - ite)$

$= u^2 + t^2 e$

$\therefore \alpha^2 + (\beta + t)^2 = |h(3)|^2$

$\leq \|3\|^2 = \|33^*\| \qquad [\because \text{A is } B^* \ algebra]$

$= \|(u^2 + t^2 e)\|$

$\leq \|u^2\| + t^2$

$\leq \|u\|^2 + t^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\beta t + t^2 \leq \|u\|^2 + t^2$

$\Rightarrow \alpha^2 + \beta^2 + 2\beta t \leq \|u\|^2, \ -\infty < t < \infty$

Since, this holds for all real values of t, so

$2\beta = 0$

$\Rightarrow \beta = 0$

$\therefore h(u) = \alpha$ is real.

Every element $x \in A$ can be expressed uniquely in the form $x = u + iv$, where $u = u^*, v = v^*$

$\therefore h(u)$ and $h(v)$ are real.

$\therefore h(x) = h(u + iv) = h(u) + ih(v)$

Then $\overline{h(x)} = h(u) - ih(v)$

$\therefore h(x^*) = h(u^* - iv^*)$

$= h(u - iv)$

$= h(u) - ih(v)$

$= \overline{h(x)}$

Hence (1) is proved.

Again, $(x^*)^\wedge(h) = h(x^*), \forall h \in \Delta$

$= \overline{h(x)}, \forall h \in \Delta$

$= \overline{\hat{x}}(h), \forall h \in \Delta$

$\Rightarrow (x^*)^\wedge = \overline{\hat{x}}$

$\therefore$ (2) is proved.

$\therefore \hat{A}$ is closed under complex conjugation.

Stone Weierstrass Theorem for complex functions :

Let L be locally compact Housforff space. Suppose A is a complex subalgebra of C(L) which is colsed under complex conjugation and strongly separates the points of L.

Then $\overline{A} = C(L)$

Applying Stone-Weierstrass Theorem, $\hat{A}$ is dense in $C(\Delta)$ ie. $\overline{\hat{A}} = C(\Delta)$

If $x \in A$ and $y = xx^*$, then

$y^* = (xx^*)^* = xx^* = y$

$\therefore \|y^2\| = \|yy^*\| = \|y\|^2$ (3)

We assume that

$\|y^{2^n}\| = \|y\|^{2^n}$

$\therefore \|y^{2^{n+1}}\| = \|y^{2^n}\|^2$

$= \|y^{2^n}\|^2$ $\left[\text{Replacing y in (3) by } y^{2^n}\right]$

$= \left(\|y^{2^n}\|\right)^2$

$= \|y\|^{2^{n+1}}$

Applying induction on n, we get

$= \|y^m\| = \|y\|^m$, where $m = 2^n, n = 1, 2, ...$

Taking mth root, as $m \to \infty$

$\lim_{x \to \infty} \|y^m\|^{1/2} = \|y\|$

$\Rightarrow P(y) = \|y\|$

$\Rightarrow \|\hat{y}\|_\infty = \|y\|$

Since $y = xx^*$

$\therefore \|\hat{y}\|_\infty = \max\left\{|\hat{y}(h)| : h \in \Delta\right\}$

$= \max\left\{|h(y)| : h \in \Delta\right\}$

$= \max\left\{|h(xx^*)| : h \in \Delta\right\}$

$= \max\left\{|h(x)h(x^*)| : h \in \Delta\right\}$

$= \max\left\{|h(x)\overline{h(x)}| : h \in \Delta\right\}$

$= \max\left\{|h(x)|^2 : h \in \Delta\right\}$

$= \left[\max\left\{|h(x)| : h \in \Delta\right\}\right]^2$

$= \left[\max\left\{|\hat{x}(h)| : h \in \Delta\right\}\right]^2$

$= \|\hat{x}\|_\infty^2$

$\therefore \|\hat{x}\|_\infty^2 = \|\hat{y}\|_\infty$

$= \|y\| = \|xx^*\| = \|x\|^2$

$\Rightarrow \|\hat{x}\|_\infty^2 = \|x\|^2$

$\Rightarrow \|\hat{x}\|_\infty = \|x\|$

$\therefore x \to \hat{x}$ is an isometry.

$\therefore$ As, 'A' is complete, So $\hat{A}$ is complete.

$\therefore \hat{A}$ must be closed i.e. $\overline{\hat{A}} = \hat{A}$

$\Rightarrow \hat{A} = C(\Delta)$

Therefore the Gelfond trandform is an isometric isomorphism A onto $C(\Delta)$

Theorem : If A is a commulative B^*-algebra which contains an element x such that the polynomials in x and x^* are dense in A, then the formula $(\psi f)^\wedge = fo\,\hat{x}$ (1)

Defines an isometric isomorphism ψ of $C(\sigma(x))$ onto A.

Moreover, if $f(\lambda) = \lambda$ on $\sigma(x)$, then $\psi f = x$.

Proof : Let Δ be the maximal ideal space of A. We know that $\hat{x}$ is a continuous function on Δ whose range is $\sigma(x)$.

Let, $h_1, h_2 \in \Delta$ and $\hat{x}(h_1) = \hat{x}(h_2)$

$\Rightarrow h_1(x) = h_2(x)$

Then, $h_1(x^*) = \overline{h_1(x)} = \overline{h_2(x)} = h_2(x^*)$

If P is any polynomial with two variables, then it follows that

$h_1(P(x, x^*)) = h_2(P(x, x^*))$, as

h_1 and h_2 are homomorphisms and

$h_1(x) = h_2(x); h_1(x^*) = h_2(x^*)$ (2)

By hypothesis, elements of the form $P(x,x^*)$ are dense in A.

Let, $y \in A$. Then $y = \lim_n Pyn(x,x^*)$

$\therefore h_1(y) = h_1\left(\lim_n Pyn(x,x^*)\right)$

$= \lim_n h_1\left(Pyn(x,x^*)\right)$ $(\because h_1$ is continuous)

$= \lim_n h_2\left(Pyn(x^*,x)\right)$, by (2)

$= h_2\left(\lim_n Pyn(x^*,x)\right)$, $(\because h_2$ is continous)

$= h_2(y)$

Thus, $h_1(y) = h_2(y) \forall y \in A$

$\Rightarrow h_1 = h_2$

$\therefore \hat{x}(h_1) = \hat{x}(h_2)$

$\Rightarrow h_1 = h_2$

$\therefore \hat{x}$ is one to one.

Theorem : Let X be a compect space and Y be a Hausdorff space. Then every bizective continuous mapping of X onto Y is a homeomorphism.

Since $\sigma(x)$ is compact and Δ is Hausdorff, so $\hat{x} : \Delta \to \sigma(x)$ is a homeomorphism.

We define, $\phi : C(\sigma(x)) \to C(\Delta)$ by $\phi(f) = fo\hat{x}$

Then ϕ is one-one and onto.

Let, $f_1, f_2 \in C(\sigma(x))$

Then $\phi(f_1, f_2) = f_1, f_2 o\hat{x}$

Now, $((f_1, f_2).\hat{x})(h) = (f_1, f_2)(\hat{x}(h))$

$= (f_1, f_2)(h(x))$

$= f_1(h(x)) f_2(h(x))$

$= f_1(\hat{x}(h)) f_2(\hat{x}(h))$

$= (f_1.\hat{x})(h)(f_2.\hat{x})(h)$

$= ((f_1.\hat{x})(f_2.\hat{x}))(h) \forall h \in \Delta$

$\Rightarrow (f_1 f_2).\hat{x} = (f_1.\hat{x})(f_2.\hat{x})$

$\Rightarrow \phi(f_1 f_2) = \phi(f_1)\phi(f_2)$

$\therefore \phi$ is a homomorphism.

Now, $\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in \sigma(x), |\lambda| \le 1\}$

$= \sup\{|f(h(x))| : h \in \Delta, |h(x)| \le 1\}$

$= \sup\{|f(\hat{x}(h))| : \|h\| \le 1, h \in \Delta\}$

$= \sup\{|fo\hat{x}(h)| : \|h\| \le 1, h \in \Delta\}$

$= \|(fo\hat{x})\|$

$= \|\phi(f)\|$

$\therefore \phi$ is an isometric isomorphism of $C(\sigma(x))$ onto $C(\Delta)$.

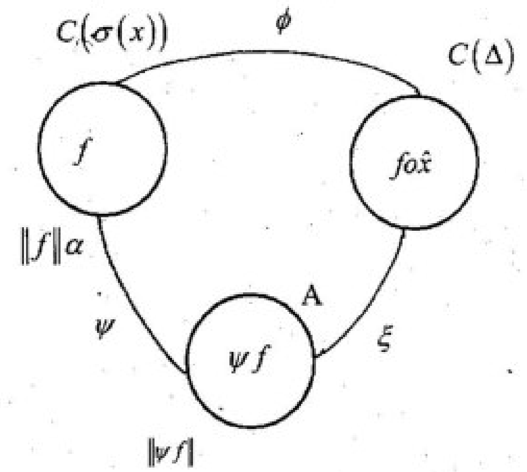
We know that, the Gelfand transform is an isometric isomorphism of A onto $C(\Delta)$

$\therefore \phi(f) = fo\hat{x}$ is the Gelfand transform of a unique element of A, which we denote by ψf.

$\therefore (\psi f)^\wedge\ fo\hat{x}$

and $\|\psi f\| = \|fo\hat{x}\| = \|f\|_\infty$

$\therefore$ (1) defines an isometric isomorphism ψ of $C(\sigma(x))$ onto A.



Let $f(\lambda) = \lambda$ on $\sigma(x)$

$\therefore (fo\hat{x})(h) = f(\hat{x}(h))$

$= f(h(x))$

$= h(x) = \hat{x}(h) \forall h \in \Delta$

$\Rightarrow fo\hat{x} = \hat{x}$

$\Rightarrow (\psi f)^{\wedge} = \hat{x} \quad \Rightarrow \psi$f=x

Centralizers :

Let S be a subset of a Banach algebra A. The centralizer of S is the set

$T(S) = \{x \in A : xs = sx$ for every s $\in$ S$\}$

S commutes if any two elements of S commute with each other.

Theorem : (a) T(S) is a closed subalgebra of A.

Proof: Clearly, $T(S) \neq \phi$, as e $\in$ T(S)

Let $x, y \in$ T(S)

$\Rightarrow xs = sx$

and $ys = sy, \forall s \in S$

$(x+y)s = xs + ys = sx + sy$

$= s(x+y)$

$(\lambda x)s = \lambda(xs)$

$= \lambda(sx)$

$= S(\lambda x)$

$(xy)s = x(ys) = x(sy) = (xs)y$

$= (sx)y$

$= s(xy)$

$\therefore x+y, \lambda x, xy \in \tau(S)$

$\therefore \tau(S)$ is a subalgebra of A.

Let $< x_n >$ be a sequence in $\tau(S)$ such that $x_n \to x$.

Since, multiplication is continuous in A, so

$x_n s \to xs$ and

$s_x n \to sx$

$\because x_n \in \tau(S)$, so for any $s \in S$,

$x_n s = s_x n \forall n$

$\Rightarrow \lim_{n\to\infty} x_n s = \lim_{n\to\infty} s x_n$

$\Rightarrow xs = sx$

$\Rightarrow x \in \tau(S)$

$\therefore \tau(S)$ is a closed subalgebra of A.

(b) $S \subset T(\Gamma(S))$

Proof : Let $x \in T(S)$. Then $xs = sx \forall s \in S$

$\therefore$ *For every*, $x \in T(S)$,

$sx = xs$

$\Rightarrow s \in T(\Gamma(S))$

$\therefore s \in S \Rightarrow s \in T(\Gamma(S))$

$\therefore S \subset T(\Gamma(S))$

(c) If S commutes, then $T(\Gamma(S))$ commutes.

Proof : Since S commutes, so $S \subset \Gamma(S)$

$\therefore T(S) \subset T(\Gamma(S))$

If $T(E) \subset E$, then T(E) commutes.

$\therefore T(\Gamma(S))$ commutes.

Theorem : Suppose A is a Banach algebra, $S \subset A, S$ commutes and $B = T(\Gamma(S))$.

Then B is a commutative Banach algebra, $S \subset B$, and $\sigma_B(x) = \sigma_A(x)$, for every $x \in B$.

Proof : We have $B = T(\Gamma(S))$ is a closed cubalgebra of A. [by (a)]
Since S commutes, so B also commutes [by (c)].
Therefore, B is a commutative Banach algebra containing S.

Let $\lambda \in \sigma_A(x) \Rightarrow \lambda e - x$ is not invertible in A.

$\Rightarrow \lambda e - x$ is not invertible in B $[\because B \subset A]$

$\Rightarrow \lambda \in \sigma_B(x)$

$\therefore \sigma_A(x) \subseteq \sigma_B(x)$

Let $x \in B$ and x is invertible in A.

$\therefore xy = yx$ for every $y \in T(S)$.

$\Rightarrow x^{-1}(xy) = x^{-1}(yx)$, for all $y \in \Gamma(S)$

$\Rightarrow y = x^{-1}yx, \forall y \in \Gamma(S)$

$\Rightarrow yx^{-1} = x^{-1}y, \forall y \in \Gamma(S)$

$\Rightarrow x^{-1} \in \Gamma(\Gamma(S)) = B$

$\therefore x$ is invertible in A

$\Rightarrow x$ is invertible in B

$\therefore x$ is not invertible in B

$\Rightarrow x$ is not invertible in A

$\therefore \lambda \in \sigma_B(x)$

$\Rightarrow \lambda e - x$ is not invertible in B.

$\Rightarrow \lambda e - x$ is not invertible in A.

$\Rightarrow \lambda \in \sigma_A(x)$

$\therefore \sigma_B(x) \subseteq \sigma_A(x)$

$\therefore \sigma_B(x) \subseteq \sigma_A(x) \forall x \in B$.

Theorem : Suppose A is a Banach algebra, $x \in A, y \in A$ and $xy = yx$

Then $\sigma(x+y) \subset \sigma(x) + \sigma(y)$

and $\sigma(xy) \subset \sigma(x) + \sigma(y)$

Proof : Let $S = \{x, y\}, B = T(\Gamma(S))$

Then $S \subset B$

$\therefore x + y, xy \in B \quad [\because \text{B is subalgebra}]$

Then $\sigma_B(z) = \sigma_A(z), \ \forall z \in B$ so, we have to prove that—

$\sigma_B(x+y) \subset \sigma_B(x) + \sigma_B(y)$

$\sigma_B(xy) \subset \sigma_B(x) \ \sigma_B(y)$

Since B is commulative, so

$\sigma_B(z)$ is the range of the Gelfand transform $\hat{z}$, for every $z \in B$.

Again, $(x+y)^\wedge = \hat{x} + \hat{y}$

$\therefore Range(x+y)^\wedge = Range \ (\hat{x} + \hat{y})$

$\subset Range \ \hat{x} + Range \ \hat{y}$

$\Rightarrow \sigma_B(x+y) \subset \sigma_B(x) + \sigma_B(y)$

$\Rightarrow \sigma(x+y) \subset \sigma(x) + \sigma(y)$

Also, $(xy)^\wedge = \hat{x}\hat{y}$

$\Rightarrow Range(xy)^{\wedge} = Range\ (\hat{x}\hat{y})$

$\subset Range\ \hat{x}.\ Range\ \hat{y}$

$\Rightarrow \sigma_B(x+y) \subset \sigma_B(x) + \sigma_B(y)$

$\Rightarrow \sigma(x+y) \subset \sigma(x) + \sigma(y)$

Normal element :

Let A be algebra with an involution. If $x \in A$ and $x^* x = x^* x$, then x is said to be normal.

A set $S \subset A$ is called normal if S commules and $x^* \in S$.

Theorem : Suppose A is a Banach algebra with an involution, and B is a normal subset of A, that is maximal w.r.t being normal.

Then

(a) B is a closed commutative subalgebra of A.

(b) $\sigma_B(x) = \sigma_A(x)\ \forall x \in B$

Proof : (a) Since $B \subset A$ is normal, so, B commules and $x^* \in B$ whenever $x \in B$.

Let $x \in A$ such that (i) $xx^* = x^* x$

(ii) $xy = yx, \forall y \in B$

Since $y \in B$, so $y^* \in B$. Hence by (ii),

$xy^* = y^* x, \forall y \in B$

$\Rightarrow (xy^*)^* = (y^* x)^*$

$\Rightarrow y^* x = xy^*, \forall y \in B$

$\therefore B \cup \{x, x^*\}$ is normal.

But, B is maximal w.r.t being normal.

$\therefore B \cup \{x, x^*\} = B$

i.e, $x \in B$

Let $x, y \in B$. Then $x^*, y^* \in B$

$\therefore (x+y)(x+y)^* = (x+y)(x^* + y^*)$

$= xx^* + xy^* + yx^* + yy^*$

$= x^* x + y^* x + x^* y + y^* y \qquad [\because B\ commutes]$

$= x^*(x+y) + y^*(x+y)$

$= (x^* + y^*)(x+y)$

$= (x+y)^*(x+y)$

For $z \in B(x+y)z = xz + yz$

$= zx + zy$

$= z(x+y)$

Therefore $x+y$ satisfies both (i) and (ii)

$\Rightarrow x + y \in B$

Similarly, $xy \in B$ and $\lambda x \in B, \lambda$ is a scalar.

$\therefore B$ is a commutative subalgebra of A.

Let $< x_n >$ be a sequence in B such that

$x_n \to x$

$\therefore$ Multiplication is continuous, so

$x_n y \to xy$

$yx_n \to yx, y \in B$

But, $x_n y = yx_n \forall n$

$\Rightarrow \lim_{n\to\infty} x_n y = \lim_{n\to\infty} yx_n$

$\Rightarrow xy = yx, \forall y \in B$

$\therefore$ (ii) is satisfied.

Now, $x^* y = (y^* x)^*$

$= (xy^*)^*$

$= yx^* \forall y \in B$

In particular, $x^* x_n = x_n x^* \forall n$

$\Rightarrow \lim_{n\to\infty} x^* x_n = \lim_{n\to\infty} x_n x^*$

$\Rightarrow x^* x = xx^*$

$\therefore$ (i) is satisfied.

$\therefore x \in B$

$\Rightarrow B$ is closed.

(b) Let $\lambda \in \sigma_A(A) \Rightarrow \lambda e - x$ is not invertible in A.

$\Rightarrow \lambda e - x$ is not invertible in B $(\because B \subseteq A)$

$\therefore \sigma_A(x) \subseteq \sigma_B(x)$

Let $x \in B$ and $x^{-1} \in A$

$\Rightarrow x^{-1}(xy)x^{-1} = x^{-1}(yx)x^{-1}$

$\Rightarrow yx^{-1} = x^{-1}y, \forall y \in B$

$\therefore xx^* = x^* x$

$\Rightarrow (xx^*)^{-1} = (x^*x)^{-1}$

$\Rightarrow (x^*)^{-1} x^{-1} = x^{-1} (x^*x)^{-1}$

$\Rightarrow (x^{-1})^* x^{-1} = x^{-1} (x^{-1})^* \quad \left[\because (x^*)^{-1} = (x^{-1})^* \right].$

$\therefore x^{-1}$ satisfies conditios (i) and (ii) of (a).

$\therefore x^{-1} \in B$

$\therefore x$ is invertible in A $\Rightarrow x$ is invertible in B

$\Rightarrow x$ is not invertible in B $\Rightarrow x$ is not invertible in A.

$\therefore \lambda \in \sigma_B(x)$

$\Rightarrow \lambda e - x$ is not invertible in B.

$\Rightarrow \lambda e - x$ is not invertible in A.

$\Rightarrow \lambda \in \sigma_A(x)$

$\therefore \sigma_B(x) \subseteq \sigma_A(x)$

$\therefore \sigma_B(x) = \sigma_A(x) \forall x \in B$

Positive element :

Let A be an involutive Banach algebra and $x \in A$

"$x \geq 0$" means $x = x^*$

and $\sigma(x) \subset [0, \infty)$.

Theorem : Let A be a b^*-algebra. Then

(a) Hermition elements have real spectra.

Proof : Let $x \in A$ such that $x = x^*$

$\therefore xx^* = xx = x^*x$

$\Rightarrow x$ is normal

$\therefore x$ is contained in some maximal normal set say B in A.

Then, B is a commulative B*-algebra. So, it is isometrically isomorphic ti ils Gelfand transform $\hat{B} = C(\Delta)$, where Δ is the maximal ideal space of B.

For $z \in B, \hat{z}(\Delta) = \sigma(z)$

If $x = x^*$, then $\hat{x}$ is real valued

$\therefore \sigma(x) = \hat{x}(\Delta)$ is real.

(b) If $x \in A$ is normal, then $P(x) = \|x\|$

Proof : We know that $\sigma(x) = \hat{x}(\Delta)$

$$P(x) = \|\hat{x}\|_\infty$$

$\because B$ *and* $\hat{B}$ are isometric, so $\|\hat{x}\|_{\infty} = \|x\|$

$$\therefore P(x) = \|x\|$$

(c) If $y \in A$, then $P(yy^*) = \|y\|^2$

Proof : Let $z = yy^*, y \in A$
Then z is hermitian.

$\therefore zz^* = zz = z^*z$
$\Rightarrow z$ is normal.

$\therefore$ By (b), $P(z) = \|z\|$

$\Rightarrow P(yy^*) = \|yy^*\| = \|y\|^2$

(d) If $u \in A, v \in A, u \geq 0, v \geq 0,$ then $u + v \geq 0$

Proof : Let $\alpha = \|u\|, \beta = \|v\|$

and $w = u + v, r = \alpha + \beta$

Now, $u \geq 0 \Rightarrow u = u^*$ and $\sigma(u) \subset [0, \infty)$

$\therefore u$ is normal as $uu^* = u^*u$

By (b), $P(u) = \|\hat{u}\|_{\infty} = \|u\| = \alpha$

$\Rightarrow \sup\{|\lambda| : \lambda \in (u)\} = x$

$\therefore \sigma(u) \subset [0, \alpha]$

Now, $\sigma(\alpha e - u) \subset \alpha\sigma(e) - \sigma(u)$

$= \sigma - \sigma(u)$

$\subset [0, \alpha]$

$\therefore \sigma(\alpha e - u) \subset [0, \alpha]$

Again, $(\alpha e - u)(\alpha e - u)^* = (\alpha e - u)(\alpha e - u^*)$

$= (\alpha e - u)(\alpha e - u)$

$= (\alpha e - u)^*(\alpha e - u)$

$\Rightarrow (\alpha e - u)$ is normal

By (b), $P(\alpha e - u) = \|\alpha e - u\|$

$\Rightarrow \|\alpha e - u\| \leq \alpha$ [by (1)] (2)

Similarly, we have

$\|\beta e - v\| \le \beta$ (3)

$\|\gamma e - v\| \le \gamma$ (4)

Now, $w^* = (u+v)^* = u + v = w$

$(\gamma e - w)^* = \gamma e - w^* = \gamma e - w$

$\therefore (\gamma e - w)$ has real spectraem.

$\therefore (4) \Rightarrow \sigma(\gamma e - w) \subset [-\gamma, \gamma]$

$\Rightarrow \sigma(w) \subset [-\gamma + \gamma, \gamma + \gamma] = [0, 2\gamma] \subset [0, \infty]$

Thus $w^* = w$ and $\sigma(w) \subset [0, \infty]$

$\therefore w \ge 0$

$\Rightarrow u + v \ge 0$

(e) If $y \in A$, then $yy^* \ge 0$

Proof : Let $x = yy^*$. Then x is hermition.

Let $B \subset A$ be the maximal normal set containing x.

Then B is a commulative B^*-algebra and B is isometrically isomorphic to $\hat{B} = C(\Delta)$ since x is hermition, so $\hat{x}$ is a real valued function. We have to show that $\hat{x} \ge 0$ on Δ

Since $|\hat{x}| - \hat{x} \in \hat{B}$, so $\exists z \in B$ such that

$\hat{z} = |\hat{x}| - \hat{x} \; on \; \Delta$ (1)

Since $\hat{z}$ is real, so $z = z^*$

Let $zy = w = u + iv$ (2), where u and v are hermition elements of A.

$\therefore ww^* = (zy)(zy)^* = zyy^*z^*$

$= z(yy^*)z^*$

$= zxz^*$

$= zzx$ $[\because$ z,x $\in$ B and B is commulative$]$

$= z^2x$

$\therefore ww^* + w^*w = (u+iv)(u-iv) + (u-iv)(u+iv)$

$= u^2 + v^2 + u^2 + v^2$

$= 2u^2 + 2v^2$

$\Rightarrow w^*w = 2u^2 + 2v^2 - w^*w$

$= 2u^2 + 2v^2 - z^2x$ (3)

Since, $u = u^*$, so $\hat{u}$ is real, i.e, $\sigma(u)$ is real.

So, by spectral mapping Theorem,

$\sigma(u^2) \subset [0,\infty)$

Similarly, $\sigma(u^2) \subset [0,\infty)$

$\therefore u^2 \geq 0, v^2 \geq 0$

If possible, let $\hat{x} < 0$

Then $\hat{z}^2 \hat{x} < 0$

$\hat{z}^2 x \in B$

$\therefore \sigma(z^2 x) = (\hat{z}^2 x)(\Delta) \subset (-\infty, 0]$

$\therefore \sigma(-z^2 x) \subset (-\infty, 0]$

$\therefore -z^2 x \geq 0$ (4)

$\therefore (3) \Rightarrow w^* w \geq 0$ [using (d)]

$\therefore \sigma(w^* w) \subset [0,\infty)$

$\therefore \sigma(w^* w) \subset \sigma(w^* w) \cup \{0\}$

$\therefore \sigma(w^* w) \subset [0,\infty)$

$\Rightarrow w^* w \geq 0$

$\Rightarrow z^2 x \geq 0$, a contradiction to (4)

$\therefore$ our assmption is wrong.

$\therefore \hat{x} \geq 0$ on Δ

$\therefore \sigma(x) = \hat{x}(\Delta) \subset [0,\infty)$

$\therefore \sigma(yy^*) \subset [0,\infty)$

$\therefore yy^* \geq 0$

6.2 Positive functional :

A positive functional is a linear functional F on a Banach algebra A with an involution which satisfies

$F(xx^*) \geq 0$

$\forall x \in A$

Theorem : Every positive functional F on a Banach algebra A with involution has the following properties :

(a) $F(x^*) = \overline{F(x)}$

Proof : Let $x, y \in A$. Let $p = F(xx^*)$, $q = F(yy^*)$

$$r = F(xy^*), \quad s = F(yx^*)$$

Let, $\alpha \in C$. Then $F\left[(x+\alpha y)(x+\alpha y)^*\right] \geq 0$

$$\Rightarrow F\left[(x+\alpha y)(x^* + \bar{\alpha}^* y)\right] \geq 0$$

$$\Rightarrow F\left(xx^* + \bar{\alpha} xy^* + \alpha yx^* + |\alpha|^2 yy^*\right) \geq 0$$

$$\Rightarrow F(xx^*) + \bar{\alpha} F(xy^*) + \alpha F(yx^*) + |\alpha|^2 F(yy^*) \geq 0$$

$$\Rightarrow p + \bar{\alpha} r + \alpha s + |\alpha|^2 q \geq 0 \qquad (1)$$

Putting, $\alpha = 1$, $(1) \Rightarrow p + r + s + q \geq 0$

$\alpha = 1$, $(1) \Rightarrow p + r + s + q \geq 0$

$$\Rightarrow (p+q) + (r+s) \geq 0 \qquad (2)$$

Putting, $\alpha = i$, $(1) \Rightarrow p - ir + is + q \geq 0$

$$\Rightarrow p + q + i(s-r) \geq 0 \qquad (3)$$

Now, (2) and (3) $\Rightarrow (s+r)$ and $i(s-r)$ are real.

Let, $s + r = a$

$s - r = ib$

$$\therefore 2s = a + ib \Rightarrow s = \frac{1}{2}(a+ib)$$

$$\therefore 2r = a - ib \Rightarrow r = \frac{1}{2}(a-ib) = \bar{s}$$

$$\Rightarrow F(xy^*) = \overline{F(yx^*)}$$

Let $y = e$. Then $F(xe^*) = \overline{F(ex^*)}$

$$\Rightarrow F(x) = \overline{F(x^*)}$$

$$\Rightarrow F(x^*) = \overline{F(x)}, \forall x \in A$$

(b) $\left|F(xy^*)\right|^2 \leq F(xx^*) F(yy^*)$

Proof : Let $p = F(xx^*), q = F(yy^*)$

$r = F(xy^*), s = F(yx^*)$

If $r = 0$, then the result is true (b)

Let $r \neq 0$, Let, $\alpha = \frac{tr}{|r|}$, where $t \in R$

$(1) \Rightarrow p + \bar{\alpha} r + \alpha s + |\alpha|^2 q \geq 0$

$\Rightarrow p + \frac{t\bar{r}}{|r|} r + \frac{tr}{|r|} s + \frac{t^2 |r|^2}{|r|^2} q \geq 0 \quad \left[when\ \alpha = \frac{tr}{|r|} \right]$

$\Rightarrow p + \frac{t\bar{r}}{|r|} r + \frac{tr}{|r|} \bar{r} + \frac{t^2 |r|^2}{|r|^2} q \geq 0 \quad [\because s = \bar{r}]$

$\Rightarrow p + 2t \frac{r\bar{r}}{|r|} + t^2 q \geq 0$

$\Rightarrow p + 2t \frac{|r|^2}{|r|} + t^2 q \geq 0$

$\Rightarrow p + 2|r| t + t^2 q \geq 0 \ (-\infty < t < \infty)$

Putting, $t = -\frac{|r|}{q}$, we get

$p + 2|r| \left(-\frac{|r|}{q} \right) + q . \frac{|r|^2}{q^2} \geq 0$

$\Rightarrow p - 2\frac{|r|^2}{q} + \frac{|r|^2}{q} \geq 0$

$\Rightarrow p - 2\frac{|r|^2}{q} + \frac{|r|^2}{q} \geq 0$

$\Rightarrow p - \frac{|r|^2}{q} \geq 0$

$\Rightarrow pq - |r|^2 \geq 0$

$\Rightarrow |r|^2 \leq pq$

$\Rightarrow |F(xy^*)|^2 \leq F(xx^*) F(yy^*)$

(c) $|F(x)|^2 \leq F(e) F(xx^*) \leq F(e)^2 \int(xx^*)$

Proof : We have, $F(ee^*) = F(e)$

By (b), when $y = e$,

$$\left|F(xe^*)\right|^2 \le F(xx^*)F(ee^*)$$

$$\Rightarrow \left|F(x)\right|^2 \le F(e)F(xx^*)$$

Let $t \in IR$ be such that $t \ge \int(xx^*)$

Then, $\sigma(te - xx^*) \subseteq t\sigma(e) - \sigma(xx^*)$

$$= t - \sigma(xx^*) \subset [0, \infty)$$

$\therefore$ There exists $u \in A$ with $u = u^*$ and $u^2 = te - xx^*$

$$\therefore F(te - xx^*) = F(u^2) \ge 0$$

$$\Rightarrow tF(e) - F(xx^*) \ge 0$$

$$\Rightarrow F(xx^*) \ge tF(e)$$

Substituting, $t = P(xx^*)$, we get

$$F(xx^*) \le P(xx^*)F(e)$$

$$\Rightarrow F(e)F(xx^*) \le F(e)^2 P(xx^*)$$

$$\therefore \left|F(x)\right|^2 \le F(e)F(xx^*)$$

$$\le F(e)^2 P(xx^*)$$

$$\therefore \left|F(x)\right|^2 \le F(e)F(xx^*) \le F(e)^2 P(xx^*)$$

(d) $\left|F(x)\right| \le F(e)P(x)$, for every normal element $x \in A$

Proof : Since x is normal, so $xx^* = x^*x$

$$\therefore \sigma(xx^*) \subset \sigma(x)\sigma(x^*)$$

$$\Rightarrow \sup\{|\lambda| : \lambda \in \sigma(xx^*)\}$$

$$\le \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

$$\sup\{|\lambda| : \lambda \in \sigma(x^*)\}$$

$$\Rightarrow P(xx^*) \le P(x)P(x^*) = P(x)P(x)$$

$$\Rightarrow P(xx^*) \le P(x)^2 \qquad (1)$$

From (C),

$$|F(x)|^2 \le F(e)^2 P(xx^*)$$

$$\le F(e)^2 P(x)^2 \text{ [using (1)]}$$

$$\Rightarrow |F(x)| \le F(e)P(x)$$

6.3 Hilbert space :

Theorem : Let H be a Hilbert space. If $T \in B(H)$ and if $< Tx, x >= 0 \forall x \in H$, then T = 0

Proof : For $x, y \in H$,

$$\langle T(x+y, x+y)\rangle = 0$$

$$\Rightarrow \langle Tx + Ty, x+y\rangle = 0$$

$$\Rightarrow \langle Tx, x\rangle + \langle Tx, y\rangle + \langle Ty, x\rangle + \langle Ty, y\rangle = 0$$

$$\Rightarrow \langle Tx, y\rangle + \langle Ty, x\rangle = 0 \qquad (1)$$

Replacing 'y' by 'iy' we get

$$\langle Tx, iy\rangle + \langle T(iy), x\rangle = 0$$

$$\Rightarrow -i\langle Tx, y\rangle + i\langle Ty, x\rangle = 0 \qquad (2)$$

Multiplying (2) by i and adding to (1), we get

$$\langle Tx, y\rangle + \langle Ty, x\rangle + \langle Tx, y\rangle - \langle Ty, x\rangle = 0$$

$$\Rightarrow 2\langle Tx, y\rangle = 0$$

$$\Rightarrow \langle Tx, y\rangle = 0$$

Taking , $y = Tx$, we get

$$\langle Tx, Tx\rangle = 0$$

$$\Rightarrow \|Tx\|^2 = 0$$

$$\Rightarrow \|Tx\| = 0$$

$$\Rightarrow Tx = 0 \forall x \in H$$

$$\Rightarrow T = 0$$

Corollary : If $S, T \in B(H)$ and $\langle Sx, x\rangle = \langle Tx, x\rangle, \forall x \in H$, then S = T.

Theorem : There is a conjugate linear isometry $y \to \Lambda$ of H onto H^* given by

1. $\Lambda x = \langle x, y\rangle, y \in H$

Proof : For $y \in H$, we have $\Lambda x = \langle x, y\rangle$

Now, $|\Lambda x| = |\langle x, y\rangle| \le \|x\|.\|y\|$

$\Rightarrow |\Lambda x| \le \|y\|.\|x\|$

$\Rightarrow |\Lambda| \le \|y\|$ (i)

$\therefore \Lambda$ is bounded.

$\Rightarrow \Lambda \in H^*$

Again $\|y\|^2 = \langle y, y\rangle = \Lambda y \le \|\Lambda\|\|y\|$

$\Rightarrow \|y\| \le \|\Lambda\|$ (ii)

From (i) and (ii), we get $\|y\| = \|\Lambda\|$

Let $f: H \to H^*$ such that $f(y) = \Lambda$

and $\Lambda x = \langle x, y\rangle$

$f(\alpha y_1 + \beta y_2) = \Lambda$

where

$\Lambda x = \langle x, \alpha y_1 + \beta y_2\rangle$

$= \bar{\alpha}\langle x, y_1\rangle + \bar{\beta}\langle x, y_2\rangle$

$= \bar{\alpha}\Lambda y_1 + \bar{\beta}\Lambda y_2$

$= \bar{\alpha} f(y_1) + \bar{\beta} f(y_2)$

$\therefore$ The mapping $y \to \Lambda$ is conjugate linear.

To show that every $\Lambda \in H^*$ has the form

(1). If $\Lambda = 0$, then we take y = 0

$\therefore \langle x, y\rangle = \langle x, 0\rangle = 0 = \Lambda x$

If $\Lambda \ne 0$, Let $N(\Lambda)$ be the null space of Λ

We know that $N(\Lambda)$ is closed. So, $\exists z \in N(\Lambda)^{\perp}, z \ne 0$

$\therefore \Lambda((\Lambda x)z - (\Lambda x))$

$= (\Lambda x)(\Lambda z) - (\Lambda z)(\Lambda x)$

$= 0$

$\Rightarrow (\Lambda x)z(\Lambda z)x \in N(\Lambda), x \in H$

$\because z \perp N(\Lambda), so \langle(\Lambda x)z - (\Lambda z)x, z\rangle = 0$

$\Rightarrow \langle(\Lambda x)z, z\rangle - \langle(\Lambda z)x, z\rangle = 0$

$\Rightarrow (\Lambda x)\langle z,z\rangle - (\Lambda z)\langle x,z\rangle = 0$

$\Rightarrow \langle z,z\rangle(\Lambda x) = (\Lambda z)\langle x,z\rangle = \langle x, (\overline{\Lambda z})z\rangle$

$\Rightarrow \langle z,z\rangle(\Lambda x) = (\Lambda z)\langle x,z\rangle$

$= \langle x, (\overline{\Lambda z})z\rangle$

$\Rightarrow \Lambda x = \langle z,z\rangle^{-1}\langle x, (\overline{\Lambda z})z\rangle$

$= \langle x, \langle z,z\rangle^{-1}(\overline{\Lambda z})z\rangle$

Taking, $y = \langle z,z\rangle^{-1}(\overline{\Lambda z})z (\in H)$, we get

$\Lambda x = \langle x, y\rangle$

Thus there exists a conjugate linear isometry $y \to \Lambda$ of H onto H* given by $\Lambda x = \langle x, y\rangle, y \in H$.

Definition : Let X be a complex vector space. A conjugate bilinear form (sequilinear) is a mapping C.

$f : X \times X \to C$

$(x, y) \to f(x, y)$ is such that

1. $x \to f(x, y)$ is linear for every $y \in X$

$f(\alpha x_1 + \beta x_2, y) = \alpha f(x_1, y) + \beta f(x_2, y)$

2. $y \to f(x, y)$ is conjugate linear for every $x \in X$

$f(x, \alpha x_1 + \beta y_2) = \overline{\alpha} f(x, y_1) + \overline{\beta} f(x, y_2)$

Theorem : If $f : H \times H \to C$ is a conjugate linear and bounded in the sense that

1. $M = \sup\{|f(x,y)| : \|x\| = \|y\| = 1\} \langle \infty$ then $\exists$ a unique $S \in B(H)$ that satisfies

2. $f(x, y) = \langle x, Sy\rangle, \quad x, y \in H$

Moreover, $\|S\| = M$

Proof : For $y \in H$, Let $Ty : H \to C$ such that

$Ty(x) = f(x, y), \quad x \in H$

$\therefore |f(x, y)| \leq M\|x\|.\|y\|$

$\Rightarrow |Ty(x)| \leq M\|x\|.\|y\|$

$\Rightarrow |Ty| \leq M\|y\|$

$\therefore$ For each $y \in H$, the mapping Ty is a bounded linear functional on H, of norm at most $M\|y\|$
To each Ty, there corresponds a unique element of H, we denote it by Sy so that

$Ty(x) = \langle x, Sy \rangle$

$\Rightarrow f(x, y) = \langle x, Sy \rangle$

$\therefore$ The condition (2) holds.

Clearly $S : H \to H$ is additive

Now, $\|Sy\| = \|Ty\| \le M\|y\|$

For $y_1, y_2 \in H$

$\langle x, S(y_1 + y_2) \rangle = f(x, y_1 + y_2)$

$= f(x, y_1) + f(x, y_2)$

$= \langle x, Sy_1 \rangle + \langle x, Sy_2 \rangle$

$= \langle x, Sy_1 + Sy_2 \rangle$

$\Rightarrow S(y_1 + y_2) = Sy_1 + Sy_2$

If $\alpha \in C$,

$\langle x, S(\alpha y) \rangle = f(x, \alpha y)$

$= \bar{\alpha} f(x, y)$

$= \bar{\alpha} \langle x, Sy \rangle$

$= S(\alpha y) = \alpha Sy$

$\therefore S$ is linear.

Now $\|Sy\| \le M\|y\|$

$\Rightarrow \|S\| \le M$ (1)

$\therefore S \in B(H)$

$\therefore |f(x, y)| = |\langle x, Sy \rangle|$

$\le \|x\| . \|Sy\|$

$\le \|x\| . \|S\| . \|y\|$

$\Rightarrow \sup\{|f(x, y)| : \|x\| = \|y\| = 1\} \le \|S\|$

$\Rightarrow M \le \|S\|$ (2)

From (1) and (2), we get $\|S\| = M$

6.4 Definition :

If $T \in B(H)$, then $\langle Tx, y\rangle$ is linear in x, conjugate linear in y and bounded, then $\exists$ a unique $S \in B(H)$ such that

$f(x, y) = \langle x, Sy\rangle$

$\Rightarrow \langle Tx, y\rangle = \langle x, Sy\rangle$

We denote S by T^*. Then

1. $\langle Tx, y\rangle = \langle x, T^*y\rangle (x, y \in H)$

2. $\|T^*\| = \|T\|$

Then T^* is called the adjoint of T and $T^* \in B(H)$

Example : Show that $T \to T^*$ is an involution on B(H)

Solution : Let $S, T \in B(H)$ and α be any scalar $(\alpha \in C)$

(i) $\langle (S+T)x, y\rangle = \langle Sx + Tx, y\rangle$

$= \langle Sx, y\rangle + \langle Tx, y\rangle$

$= \langle x, S^*y\rangle + \langle x, T^*y\rangle$

$= \langle x, (S^* + T^*)y\rangle$

$\Rightarrow \langle (S+T)x, y\rangle = \langle x, (S^* + T^*)y\rangle$

$\Rightarrow \langle x, (S+T)^* y\rangle = \langle x, (S^* + T^*)y\rangle$

$\therefore \; (S+T)^* = S^* + T^*$

(ii) $\langle (\alpha T)x, y\rangle = \langle \alpha Tx, y\rangle$

$= \alpha \langle Tx, y\rangle$

$= \alpha \langle x, T^*y\rangle$

$= \langle x, (\alpha T)^* y\rangle = \langle x, \bar{\alpha} T^* y\rangle$

$\Rightarrow (\alpha T)^* = \bar{\alpha} T^*$

(iii) $\langle (ST)x, y\rangle = \langle S(Tx), y\rangle$

$= \langle Tx, S^*y\rangle$

$= \langle x, T^*S^*y\rangle$

$\Rightarrow \langle x, (ST)^* y \rangle = \langle x, T^* S^* y \rangle$

$\Rightarrow (ST)^* = T^* S^*$

(iv) $\Rightarrow \langle Tx, y \rangle = \langle x, T^* y \rangle$

$= \overline{\langle T^* y, x \rangle}$

$= \overline{\langle y, T^{**} x \rangle}$

$= \overline{\langle T^{**} x, y \rangle}$

$\Rightarrow T = T^{**}$

$\therefore T \to T^*$ is an involution on B(H)

Again, $\|Tx\|^2 = \langle Tx, Tx \rangle$

$= \langle T^*, Tx, x \rangle$

$\leq \|T^* T\| \|x\|^2 \; \forall x \in H$

$\Rightarrow \|Tx\| \leq \|T^* T\|^{1/2} \|x\| \forall x \in H$

$\Rightarrow \|T\| \leq \|T^* T\|^{1/2}$

$\Rightarrow \|T\|^2 \leq \|T^* T\|$

Next, $\|T^* T\| \leq \|T^*\| . \|T\| = \|T\|^2$

$\therefore \|T^* T\| = \|T\|^2, \forall T \in B(H)$

$\therefore B(H)$ is a B^*-algebra, relative to the involution $T \to T^*$

6.5 Definition : A C^*-algebra is a closed star subalgebra of B(H) for some Hilbert space H. Every C^*-algebra is a B^*-algebra.

Examples :

1. Let $\{Ai\}_i$ be a family of C^*-algebras.

Let A be the set of $(xi)_{i \in I} \left[(xi) = (x_1, x_2, ...xi, ...)_{i \in I} \right]$

Such that $xi \in Ai$

$\sup_{i \in I} \|xi\| < \infty$

We define, $(xi) + (yi) = (xi + yi)$

$(xi)(yi) = (xiyi)$

$\lambda(xi) = (\lambda xi)$

$(xi)^* = (xi^*)$

$\|xi\| = \sup_{i \in I} \|xi\|$

Then A is a C^*-algebra, called the product C^*-algebra of Ai's.

2. Let A be a C^*-algebra. Let A^0 be the algebra obtained from A replacing the multiplication $(x, y) \to xy$ *in A by*

$(x, y) \to yx,$ *with all*

other algebraic operations and the norm same as A.

Then A^0 is a C^*-algebra called the reversed normed involutive algebra of A.

6.6 Homomorphisms :

Let A and B be two involution algebras. A mapping $\phi : A \to B$ is a homomorphism if

1. $\phi(x+y) = \phi(x) + \phi(y)$

2. $\phi(\lambda x) = \lambda \phi(x)$

3. $\phi(xy) = \phi(x)\phi(y)$

4. $\phi(x^*) = \phi(x)^*, \forall x, y \in A, \lambda \in \square$

Proposition :

Let A be an involutive B^*-algebra; B a C^*-algebra and Π is a homomorphism of A into B.

Then $\|\Pi(x)\| \le \|x\|, \forall x \in A$

Proof : For each Hermition element y of B, we have

$\|y^2\| = \|y^* y\| = \|y\|^2$

By induction,

$\|y^{2^n}\| = \|y\|^{2^n}, n = 1, 2, 3, ...$

$\Rightarrow \|y^m\|^{1/m} = \|y\|, \; m = 2^n, n = 1, 2, 3, ...$

$\Rightarrow P(y) = \|y\|$ (1)

Let $\lambda \in \sigma_B(\Pi(x)), x \in A$

$\Rightarrow \lambda e' - \Pi(x)$ is not invertible in B. (e' is the unit element of B)

$\Rightarrow \lambda \Pi(e) - \Pi(x)$ is not invertible in B. (e is the unit element of A)

$\Rightarrow \Pi(\lambda e - x)$ is not invertible in B.

Let $y \in A$ such that $y^{-1} \in A$

$\therefore \Pi(y^{-1}) \in B$

and $\Pi(y)\Pi(y^{-1}) = \Pi(yy^{-1}) = \Pi(e) = e'$

Similarly, $\Pi(y^{-1})\Pi(y) = e'$

$\therefore [\Pi(y)]^{-1} = \Pi(y^{-1}) \in B$

Now, $\Pi(\lambda e - x)$ is not invertible in B.

$\Rightarrow \lambda e - x$ is not invertible in A.

$\Rightarrow \lambda \in \sigma_A(x)$

$\therefore \sigma_B(\Pi(x)) \subseteq \sigma_A(x)$

$\therefore P(\Pi(x)) \leq P(x) \leq \|x\| \qquad (2)$

Now, $\|\Pi(x)\|^2 = \|\Pi(x)\Pi(x)^*\|$

$= \|\Pi(x)\Pi(x^*)\|$

$\Rightarrow \|\Pi(x)\|^2 = \|\Pi(xx^*)\|$

$\Rightarrow \|\Pi(x)\|^2 = P(\Pi(xx^*)) \leq \|xx^*\| \qquad$ [by (2)]

$= \|x\|^2$

$\Rightarrow \|\Pi(x)\| \leq \|x\| \; \forall x \in A$

6.7 Definition :

Let $T \in B(H)$. Then T is

(i) normal if $TT^* = T^*T$
(ii) self-adjoint if $T = T^*$
(iii) unitary, if $TT^* = T^*T = I$
(iv) projection if $T^2 = T$, where I is the identity operator on H.

Theorem : Let $T \in B(H)$. Then

(i) If T is self-adjoint, then $\langle Tx, x\rangle$ is real, $\forall x \in H$ and conversely.

Proof : (i) First suppose that T is self-adjoint. Then for all $x \in H$,

$\langle Tx, x\rangle = \langle x, T^*x\rangle$

$= \langle x, Tx\rangle, \; as T^* = T$

$= \overline{\langle Tx, x\rangle}$

$\therefore \langle Tx, x\rangle$ *is real* $\forall x \in H$

Conversely, suppose that $\langle Tx, x\rangle$ is real $\forall x \in H$.

Then $\langle Tx, x\rangle = \overline{\langle Tx, x\rangle}$

$= \overline{\langle x, T^* x\rangle}$

$= \langle T^* x, x\rangle$

$\Rightarrow \langle (T - T^*)x, x\rangle = 0 \ \forall x \in H$

$\Rightarrow (T - T^*)x = 0 \ \forall x \in H$

$\Rightarrow T - T^* = 0$

$\Rightarrow T = T^*$

$\therefore T$ is self-adjoint.

Theorem : Let *{Tn}* be a sequence of bounded self-adjoint linear operators in B(H) and $Tn \to T$. Then *T* is a self-adjoint operator in *B(H)*.

Proof : $\|Tn^* - T^*\| = \|(Tn - T)^*\|$

$= \|Tn - T\| \ [\because \|T^*\| = \|T\|]$

$\therefore T - T^* = T - Tn + Tn - Tn^* + Tn^* - T^*$

$= (T - Tn) + (Tn - Tn^*) + (Tn^* - T^*)$

$= (T - Tn) + (Tn^* - T^*)$, *as* $Tn = Tn^*$

$\therefore \|T - T^*\| \leq \|T - Tn\| + \|Tn^* - T^*\|$

$= 2\|Tn - T\| \to 0$ *as* $n \to \infty$

$\Rightarrow \|T - T^*\| = 0 \ \Rightarrow \mathrm{T} - \mathrm{T}^* = 0$

$\Rightarrow T = T^*$

$\Rightarrow T$ is self-adjoint.

Theorem : An operator $T \in B(H)$ is unitary iff T is isometric and surjective.

Proof : Let T be isometric and surjective.

Now, T is isometric $\Rightarrow \|Tx\| = \|x\|$

Now, $\Rightarrow Tx = 0 \Rightarrow \|Tx\| = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$

$\therefore T$ is bijective.

Now, $\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 = \|x\|^2 = \langle x, x\rangle$

$\Rightarrow \langle (T^*T - I)x, x\rangle = 0 \ \forall x \in H$

$\Rightarrow (T^*T - I)(x) = 0 \ \forall x \in H$

$\Rightarrow T^*T - I = 0 \Rightarrow T^*T = I$

Again, $TT^* = TT^*(TT^{-1})$

$= T(T^*T)T^{-1}$

$= TIT^{-1}$

$= TT^{-1}$

$= I$

$\Rightarrow T^*T = TT^* = I \text{ and } T^* = T^{-1}$

$\therefore T$ is unitary.

Conversely suppose that T is a unitary operator on H. Then T is invertible and therefore T is onto.

Also $T^*T = TT^* = I$.

Thus we get $\|Tx\| = \|x\|, \ \forall x \in H$

$\therefore T$ is isometric and surjective.

Theorem : Let P be a projection on a Hilbert space X. Then

(i) I – P is a projection on X.

(ii) $R(P) = \{x \in X : Px = x\}$

(iii) $R(P) = N(I - P)$

(iv) $X = R(P) \oplus R(I - P)$

(v) If P is bounded, then R(P) and and R(I-P) are closed.

Proof : (i) Let P be a projection on X.

Then $P^2x = Px \ \forall x \in X$

Now, $(I - P)^2 x = (I - P)(I - P)(x)$

$= (I - P - P + P^2)(x)$

$= (I - P - P + P)(x)$

$= (I - P)(x) \ \forall x \in X$

$\therefore (I - P)$ is a projection on X.

(ii) Cleary, $\{x \in X : Px = x\} \subseteq R(P)$

Let $y \in R(P) \Rightarrow y = Px$ for some $x \in X$

$\therefore Py = P(Px) = P^2 x = Px = y$

$\therefore y \in \{x \in X : Px = x\}$

$\therefore R(P) \subseteq \{x \in X : Px = x\}$

$\therefore R(P) = \{x \in X : Px = x\}$

(iii) Let $x \in R(P)$

$\Leftrightarrow x = Px$ [by (ii)]

$\Leftrightarrow Ix = Px$

$\Leftrightarrow (I - P)x = 0$

$\Leftrightarrow x \in N(I - P)$

$\therefore R(P) = N(I - P)$

(iv) Let $x \in X$ Clearly, $R(P) + R(I - P) \subseteq X$

Now, $x = Ix = Px + Ix - Px$

$= Px + (I - P)x$

$\Rightarrow x \in R(P) + R(I - P)$

$\therefore X \subseteq R(P) + R(I - P)$

$\therefore X = R(P) + R(I - P)$

Let $y \in R(P) \cap R(I - P)$

$\Rightarrow y = Py = (I - P)y$

$\Rightarrow y = P^2 y = P(Py)$

$= P(I - P)y$

$= Py - P^2 y$

$= Py - Py$

$= 0$

$\therefore R(\rho) \cap R(I - P) = (\{0\})$

$\therefore X = R(\rho) \oplus R(I - P)$

(v) $R(P) = N(I - P) = (I - P)^{-1}(\{0\})$

Here $\{0\}$ is closed. Again, P is bounded. Thn (I – P) is bounded and so $(I-P)^{-1}$ is bounded.

Hence $(I-P)^{-1}$ is continuous.
Now continuous image of a closed set is closed.

$\Rightarrow R(P) = (I-P)^{-1}(\{0\})$ is closed.

Similarly, $R(I-P) = P^{-1}(\{0\})$ is closed.

Theorem : If $T \in B(H)$, then $N(T^*) = R(T)^{\perp}$ and $N(T) = R(T^*)^{\perp}$

Proof : Let $y \in N(T^*) \Leftrightarrow T^* y = 0$

$\Leftrightarrow \langle x, T^* y \rangle = 0 \ \forall x \in H$

$\Leftrightarrow \langle Tx, y \rangle = 0 \ \forall x \in H$

$\Leftrightarrow y^{\perp} Tx, \ \forall x \in H$

$\Leftrightarrow y^{\perp} R(T)$

$\Leftrightarrow y \in R(T)^{\perp}$

$\therefore N(T^*) = R(T)^{\perp}$

Replacing T by T^*, we get

$N(T^{**}) = R(T)^{\perp}$

$\Rightarrow N(T) = R(T^*)^{\perp}$

Theorem : If $T \in B(H)$, then

(a) T is normal iff $\|Tx\| = \|T^* x\| \ \forall x \in H$

Proof : Let $T \in B(H)$ be normal

$\therefore TT^* = T^* T$

Now, $\|Tx\|^2 = \langle Tx, Tx \rangle$

$= \langle T^* Tx, x \rangle$

$= \langle TT^* x, x \rangle$

$= \langle T^* x, T^* x \rangle$

$\Rightarrow \|Tx\| = \|T^* x\| \ \forall x \in H$

Conversely, $\|Tx\| = \|T^* x\|$

$\Rightarrow \|Tx\|^2 = \|T^* x\|^2$

$\Rightarrow \langle Tx, Tx\rangle = \langle T^*x, T^*x\rangle$

$\Rightarrow \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle \ \forall x \in H$

$\Rightarrow T^*T = TT^*$

$\Rightarrow T$ is normal.

(b) If T is normal, then $N(T) = N(T^*) = R(T)^1$

Proof : Since T is normal, so

$\Rightarrow \|Tx\| = \|T^*x\|, \ \forall x \in H$

Let $y \in N(T) \Leftrightarrow T(y) = 0$

$\Leftrightarrow \|Ty\| = 0$

$\Leftrightarrow \|T^*y\| = 0$

$\Leftrightarrow T^*y = 0$

$\Leftrightarrow y \in N(T^*)$

$\therefore N(T) = N(T^*) = R(T)^\perp$

(c) If T is normal and $Tx = \alpha x$, for some $x \in H, \alpha \in C$, then $T^*x = \bar{\alpha}x$.

Proof : Since T is normal, so $TT^* = T^*T$

Now, $(T-\alpha I)(T-\alpha I)^* = (T-\alpha I)(T^* - \bar{\alpha} I)$.

$= TT^* - \bar{\alpha}T - \alpha T^* + \alpha\bar{\alpha}I$

$= T^*T - \alpha T^* - \bar{\alpha}T + \bar{\alpha}\alpha I$

$= T^*(T-\alpha I) - \bar{\alpha}(T-\alpha I)$

$= (T^* - \bar{\alpha}I)(T-\alpha I)$

$= (T-\alpha I)^*(T-\alpha I)$

$\therefore (T-\alpha I)$ is also a normal operator.

Given that,

$Tx = \alpha x$, for some $x \in H, \alpha \in$□

$\Rightarrow (T-\alpha I)x = 0$

$\Rightarrow x \in N(T-\alpha I) = N((T-\alpha I)^*)$

$\Rightarrow x \in N(T^* - \bar{\alpha}I)$

$\Rightarrow (T^* - \bar{\alpha}I)x = 0$

$\Rightarrow T^{*}x = \bar{\alpha}x$

(d) If T is normal and If α and β are district eigenvalues of T, then the corresponding eigen-spaces are orthogonal to each other.

Proof : Since α and β are eigenvalues of T, so $\exists$ non zero vectors x and y such that

$Tx = \alpha x,\ Ty = \beta y$

Let E_α and E_β be the eigenspace corresponding to the eigenvalues α and β respectively.

$\therefore Tx = \alpha x,\ \forall x \in E_\alpha$

$Ty = \beta y,\ \forall y \in E_\beta$

For, $x \in E_\alpha,\ y \in E_\beta$

$\alpha\langle x, y\rangle = \langle \alpha x, y\rangle$

$= \langle Tx, y\rangle$

$= \langle x, T^{*}y\rangle$

$= \langle x, \bar{\beta}y\rangle$

$= \beta\langle x, y\rangle$

$\Rightarrow (\alpha - \beta)\langle x, y\rangle = 0$

$\Rightarrow \langle x, y\rangle = 0 \quad (\because \alpha \neq \beta)$

$\Rightarrow x \perp y$

$\therefore E_\alpha \perp E_\beta$

Theorem : If $U \in \beta(H)$, then the following statements are equivalent

(a) U is unitary.

(b) R(U) = H and $\langle Ux, Uy\rangle = \langle x, y\rangle\ \forall x, y \in H$

(c) R(U) = H and $\|Ux\| = \|x\|,\ \forall x, y \in H$

Proof : $(a) \Rightarrow (b)$:

U is unitary, so $U^{-1} = U^{*}$ and u is bijictive.

$\therefore R(U) = H$

And, $\langle Ux, Uy\rangle = \langle x, U^{*}Uy\rangle$

$= \langle x, U^{-1}Uy\rangle$

$= \langle x, Iy\rangle = \langle x, y\rangle,\ \forall x, y \in H.$

$(b) \Rightarrow (c)$: R(U) = H and $\langle Ux, Uy\rangle = \langle x, y\rangle,\ \forall x, y \in H.$

In particular, for y = x,

$\langle Ux, Ux\rangle = \langle x, x\rangle$

$\Rightarrow \|Ux\|^2 = \|x\|^2$

$\Rightarrow \|Ux\| = \|x\|, \ \forall x \in H$

$(c) \Rightarrow (a)$: R(U) = H and $\|Ux\| = \|x\|, \ \forall x \in H$

$\Rightarrow \|Ux\|^2 = \|x\|^2$

$\Rightarrow \langle Ux, Ux\rangle = \langle x, x\rangle$

$\Rightarrow \langle U^*Ux, x\rangle = \langle x, x\rangle$

$\Rightarrow \langle (U^*U - I)x, x\rangle = 0 \ \forall x \in H$

$\Rightarrow U^*U - I = 0$

$\Rightarrow U^*U = I$

Similarly, $UU^* = I$

$\therefore U$ is unitary.

Theorem : Let $P \in \beta(H)$ be a projection. Then the following are equivalent.

(a) P is self-adjoint.

(b) P is normal

(c) $R(P) = N(P)^{\perp}$

(d) $\langle Px, x\rangle = \|Px\|^2, \ \forall x \in H$

Proof : $(a) \Rightarrow (b)$:

$P^* = P$

$\Rightarrow PP^* = P^*P$

$\Rightarrow P$ is normal.

$(b) \Rightarrow (c)$: P is normal

$\Rightarrow N(P) = N(P^*) = R(P)^{\perp}$

$\Rightarrow N(P)^{\perp} = \left(R(P)^{\perp}\right)^{\perp}$

$\Rightarrow N(P)^{\perp} R(P) \quad [\because R(P) \textit{ is closed}]$

$\therefore R(P) = N(P)^{\perp}$

$(c) \Rightarrow (d)$: $R(P) = N(P)^{\perp}$

Now, $N(P) = R(I - P)$

and $H = R(P) \oplus R(I-P)$

$= R(P) \oplus N(P)$

$= N(P) \oplus R(P)$

$\therefore$ Every element $x \in H$ has the form $x = y+z$, where $y \perp z, y \in N(P) z \in R(P)$

$\therefore \langle y,z \rangle = 0$ and $Py = 0$

Since $z \in R(P)$, so $z = Pu$, where $u \in H$

$\therefore Pz = P(Pu) = P^2u = Pu = z$

$\therefore Px = P(y+z) = P(y)+P(z) = 0+z = z$

$\therefore \langle Px, x \rangle = \langle z, y+z \rangle$

$= 0 + \|z\|^2$

$= \|z\|^2$

$\therefore \|Px\|^2 = \langle Px, Px \rangle = \langle z, z \rangle = \|z\|^2$

$\therefore \langle Px, Px \rangle = \|Px\|^2$, $\forall x \in H$

$(d) \Rightarrow (a)$: $\|Px\|^2 = \langle Px, Px \rangle$, $\forall x \in H$

$= \langle x, P^* x \rangle$

$= \langle P^* x, x \rangle \left[\because \langle x, P^* x \rangle \textit{is real} \right]$

$\Rightarrow \langle Px, x \rangle = \langle P^* x, x \rangle \ \forall x \in H$

$\Rightarrow \langle (P - P^*) x, x \rangle = 0, \ \forall x \in H$

$\Rightarrow P - P^* = 0$

$\Rightarrow P = P^*$

$\therefore P$ is self-adjoint.

Theorem : Suppose $S \in \beta(H)$ and S is self-adjoint. Then $ST = 0 \Leftrightarrow R(S) \perp R(T), T \in \beta(H)$

Proof : Given that S is self-adjoint.

$\Rightarrow S = S^*$

Let ST = 0. Let $x, y \in H$

$\therefore \langle Sx, Ty \rangle = \langle x, S^* Ty \rangle$

$= \langle x, STy \rangle = 0 \quad (\because ST = 0)$

$\Rightarrow Sx \perp Ty$ for since $Sx \in R(S)$ and $Ty \in R(T)$

$\Rightarrow R(S) \perp R(T)$

Conversely let $R(S) \perp R(T)$

Then $\exists x, y \in H$ such that

$Sx \perp Ty$

$\Rightarrow \langle Sx, Ty \rangle = 0$

$\Rightarrow \langle x, S^{*}Ty \rangle = 0$

$\Rightarrow \langle x, STy \rangle = 0 \ \forall x, y \in H$

$\Rightarrow STy = 0 \ \forall y \in H$

$\Rightarrow ST = 0$

OR

$ST = 0 \Leftrightarrow \text{STy=0} \ \forall \text{y} \in \text{H}$

$\Leftrightarrow \langle \text{x,STy} \rangle \text{=0} \ \forall x, \text{y} \in \text{H}$

$\Leftrightarrow \langle x, S^{*}Ty \rangle = 0$, as $S^{*}\text{=S}$

$\Leftrightarrow \langle Sx, Ty \rangle = 0$

$\Leftrightarrow Sx \perp Ty \ \forall x, y \in H$

$\Leftrightarrow R(S) \perp R(T)$

Thus $ST = 0 \Leftrightarrow R(S) \perp R(T), \ \forall T \in \beta(H)$

Theorem : Let $M, N, T \in \beta(H)$, M and N are normal and MT = TN. Then $M^{*}T = TN^{*}$

Proof : Let, $S \in \beta(H)$. Let

$V = S - S^{*}$. We define

$Q = e \times P(V) = \sum_{n=0}^{\infty} \frac{1}{\lfloor n} V^{n}$

Then $V^{*} = (S - S^{*})^{*} = S^{*} - S = (S - S^{*}) = -V$

$Q^{*} = e \times P(V^{*}) = e \times P(-V)$

$= \frac{1}{e \times P(-V)}$

$= Q^{-1}$

$\therefore Q$ is unitary.

$\therefore QQ^* = I \Rightarrow \|QQ^*\| = \|I\| = 1$

$\Rightarrow \|Q\|^2 = 1 \left[\because \beta(H) \text{ is a } B^* - a\lg ebra\right]$

$\Rightarrow \|Q\| = 1$

$\Rightarrow \|e \times P(V)\| = 1$

$\Rightarrow \|e \times P(S - S^*)\| = 1, \ \forall S \in \beta(H)$ (1)

Given that

$MT = TN, \ T \in \beta(H)$

$\Rightarrow M(MT) = M(TN)$

$\Rightarrow (MM)T = (MT)N = (TN)N$

$\Rightarrow (MM)T = T(NN)$

$\Rightarrow M^2T = TN^2$

Let, $M^nT = TN^n$

$\therefore M^{n+1}T = M(M^nT)$

$= M(TN^n)$

$= (MT)N^n$

$= (TN)N^n$

$= TN^{n+1}$

$\therefore$ By induction, $M^nT = TN^n, \ \forall n = 1, 2, ...$

Now, $\left(I + M + \frac{M^2}{L^2} + ...\right)T$

$= T + MT + \frac{M^2T}{L^2} + ...$

$= T + TN + \frac{TN^2}{L^2} + ...$

$= T\left(I + N + \frac{N^2}{L^2} + ...\right)$

$\Rightarrow e \times P(M)T = Te \times P(N)$

$\Rightarrow T = e \times P(-M)Te \times P(N)$ (3)

Let $u_1 = e \times P\left(M^* - M\right)$

$u_2 = e \times P\left(N - N^*\right)$

Since, M and N are normal, by (3)

$u_1 T u_2 = e \times P\left(M^* - M\right) T e \times P\left(N - N^*\right)$

$= e \times P\left(M^*\right) T e \times P\left(-N^*\right) \qquad (4)$

$(1) \Rightarrow \|u_1\| = \left\|e \times P\left(M^* - M\right)\right\| = 1$

$\|u_2\| = \left\|e \times P\left(N - N^*\right)\right\| = 1$

$(4) \Rightarrow \left\|e \times P\left(M^*\right) T e \times P\left(-N^*\right)\right\| = \|u_1 T u_2\|$

$\leq \|u_1\| \|T\| \|u_2\|$

$= \|T\| \qquad (5)$

We define, $f : \mathbb{C} \rightarrow \beta(H)$ by

$f(\lambda) = e \times P\left(\lambda M^*\right) T e \times P\left(-\lambda N^*\right), \lambda \in \mathbb{C}$

$\therefore \left(\bar{\lambda} M\right)\left(\bar{\lambda} M\right)^* = \bar{\lambda} M \lambda M^*$

$= \bar{\lambda} \lambda M M^*$

$= \lambda \bar{\lambda} M^* M$

$= \left(\lambda M^*\right)\left(\bar{\lambda} M\right)$

$= \left(\lambda M\right)^* \left(\bar{\lambda} M\right)$

$\therefore \bar{\lambda} M$ is normal

Similarly, $\bar{\lambda} N$ is also normal.

Applying (5) to $\bar{\lambda} M$ and $\bar{\lambda} N$, we get

$\Rightarrow \left\|e \times P\left(\left(\bar{\lambda} M\right)^*\right) T e \times P\left(-\left(\bar{\lambda} N\right)^*\right)\right\| \leq \|T\|$

$\Rightarrow \left\|e \times P\left(\lambda M^*\right) T e \times P\left(-\lambda N^*\right)\right\| \leq \|T\|$

$\Rightarrow \|f(\lambda)\| \leq \|T\|, \ \forall \lambda \in \mathbb{C}$

$\therefore f$ is bounded entire function on $\mathbb{C}$. So by Liouville's theorem, f is constant.

But *f(0) = T*

$\therefore f(\lambda) = T, \ \forall \lambda \in \mathbb{C}$

$\Rightarrow e \times P\left(\lambda M^*\right) T e \times P\left(-\lambda N^*\right) = T, \ \forall \lambda \in \mathbb{C}$

$$\Rightarrow Te\times P(\lambda N^*) = e\times P(\lambda M^*)T,\ \forall \lambda \in \mathbb{C}$$

$$\Rightarrow T\left(I + \lambda N^* + \frac{(\lambda N^*)^2}{\lfloor 2} + \ldots\right)$$

$$= \left(I + \lambda M^* + \frac{(\lambda M^*)^2}{\lfloor 2} + \ldots\right)T\ \forall \lambda \in \mathbb{C}$$

Equating coefficient of λ, we get $TN^* = M^*T$

Exercise :

1. Suppose * is an involution in a complex algebra A, q is an invertible element of A such that q^* = q and $x \neq$ defined by $x \neq q^{-1}x^*q,\ \forall x \in A$

Show that $\neq$ is an involution on A.

2. Let A be the algebra of all complex 4×4 matrices. If $M = (m_{ij}) \in A$, let M^* be the conjugate transpose of M.

Let $Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Define, $M \neq Q^{-1}M^*Q, M \in A$,

Show that S and T are normal w.r.t the involution $\neq$, that ST = TS, but

$ST^{\neq} \neq T^{\neq}S$

6.8 Resolution of the Identity :

Let M be a σ-algebra in a set Ω and H be a Hilbert space. a resolution of the identity properties :

(i) $E(\phi) = 0; E(\Omega) = I$

(ii) Each E(w) is a self-adjoint projection $(w \in M)$

(iii) $E(w' \cap w'') = E(w')E(w'')$

(iv) If $w' \cap w'' = \phi$, then $E(w' \cup w'') = E(w') + E(w'')$

(v) For every $x \in H$ *and* $y \in H$, the set function Ex,y defined by $Ex,y(w) = \langle E(w)x, y\rangle$ is a complex measure on M.

Properties :

(1) Ex,x is a positive measure on M.

Proof : Since each *E(w)* is a self-adjoint projection,

$Ex,x(w) = \langle E(w)x, x\rangle$

$= \langle E(w)^2 x, x\rangle$

$= \langle E(w)E(w)x, x\rangle$

$= \langle E(w)x, E(w)x\rangle$, as *E(w)* is self-adjoint

$= \langle E(w)x\rangle \geq 0$

$\Rightarrow Ex,x$ is a positive measure on M.

2. Any two projections *E(w)* commute with each other.

Proof : $E(w' \cap w'') = E(w')E(w'')$

$E(w'' \cup w') = E(w'')E(w')$

Now, $w' \cap w'' = w'' \cap w'$

$\Rightarrow E(w' \cap w'') = E(w'' \cap w')$

$\Rightarrow E(w')E(w'') = E(w'')E(w')$

3. If $w' \cap w'' = \phi$, then $E(w' \cap w'') = E(\phi) = 0$

$\Rightarrow E(w')E(w'') = 0$

$E(w')$ is a self-adjoint.

$\therefore R(E(w')) \perp R(E(w''))$

6.9 Spectral decomposition :

Theorem : If $T \in \beta(H)$ and T is normal, then there exists a unique resolution of the identity, E on the Borel subsets of $\sigma(T)$, which satisfies

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

i.e, $\langle Tx, y\rangle = \int_{\sigma(T)} \lambda dE_{x,y}(\lambda)$, $x, y \in H$

Moreover, every projection *E(w)* commutes with every $S \in \beta(H)$ which commutes with T.

E is called the spectral decomposition of T.

6.10 Definition :

If E is the spectral decomposition of a normal operator $T \in \beta(H)$ and if *f* is a bounded Borel function on $\sigma(T)$, then the operator

$\psi(f) = \int_{\sigma(T)} f dE$ is denoted by *f(T)*

Theorem : Let $T \in \beta(H)$. Then the following are equivalent.

(a) $\langle Tx, x \rangle \geq 0, \ \forall x \in H$

(b) $T = T^*$ and $\sigma(T) \subset [0, \infty)$

Proof : $(a) \Rightarrow (b)$:

If $\langle Tx, x \rangle \geq 0, \ \forall x \in H$, then

$\langle Tx, x \rangle$ *is real* $\forall x \in H$

$\therefore \langle Tx, x \rangle = \overline{\langle Tx, x \rangle}$

$= \langle x, Tx \rangle$

$= \langle T^* x, x \rangle$

$\Rightarrow \langle (T - T^*) x, x \rangle = 0 \ \forall x \in H$

$\Rightarrow T - T^* = 0 \ \Rightarrow T = T^*$

$\therefore \hat{T}$ is real valued.

$\therefore \sigma(T) = Range \ \hat{T}$ is real.

Let, $\lambda > 0$. Then $\lambda \|x\|^2 = \lambda \langle x, x \rangle$

$= \langle \lambda x, x \rangle$

$\leq \langle \lambda x, x \rangle + \langle Tx, x \rangle$

$= \langle (T + \lambda I) x, x \rangle$

$\leq \|(T + \lambda I) x\| \|x\|$

$\Rightarrow \lambda \|x\| \leq \|(T + \lambda I) x\|$ (1)

$\therefore (T + \lambda I)$ is bounded below.

$\therefore (T + \lambda I)^* = T^* + \overline{\lambda} I = \lambda I$

$\Rightarrow (T + \lambda I)^*$ is bounded below.

Lemma : Let X be a Banach space, Y be a normed space and let $T \in \beta(X,Y)$. Then the followings are equivalent.

(a) T is invertible

(b) T^* is invertible

(c) R(T) is dense in Y and T is bounded below.

(d) T and T^* are both bounded below.

Hence $T + \lambda I$ is invertible.

i.e $T + \lambda I = T - (-\lambda) I$ is invertible.

$\therefore -\lambda \notin \sigma(T)$

So, for any $\lambda > 0, -\lambda \notin \sigma(T)$

$\therefore \sigma(T) \subset [0, \infty)$

$(b) \Rightarrow (a)$: Let $T = T^*$ and $\sigma(T) \subset [0, \infty)$

Now T is normal.

Let E be the spectral decomposition of T

$$\therefore \langle Tx, y \rangle = \int_{\sigma(T)} \lambda dE_{x,y}(\lambda), \; x \in H$$

We have, $E_{x,x}$ is a positive measure on $\sigma(T)$

Also $\lambda \geq 0$ on $\sigma(T)$.

Hence $\int_{\sigma(T)} \lambda dE_{x,x}(\lambda) \geq 0 \; \forall x \in H$

$\Rightarrow \langle Tx, x \rangle \geq 0 \; \forall x \in H$

Note : If $T \in \beta(H)$ satisfies any of the conditions (a) and (b), then T is called a positive operator and is denoted by $T \geq 0$

A self adjoint operator 'S' is called a square root of a positive operator T if $S^2 = T$.

Theorem : Every positive operator $T \in \beta(H)$ has a unique positive square root $S \in \beta(H)$. Again if T is invertible, then so is 'S'.

Proof : Let S be the collection of all self-adjoint operators on H.

We define a relation $\leq$ on 'S' by

If $T_1, T_2 \in S$, then

$T_1 \leq T_2 \Leftrightarrow \langle T_1 x, x \rangle \leq \langle T_2 x, x \rangle \; \forall x \in H$

$\therefore T_1 \leq T_2 \Rightarrow \alpha T_1 \leq \alpha T_2, \alpha > 0$

Let $T \in \beta(H)$ be a positive operator. Then T is self-adjoint.

Without loss of generality, we can assume that $T \leq I$.

Let $S_0 = 0$ and $S_{n+1} = S_n + \frac{1}{2}\left(T - S_n^2\right), n = 0,1,2,....\}$ (1)

$\therefore S_1 = S_0 + \frac{1}{2}\left(T - S_0^2\right)$

$= 0 + \frac{1}{2}\left(T - S_0^2\right) = \frac{1}{2}T$

$\therefore S_1^* = \left(\frac{1}{2}T\right) = \frac{1}{2}T^* = \frac{1}{2}T = S_1$

$\therefore S_2 = S_1 + \frac{1}{2}\left(T - S_1^2\right)$

$= \frac{1}{2}T + \frac{1}{2}\left(T - \frac{1}{4}T^2\right)$

$= T - \frac{1}{8}T^2$

$= T\left(T - \frac{1}{8}T\right).$

$\therefore \langle S_1x, x\rangle = \left\langle \frac{1}{2}Tx, x\right\rangle = \frac{1}{2}\langle Tx, x\rangle \geq 0$

$\Rightarrow S_1 \geq 0$

Again, $\langle S_2x, x\rangle = \left\langle\left(T - \frac{1}{8}T^2\right)x, x\right\rangle$

$\langle Tx, x\rangle - \frac{1}{8} = \langle T^2x, x\rangle$

Since, product of two positive operators is positive, so $T\left(I - \frac{1}{8}\right) \geq 0$

$\Rightarrow S_2 \geq 0$

Let $P \in \beta(H)$ commutes with T, i.e $PT = TP$

Then, $S_1P = \frac{1}{2}TP = \frac{1}{2}PT = P\left(\frac{1}{2}T\right) = PS_1$

Similarly, $S_2P = PS_2$

Thus the operators *Sn's* are positive and they commute with every operator P which commutes with T.

In particular, *Sn's* commut with each other, i.e., $S_nS_m = S_mS_n$

Now, From (1),

$$I - S_{n+1} = I - S_n - \frac{1}{2}\left(T - S_n^2\right)$$

$$= \frac{1}{2}I - S_n + \frac{1}{2}S_n^2 + \frac{1}{2}(I - T)$$

$$= \frac{1}{2}(I - S_n)^2 + \frac{1}{2}(I - T) \qquad (2)$$

By (1), we get

$$S_n = S_{n-1} + \frac{1}{2}\left(T - S_{n-1}^2\right)$$

$$\therefore S_{n+1} - S_n = S_n - S_{n-1} + \frac{1}{2}\left(S_{n-1}^2 - S_n^2\right)$$

$$= \left[I - \frac{1}{2}(S_n - S_{n-1})\right](S_n - S_{n-1})$$

$$\Rightarrow S_{n+1} - S_n = \frac{1}{2}\left[(I - S_{n-1}) + (I - S_n)\right](S_n - S_{n-1}) \qquad (3)$$

$$(2) \Rightarrow I - S_{n-1} \geq 0$$

$$\Rightarrow S_{n+1} \leq I,\ n = 0,1,2,....$$

$$\therefore S_n \leq I,\ n = 0,1,2,....$$

$$(3) \Rightarrow S_{n-1} - S_n \geq 0 \text{ iff } S_n - S_{n-1} \geq 0,\ n = 1,2,....$$

But, $S_1 = \frac{1}{2}T \geq 0 = S_0$

$$\therefore S_n \leq S_{n+1},\ n = 0,1,2,....$$

$\therefore \{S_n\}$ is a bounded increasing sequence of positive self-adjoint operators

$$\therefore S_1 \leq S_2 \leq S_3 \leq ..... \leq S_n \leq S_{n+1} \leq ....$$

$$\Rightarrow \langle S_1x, x\rangle \leq \langle S_2x, x\rangle \leq ..... \leq \langle S_nx, x\rangle \leq \langle S_{n+1}x, x\rangle \leq .... \forall x \in H$$

$\therefore$ The sequence $\{\langle S_nx, x\rangle\}$ converges to a limit say $\langle Sx, x\rangle, x \in \beta(H)$

$$\therefore \lim_{n\to\infty}\langle S_nx, x\rangle = \langle Sx, x\rangle, \forall x \in \beta(H)$$

$$\Rightarrow \left\langle \lim_{n\to\infty} S_nx, x\right\rangle = \langle Sx, x\rangle, \forall x \in H$$

$$\Rightarrow \left\langle \left(\lim_{n\to\infty} S_n - S\right)x, x\right\rangle = 0\ \forall x \in H$$

$$\Rightarrow \lim_{n\to\infty} S_n - S = 0$$

$\Rightarrow \lim_{n\to\infty} S_n = S$

$\therefore (1) \Rightarrow \lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} S_n + \frac{1}{2}\left(T - \lim_{n\to\infty} S_n^2\right)$

$\Rightarrow S = S + \frac{1}{2}\left(T - S^2\right)$

$\Rightarrow \frac{1}{2}\left(T - S^2\right) = 0$

$\Rightarrow S^2 = T$

$\therefore T$ has a positive square root S.

To show that S is unique.

Let S' be another square root of T.

$\therefore S'^2 = T$

$\therefore S'T = S'S'^2 = S'^3 = S'^2S' = TS'$

$\therefore S'$ commutes with T

$\therefore S$ commutes with S'

i.e $SS' = S'S$

$\therefore (S'S)^2 = S'^2S^2 = T^2 = (SS')^2$

Let $x \in H$ and $y = (S - S')x$

$\langle Sy, y\rangle + \langle S'y, y\rangle$

$= \langle (S + S')y, y\rangle$

$= \langle (S^2 - S'^2)x, y\rangle$

$= \langle (T - T)x, y\rangle$

$= \langle 0, y\rangle$

$= 0$

$\because S$ and S' are positive so,

$\langle Sy, y\rangle \geq 0, \langle S'y, y\rangle \geq 0$

$\therefore \langle Sy, y\rangle = 0,\ \langle S'y, y\rangle = 0$

$\because S$ is a positive operator, so S has a positive square root (say) Q, i.e $Q^2 = S$.

$\therefore Q$ is self-adjoint.

Now, $\|Q(y)\|^2 = \langle Qy, Qy\rangle$

$= \langle Q^*Qy, y\rangle$

$= \langle Q^2 y, y \rangle$

$= \langle Sy, y \rangle$

$= 0$

$\Rightarrow Qy = 0$

$\therefore Sy = Q^2 y = Q(Qy) = Q(0) = 0$

Similarly, $S'y = 0$

$\therefore \|Sx - S'x\|^2 = \langle Sx - S'x, Sx - S'x \rangle$

$= \langle (S - S')x, (S - S')x \rangle$

$= \langle (S - S')^* (S - S')x, x \rangle$

$= \langle (S - S')^2 x, x \rangle$

$= \langle (S - S')(S - S')x, x \rangle$

$= \langle (S - S')y, x \rangle$

$= \langle 0, x \rangle$

$= 0$

$\Rightarrow \|Sx - S'x\| = 0 \ \forall x \in H$

$\Rightarrow Sx - S'x = 0 \ \forall x \in H$

$\Rightarrow Sx = S'x \ \forall x \in H$

$\Rightarrow S = S'$

Let T be invertible.

$\therefore ST = TS$

$\Rightarrow STT^{-1} = TST^{-1}$

$\Rightarrow T^{-1}S(TT^{-1}) = (T^{-1}T)ST^{-1}$

$\Rightarrow T^{-1}S = ST^{-1}$

$\therefore S(T^{-1}S) = S(ST^{-1}) = S^2T^{-1} = TT^{-1} = I$

Similarly, $(T^{-1}S)S = T^{-1}S^2 = T^{-1}T = I$

$\therefore S$ is invertible and $S^{-1} = T^{-1}S$

6.11 Polar decomposition :

Every complex number λ can be factorized in the way $\lambda = \alpha\,|\lambda|$, where $|\alpha| = 1$.

If $T \in \beta(H)$, then if T can be factorized in the way,

$T = UP$, where U is unitary and

$$P \geq 0,$$

then UP is called the polar decomposition of T.

Theorem :

(a) If $T \in \beta(H)$ is invertible, then T has a unique polar decomposition $T = UP$

Proof : T is invertible $\Rightarrow T\,T^{-1} = I = T^{-1}T$

$\Rightarrow (TT^{-1})^* = I = (T^{-1}T)^*$

$\Rightarrow (T^{-1})^*T^* = I = T^*(T^{-1})^*$

$\Rightarrow T^*$ is invertible.

Thus T and T^* are invertible

$\Rightarrow T^*T$ is invertible

Again, T^*T is hermitian.

$\therefore \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0$

$\therefore T^*T$ is positive.

$\therefore T^*T$ has a unique positive square root, say P.

$\therefore P^2 = T^*T$

$\therefore T^*T$ is invertible $\Rightarrow$ P is invertible

Let $U = TP^{-1}$

$\because$ T and P^{-1} are invertible, so U is also invertible.

Now,

$U^*U = (TP^{-1})^*(TP^{-1})$	$P^* = P$
$= (P^{-1})^*\,T^*T\,P^{-1}$	$(P^{-1})^* = (P^*)^{-1}$
$= (P^{-1})^*\,P^2P^{-1}$	$= P^{-1}$
$= P^{-1}\,P\,P\,P^{-1}$	
$= I$	

Similarly, $UU^* = I$

$\therefore$ U is unitary.

Again,

$U = TP^{-1}$

$\Rightarrow T = UP$, P is positive and U is unitary.

Therefore UP is a polar decomposition of T.

To show that U is unique

Let if possible U' be another unitary operator such that

$$T = U'P$$

$$\therefore\ UP = U'P$$

$\Rightarrow (UP)P^{-1} = (U'P)P^{-1}$

$\Rightarrow U = U' \Rightarrow U$ is unique.

$\therefore$ T has a unique polar decomposition.

(b) If $T \in \beta(H)$ is normal, then T has a polar decomposition $T = UP$, in which U and P commute with each other and also with T.

Proof : Let $T \in \beta(H)$ be normal operator.

Let $p(\lambda) = |\lambda|$

$$u(\lambda) = \begin{cases} \dfrac{\lambda}{|\lambda|} & , \quad \lambda \neq 0 \\ 1 & , \quad \lambda = 0 \end{cases}$$

Then p and u are bounded Borel functions on σ (T)

Let $P = p(T)$, $U = u(T)$

Since $p \geq 0$, $P \geq 0$

since $u\bar{u} = 1$, so $UU^* = I$

$\bar{u}u = 1$, so $U^*U = I$

$\because \lambda = u(\lambda)\ p(\lambda)$, so $T = UP$

$\therefore$ T has a polar decomposition.

$\because u(\lambda)\ p(\lambda) = p(\lambda)\ u(\lambda)$, so

$UP = PU$

and since $\lambda u(\lambda) = u(\lambda)\lambda$ and $\lambda p(\lambda) = \rho(\lambda)\lambda$

so $TU = UT$ and $TP = PT$.

Theorem : Let M, N, T $\in \beta(H)$, M and N are normal, T is invertible and $M = TNT^{-1}$.

If T = UP is the polar dcomposition of T, then $M = UNU^{-1}$ [M and N are unitarily equivalent]

Proof : Given that $M = TNT^{-1}$ ......(1)

$\because$ T = UP is the polar decomposition of T, so U is a unitary operator.

Hence form (1), MT = TN

Then, $M^*T = TN^*$

$\therefore T^*M = (M^*T)^* = (TN^*)^*$

$= NT^*$

Again, P is the positive square root of T^*T.

$\therefore NP^2 = NT^*T = T^*MT$

$= T^*TN$

$= P^2N.$

$\therefore$ N commutes with $f(P^2)$ for every $f \in C(\sigma(P^2))$

Now, $\sigma(P^2)$ is a non-empty compact subset of R [$\because P^2$ is positive, $\sigma(p^2) \subset [0, \infty)$]

If $f(\lambda) = \lambda^{1/2}$ on $\lambda(P^2)$, then

$$Nf(P^2) = f(P^2)N$$

$\Rightarrow NP = PN$

$\Rightarrow N = PNP^{-1}$

Now, $M = TNT^{-1} = (UP)N(UP)^{-1}$

$= UPNP^{-1}U^{-1}$

$= UNU^{-1}$

$\Rightarrow M = UNU^{-1}$

Theorem : If A is a B* algebra and if $z \in A$, then there exists a positive functional F on A such that $F(e) = 1$ and $F(zz^*) = \|z\|^2$

Proof : We fix $z \in A$. Let A_r be the real vector space that consists of Hermitian elements of A.

Let P be the set of all $x \in A_r$ such that

$\sigma(x) \subset [0, \infty)$

$\therefore x \in P$ iff $x \geq 0$

Now, $x \in P, y \in P \Rightarrow cx \in P$, where c is a positive scalar

and $x + y \in P$.

Also, P contains all the elements of the form xx^*, for every $x \in A$ ($\because xx^* \geq 0$)

so, in order to prove the theorem, we have to find out a real linear functional 'f' on A_r that satisfies the given conditions and

$f(x) \geq 0, \ \forall \ x \in P$ .....(1)

Let M_0 be the subspace of Ar generated by e and zz*.

We define f_0 on M_0 by

$f_0(\alpha e + \beta zz^*) = \alpha + \beta \|zz^*\|, \alpha, \beta \in R.$

Clearly, f_0 is well-defined and linear.

Also, zz* is a positive element, so

$\rho(zz^*) = \|zz^*\|$

$\therefore \|zz^*\| \in \sigma(zz^*)$

By Spectral mapping theorem,

$\sigma(\alpha e + \beta zz^*) = \alpha + \beta\sigma(zz^*)$

$\therefore \alpha + \beta \|zz^*\| \in \sigma(\alpha e + \beta zz^*)$

$\therefore f_0(x) \in \sigma(x)$, if $x \in M_0$.

$\therefore f_0(x) \geq 0$ if $x \in P \cap M_0$

Also, $f_0(e) = 1$ and $f_0(zz^*) = \|zz^*\|$

$= \|z\|^2$

$[\because$ A is a B* algebra$]$

Let f_0 be extended to a real linear functional f_1 on a subspace M_1 of A_r such that

$f_1(x) \geq 0 \ \forall \ x \in P \cap M_1$

Let $y \in A_r$ be such that $y \notin M_1$

Let $M_2 = < M_1, y >$, is the subspace generated by M_1 and y.

$\therefore M_2 = \{x + y : x \in M_1, \alpha \in R\}$

Let $E' = M_1 \cap (Y - P)$

$E'' = M_1 \cap (Y + P)$

If $x' \in E'$, then $x' \in M_1 \cap (y - P)$

$\Rightarrow x' \in M_1$ and $x' \in y - P$

$\Rightarrow x' \in M_1$ and $y - x' \in P$.

Similarly, if $x'' \in E''$, then $x'' \in M_1 \cap (y + P)$

$\Rightarrow x'' \in M_1$ and $x'' \in y + P$

$\Rightarrow x'' \in M_1$ and $-y + x'' \in P$.

$\therefore (y - x') + (-y + x'') \in P$

$\Rightarrow x'' - x' \in P.$

$\Rightarrow x'' - x' \in M_1 \cap P.$

$\therefore f_1(x'' - x') \geq 0 \Rightarrow f_1(x'') - f_1(x') \geq 0$

$\Rightarrow f_1(x') \leq f_1(x'').$

Let c be a real number that satisfies

$f_1(x') \leq c \leq f_1(x'')$ ....(2)

We define, f_2 on M_2 by $f_2(x + \alpha y) = f_1(x) + \alpha C$,

$x \in M_1, \alpha \in R.$

Since f_1 is linear, so f_2 is also linear.

For $x \in M_1$, if $x + y \in P$, then $- x \in y - P$

$\Rightarrow -x \in M_1 (y - P)$

$\Rightarrow -x \in E'$

$\Rightarrow f_1(-x) \leq c$ [by (2)]

$\Rightarrow f_1(x) \geq - c$

$\therefore f_2(x + y) = f_1(x) + c \geq - c + c = 0$

$\Rightarrow f_2(x + y) \geq 0$

Again, if $x - y \in P$, then $x \in y + P$.

$\Rightarrow x \in M_1 \cap (y + P)$

$\Rightarrow x \in E''$

$\Rightarrow f_1(x) \geq c.$

$\therefore f_2(x - y) = f_1(x) - c \geq c - c = 0$

$\Rightarrow f_2(x - y) \geq 0$

$\therefore f_2(x + \alpha y) \geq 0 \ \forall \ x + \alpha y \in P \cap M_2$

If $M_2 = A_r$, then $P \cap M_2 = P$, and taking

$f = f_2$, we get $f(x) \geq 0 \ \forall \ x \in P$

i.e. (1) holds.

If $M_2 \neq A_r$, then proceeding in the same way, and finally applying zorn's Lomma, in the partially ordered set of all classes (f_i, M_i), we can conclude that there exists a real linear functional f on Ar such that $f(x) \geq 0 \ \forall \ x \in P$

and $f(e) = 1$, $f(zz^*) = \|z\|^2$

Any $x \in A$ can be put in the form $x = u + iv$, $u, v \in A_r$.

We define F on A such that

$F(x) = f(u) + i\, f(v)$, i f $x = u + iv$.

Let $y = u' + iv'$

$$\begin{aligned} \therefore\ F(x+y) &= F((u+iv)+(u'+iv')) \\ &= F((u+u')+i(v+v')) \\ &= f(u+u') + i\, f(v+v') \\ &= f(u)+f(u')+i[f(v)+f(v')] \\ &= [f(u)+i\, f(v)] + [f(u')+i\, f(v')] \\ &= F(x)+F(y) \end{aligned}$$

$$\begin{aligned} \text{Again, } F(ix) = F(iu - v) &= f(-v) + i\, f(u) \\ &= -f(v) + i\, f(u) \\ &= i\,[f(u) + i\, f(v)] \\ &= i\, F(x) \end{aligned}$$

$\therefore$ F is linear.

$$\begin{aligned} \text{Now, } xx^* &= (u+iv)\,)u - iv) \\ &= u^2 + v^2 + i\,(vu - uv) \end{aligned}$$

$$\begin{aligned} \therefore\ F(xx^*) &= f(u^2+v^2) + i\, f(vu - uv) \\ &= f(u^2) + f(v^2) + i\, f[vu - (vu)^*] \end{aligned}$$

$[\because\ (vu)^* = u^*v^* = uv]$

$$\begin{aligned} &= f(uu^*) + f(vv^*) + i[f(vu) - f[(vu)^*) \\ &= f(uu^*) + f(vv^*) + i[f(vu) - f(vu)] \\ &= f(uu^*) + f(vv^*) \geq 0 \end{aligned}$$

$[\because\ f((vu)^*) = \overline{f((vu))} = f(vu)]$

$\therefore$ F is a positive functional

Hence $F(e) = 1$, $F(zz^*) = ||z||^2$

6.12 Representation :

Let A be an involutive algebra and H, a Hilbert space.

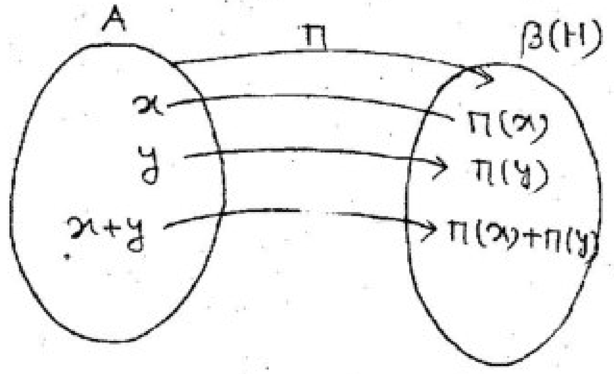
A repesentation of A in H is a mapping Π of A into $\beta(H)$ such that

$\Pi\,(x+y) = \Pi(x) + \Pi(y)$

$\Pi(xy) = \Pi(x)\,\Pi(y)$

$\Pi(x) = \Pi(x)$

$\Pi(x^*) = \Pi(x)^*$, x, y $\in$ A, $\lambda \in$ C.



State : Let A be an involutive algebra with a norm. A state of A is a continuous positive linear functional 'f' on A such that $||f|| = 1$

Theorem : If A is a B* algebra and if u $\in$ A, u $\neq$ 0; then there exists a Hilbert space Hu and a homomorphism T_u of A into $B(H_u)$ that satisfies $T_u(e) = I$

1. $T_u(x^*) = T_u(x)^*$, $x \in A$

2. $||T_u(x)|| \leq ||x||$, $\forall\ x \in A$

and $||T_u(u)|| = ||u||$

Proof : We take 'u' as a finxed element.

Then $\exists$ a positive functional F on A such that

$F(e) = 1$ and $F(uu^*) = ||u||^2$ ....(3)

Let $Y = \{y \in A : F(xy) = 0,\ \forall\ x \in A\}$ ....(4)

$\because$ F is continuous, so Y is a closed subspace of A.

We denote the elements of A/Y be

$$x' = x + Y,\ x \in A. \quad .....(5)$$

We claim that $< a', b' > = F(b^*, a)$ ....(6)

defines an inner product on A/Y.

Remark :

Suppose $(a', b') = 0$

$\Rightarrow a' = 0$ or $b' = 0$

$\Rightarrow a \in Y$ or $b \in Y$.

$\therefore <a', b'> = 0$

$\Rightarrow F(b^*a) = 0$

To show that $< a', b' >$ is well =defined, we have to show that $F(b^*a) = 0$, if at least one of a and b lies in Y.

If $a \in Y$, then by definition, $F(b^*a) = 0$

If $b \in Y$, then $F(a * b) = 0$

$$\Rightarrow \overline{F((a * b)^*)} = 0$$

$$\Rightarrow \overline{F(b^*a)} = 0$$

$$\Rightarrow F(b^*a) = 0$$

$\therefore \langle a', b' \rangle$ is well-defined.

Again,

(i) $\langle a_1', b' \rangle + \langle a_2', b' \rangle = F(b^*a_1) + F(b^*a_2)$

$$= F(b^*a_1 + b^*a_2)$$

$$= F(b^*(a_1 + a_2))$$

$$= \langle (a_1 + a_2)', b' \rangle$$

$$= \langle a_1' + a_2', b' \rangle$$

(ii) $\langle a', b' \rangle = F(b^*a) = \overline{F(a^*b)} = \overline{\langle b', a' \rangle}$

(iii) $\langle a', a' \rangle = F(a^*a) \geq 0$

(iv) $\langle a', a' \rangle = 0 \Leftrightarrow F(a^*a) = 0$

$$\Leftrightarrow F(xa) = 0 \;\; \forall \; x \in A$$

$$\Leftrightarrow a \in Y$$

$$\Leftrightarrow a + Y = Y$$

$$\Leftrightarrow a' = 0$$

$\therefore \langle a', a' \rangle = 0 \Leftrightarrow a' = 0$

$\therefore \langle a', b' \rangle = F(b^*a)$ is a well-defined inner product on A/Y.

$\therefore \|a'\| = \langle a', a' \rangle^{\frac{1}{2}} = F(a^*a)^{\frac{1}{2}}$

Let H be the completion of A/Y w.r.t this norm. Then H is a Hilbert space.

For each $x \in A$, we define

$T_x : A/Y \rightarrow A/Y$ such that

$T_x(a') = (xa)'$

i.e. $T_x(a + Y) = xa + Y$

Then T_x is well-defined.

Now,

(i) $T_x(a' + b') = (x(a + b))'$

$$= x(a + b) + Y$$

$$= (xa + Y) + (xb + Y)$$

$= (xa)' + (xb)'$

$= T_x(a') + T_x(b')$

(ii) $T_{x_1+x_2}(a') = ((x_1 + x_2)a)'$

$= (x_1 + x_2)a + Y$

$= (x_1a + Y) + (x_2a + Y)$

$= T_{x_1}(a') + T_{x_2}(a')$

$= (T_{x_1} + T_{x_2})(a')$

$\Rightarrow T_{x_1+x_2} = T_{x_1} + T_{x_2}$

(iii) $T_{x_1x_2}(a') = ((x_1x_2)a)'$

$= (x_1x_2)a + Y$

and $T_{x_1}(T_{x_2}(a') = T_{x_1}((x_2a)')$

$= (x_1(x_2a))'$

$= x_1(x_2a) + Y$

$= (x_1x_2)a + Y$

$\therefore T_{x_1x_2} = T_{x_1}T_{x_2}$

(iv) $T_{\alpha x}(a') = ((\alpha x)a)'$

$= (\alpha x)a + Y$

$= (\alpha xa + Y)$

$= (\alpha T_x)(a')$

$\Rightarrow T_{\alpha x} = \alpha T_x$.

and

$T_e(a') = (ea)' = ea + Y = a + Y = a' = I(a')$

$\Rightarrow T_e = I$

We define,

$\phi : A \to \beta(A/Y)$ by $\phi(x) = T_x$

$\therefore \phi$ is well defined.

Hence,

$\phi(x + y) = T_{x+y} = T_x + T_y = \phi(x) + \phi(y)$

$\phi(\alpha x) = T_{\alpha x} = \alpha T_x = \alpha\phi(x)$

$\phi(xy) = T_{xy} = T_x T_y = \phi(x)\phi(y)$

$\therefore \phi$ is a homomorphism.

$\therefore \|T_x(a')\|^2 = \|(xa)'\|^2$

$= < (xa)', (xa)' >$

$= F(xa) * (xa))$

$= F(a^*x^*xa)$ ....(7)

For a fixed $a \in A$, we define

$$G(x) = F(a^*xa)$$

Then G is a positive functional on A and

$G(x^*x) \leq G(e)\, \|x\|^2$

$$\|T_x(a')\|^2 = G(x^*x)$$

$$\leq G(e)\, \|x\|^2$$

$$= F(a^*a)\, \|x\|^2$$

$$= < a', a' > \|x\|^2$$

$$= \|a'\|^2 \,.\, \|x\|^2$$

$\Rightarrow \|T_x(a')\| \leq \|a'\|\, \|x\|$

$\Rightarrow \|T_x\| \leq \|x\|, \ \forall \ x \in A.$

Again,

$$< T_{x^*}(a'), b' > = < (x^*a)', b' >$$

$$= F(b^*\, x^*a)$$

$$= F((xb)^*\, a)$$

$$= < a/\ (xb)/ >$$

$$= (a/\ T_x(b/) >$$

$$= < T_x^*\ (a/), b/ >$$

$\Rightarrow T_{x^*} = T_x^*$ ....(8)

Since A/γ is dense in H, so

$(T_x^*)\ (a') = (Tx)^*\ (a'), \ \forall \ T_x \in \beta(H)$

i.e. $T_{x^*} = T_x^*$ and $\|T_x\| \leq \|x\|, \ \forall \ x \in A,$

$$T_x \in \beta(H)$$

Replacing T_x by $T_u(x)$, we get

(i) $T_e = T_u(e) = I$

(ii) $T_u\ (x^*) = T_u\ (x)^*, \ x \in A$

(iii) $\|T_u(x)\| \leq \|x\|, \ x \in A.$

(iv) $\|u\|^2 = F\ (u^*u)$

$$= F((ue)^*\ ue)$$

$= \|T_u(e')\|^2$ [using (7)]

$\leq \|T_u\|^2 \|e'\|^2$

$\{\|e'\|^2 = F(e^*e) = F(e) = 1\}$

$\therefore \|u\|^2 \leq \|T_u\|^2$

$\Rightarrow \|u\| \leq \|T_u\|$

But,

$\|T_x\| \leq \|x\|, \ \forall \ x \in A$

$\therefore \|T_u\| \leq \|u\|$

$\therefore \|T_u(u)\| = \|u\|$

Theorem : Let A be a B* algebra. Then ∃ an isometric *isomorphism of A onto a closed subalgebra of β(H) where H is a suitably chosen Hilbert space.

Proof : Let u be an arbitrary element of A ($u \neq 0$). Then u gives rise to a Hilbert space H_u.

Let H be the direct sum of all the Hilbert spaces H_u, $u \in A$.

Let $\pi_u(v)$ be the H_u coordinate of an element 'v' of the cartesian product of the spaces Hu.

Then, by definition,

$v \in H$ if and only if $\sum_u \|\pi_u(v)\|^2 < \infty$ ....(1)

where $\|\pi_u(v)\|$ denotes the H_u-norm of $\pi_u(v)$.

The convergence of (1) implies that at most countably many $\pi_u(v)$'s are different from 0.

The inner product in H is given by

$$< v', v'' > = \sum_u < \pi_u(v'), \pi_u(v'') > \text{ ....(2)}$$

$$v', v'' \in H.$$

Now,

(i) $< v, v > = \sum_u < \pi_u(v), \pi_u(v) >$

$= \sum_u \|\pi_u(v)\|^2$

$= \|v\|^2$

$\therefore < v, v > = 0 \Leftrightarrow \|v\|^2 = 0 \Leftrightarrow \|v\| = 0 \Leftrightarrow v = 0.$

(ii) $\overline{< v, v' >} = \overline{\sum_u < \pi_u(v), \pi_u(v')>}$

$= \sum_u < \pi_u(v'), \pi_u(v)>$

$= < v', v >$

(iii) $< \alpha v', v'' > = \sum_u < \pi_u(\alpha v'), \pi_u(v'')>$

$= \sum_u < \alpha\, \pi_u(v'), \pi_u(v'') >$

$= \alpha \sum_u < \pi_u(v'), \pi_u(v'') >$

$= \alpha < v', v'' >$

(iv) $< v' + v'', v''' > = \sum_u < \pi_u(v' + v''), \pi_u(v''') >$

$= \sum_u < \pi_u(v') + \pi_u(v''), \pi_u(v''') >$

$= \sum_u < \pi_u(v'), \pi_u(v''') >$

$+ \sum_u < \pi_u(v''), \pi_u(v''') >$

$= < v', v''' > + < v'', v''' >$

Also, H is complete. So, H is a Hilbert space.

Let $S_u \in \beta(H_u)$ and let $\|S_u\| \leq M, \ \forall\ u$

Let S_v be defined as the vector whose co-ordinate in H_u is

$\pi_u(S_v) = S_u\, \pi_u(v)$ ....(3)

If $v \in H$, then

$$\sum_u \| \pi_u(v)\|^2 < \infty$$

$$\therefore \sum_u \|S_u\, \pi_u(v)\|^2 \leq \sum_u \|S_u\|^2\, \|\pi_u(v)\|^2$$

$$\leq M^2 \sum_u \|\pi_u(v)\|^2$$

$$< \infty$$

Using (3),

$$\sum_u \|\pi_u(S_v)\|^2 < \infty$$

$\therefore S_v \in H$ if $v \in H$.

Again,

$$\|S_v\|^2 = \sum_u \|\pi_u(S_v)\|^2$$

$$= \sum_u \|S_u\, \pi_u(v)\|^2$$

$$\leq \sum_u \|S_u\|^2\, \|\pi_u(v)\|^2$$

$$\leq \sum_u \{\sup_u \|S_u\|\}^2 \|\pi_u(v)\|^2$$

$$= \{\sup_u \|S_u\|\}^2 \sum_u \|\pi_u(v)\|^2$$

$$= \{\sup_u \|S_u\|\}^2 \|v\|^2$$

$$\Rightarrow \|S_v\| \leq \sup_u \|S_u\| \cdot \|v\|$$

$$\Rightarrow \|S\| \leq \sup_u \|S_u\|$$

Let $u \in H_u$ with $\|u\| = 1$

Now,

$$\|S\| \geq \|S_u\|$$

$$= \sqrt{\sum_u \|\pi_u(S_u)\|^2}$$

$$= \sqrt{\sum_u \|S_u\ \pi_u(u)\|^2}$$

$$= \sqrt{\|S_u\ (u)\|^2}$$

$$= \|S_u(u)\|$$

$\Rightarrow \|S_u(u)\| \leq \|s\|\ \forall\ u \in H_u$ with $\|u\| = 1$

$\therefore \sup \{\|S_u(u)\| : \|u\| = 1\} \leq \|s\|$

$\Rightarrow \|S_u\| \leq \|s\|\ \forall\ u$

$$\therefore \sup_u \|S_u\| \leq \|S\|$$

$$\therefore \|S\| = \sup_u \|S_u\| \quad ....(4)$$

Again, for $u\ (\neq 0) \in A$, $\exists$ a homomorphism

$T_u : A \rightarrow \beta\ (H_u)$

To each $x \in A$, we associate an operator

$T(x) \in \beta(H)$ by

$\pi_u\ (T(x)v) = (T_u(x))\ (\pi_u(v))$

Now,

(i) $\pi_u(T(x_1 + x_2)v) = (T_u(x_1 + x_2))\ (\pi_u(v))$

$= [T_u(x_1) + T_u(x_2)]\ (\pi_u(v))$

$= (T_u(x_1))\ (\pi_u(v)) + T_u(x_2))\ (\pi_u(v))$

$= \pi_u\ (T_u(x_1)v) + \pi_u\ (T(x_2)v)$

Similarly,

(ii) $\pi_u\ (T\ (\alpha x)v) = \alpha\pi_u(T(x)v)$

Again

(iii) $\pi_u(T(x)\ (v_1 + v_2)) = T_u(x)(\pi_u(v_1 + v_2))$

$= (T_u(x)) [\pi_u(v_1) + \pi_u (v_2)]$

$= (T_u(x)) (\pi_u (v_1) + (T_u(x))$
$(\pi_u(v_2))$

$= \pi_u (T(x)v_1) + \pi_u((Tx)v_2)$

(iv) $\pi_u(T(x) (\alpha v)) = \alpha\pi_u ((Tx)v)$

We define

$\phi : A \to \beta(H)$ by

$\phi(x) = T(x) \ \forall \ x \in H$

Now,

(i) $\phi(x_1 + x_2) = T(x_1 + x_2)$
$= T(x_1) + T(x_2)$
$= \phi(x_1) + \phi(x_2)$

(ii) $\phi(\alpha x) = T(\alpha x)$
$= \alpha T(x)$
$= \alpha\phi(x)$

Again,

$\|T_u(x)\| \leq \|x\| = \|T_x(x)\|$

$\therefore \|T(x)\| = \sup_u \|T_u(x)\|$ [by (4)]
$\leq \sup_u \|x\| = \|x\|$

$\Rightarrow \|T(x)\| \leq \|x\|$

Again,

$\|T(x)\| = \sup_u \|Tu(x)\|$
$\geq \|T_x(x)\| = \|x\|$

$\Rightarrow \|T(x)\| \geq \|x\|$

$\therefore \|T(x)\| = \|x\|$

$\Rightarrow \|\phi(x)\| = \|x\|$

Also, $\phi(x_1) = 0 \Rightarrow \|\phi(x_1) = 0$
$\Rightarrow \|x_1\| = 0$
$\Rightarrow x_1 = 0 \Rightarrow \phi$ is one-one

$\therefore \phi$ is a isometry

Again,

$\phi(x^*) = T(x^*)$
$= T(x)^*$
$= \phi(x)^*$

$\therefore \phi$ is an isometric * isomorphism of A onto a closed subalgebra of $\beta(H)$.
